PICTURE FUZZY SOFT-HYPERSOFT SETS, INFORMATION MEASURES & AGGREGATION OPERATORS IN DECISION-MAKING APPLICATIONS

Thesis submitted in partial fulfillment of the requirements for the Degree of

DOCTOR OF PHILOSOPHY

IN

MATHEMATICS

BY

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* * *

Dedicated to

My beloved parents and brother,

This work is a testament to your belief in me.

Thank you !!

* * *

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DECLARATION BY THE SCHOLAR

I hereby declare that the work reported in the Ph.D. thesis entitled, **"Picture Fuzzy Soft-Hypersoft Sets, Information Measures & Aggregation Operators in Decision-Making Applications"** submitted at **Jaypee University of Information Technology, Waknaghat, India**, is an authentic record of my work carried out under the supervision of **Prof. (Dr.) Rakesh Kumar Bajaj**. I have not submitted this work elsewhere for any other degree or diploma. I am fully responsible for the content of my Ph.D. Thesis.

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SUPERVISOR'S CERTIFICATE

This is to certify that the thesis entitled, **"Picture Fuzzy Soft-Hypersoft Sets, Information Measures & Aggregation Operators in Decision-Making Applications"** submitted by **Himanshu Dhumras** at **Jaypee University of Information Technology, Waknaghat, India**, is bonafide record of his original work carried out under my supervision. This work has not been submitted elsewhere for any other degree or diploma.

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(Himanshu Dhumras)

Abstract

The thesis titled "*Picture Fuzzy Soft-Hypersoft Sets, Information Measures & Aggregation Operators in Decision-Making Applications*" explores picture fuzzy information within the framework of soft-hypersoft sets and their applications in decision-making using various information measures and aggregation operators. It introduces extensions of picture fuzzy sets, including bi-parametric discriminant measures, picture fuzzy soft sets, picture fuzzy hypersoft sets/matrices, and q-rung picture fuzzy sets, with applications in hydrogen fuel cell technology, sustainable agrifarming, renewable energy source selection, and green supply chain management. The thesis begins with a comprehensive background on picture fuzzy sets and their extensions, including definitions, operations, and a literature survey. A bi-parametric picture fuzzy discriminant measure is proposed, mathematically validated, and integrated with modified VIKOR and TOPSIS methods to assess hydrogen fuel cell technologies. Modified picture fuzzy soft Dombi aggregation operators and their algebraic properties are introduced and applied within the EDAS methodology to prioritize factors for sustainable agrifarming. Furthermore, the concept of picture fuzzy hypersoft sets and similarity measures is developed, with the proposed properties validated through numerical illustrations and comparative analyses. Picture fuzzy hypersoft matrices are constructed to organize information, and new choice and value matrices are introduced to address renewable energy source selection problems. In addition this, a modified q-rung picture fuzzy AHP/WASPAS methodology is presented, overcoming restrictions on uncertainty components. This methodology is applied to green supply chain management for strategic planning in the energy sector. The thesis concludes by summarizing its findings and contributions, highlighting the theoretical advancements and practical applicability of the proposed methodologies. Additionally, potential directions for future work are discussed, including further generalizations of picture fuzzy hypersoft sets and their applications to more complex multi-criteria decision-making problems across diverse domains.

Keywords: Picture Fuzzy Sets, Soft Sets, Hypersoft Sets, Aggregation Operators, Information Measures, q-Rung Picture Fuzzy Sets, Sustainable Development, Renewable Energy, Green Supply Chain Management, Multi-Criteria Decision-Making.

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Journal Publications

1. H. Dhumras, R. K. Bajaj, "On prioritization of hydrogen fuel cell technology utilizing bi-parametric picture fuzzy information measures in VIKOR & TOPSIS decision-making approaches", *International Journal of Hydrogen Energy*, vol. 48, no. 96, pp. 37981-37998, 2022. **(SCIE (Q1), SCOPUS, IF-8.1)**

2. H. Dhumras, R. K. Bajaj, "Modified EDAS method for MCDM in robotic agri-farming with picture fuzzy soft dombi arithmetic-geometric aggregation operators", *Soft Computing*, vol. 27, no. 8, pp. 5077-5098, 2023. **(SCIE (Q2), SCOPUS, IF-3.1)**

3. H. Dhumras, R. K. Bajaj, "On various aggregation operators for picture fuzzy hypersoft information in decision making application", *Journal of Intelligent & Fuzzy Systems*, vol. 44, no. 5, pp. 7419-7447, 2023. **(SCIE (Q3), SCOPUS, IF-1.7)**

4. H. Dhumras, R. K. Bajaj, "On Renewable Energy Source Selection Methodologies Utilizing Picture Fuzzy Hypersoft Information with Choice and Value Matrices", *Scientia Iranica*, 2022. https://doi.org/10.24200/SCI.2022.60529.6847. **(SCIE (Q2), SCOPUS, IF-1.4)**

5. H. Dhumras, R. K. Bajaj, "On potential strategic framework for green supply chain management in the energy sector using q-rung picture fuzzy AHP & WASPAS decision-making model", *Expert Systems with Applications*, vol. 237, no. B, p. 121550, 2023. **(SCIE (Q1), SCOPUS, IF-7.5)**

Conference Publication

- **H. Dhumras** and R. K. Bajaj, "On some new similarity measures for picture fuzzy hypersoft sets with application in medical diagnosis", *in Proceedings of the 2nd International Conference, EmergentConverging Technologies and Biomedical Systems (ETBS 2022)*, JUIT, Solan, India, 2023, 119–130 **(SCOPUS)**.

Conference Paper Presented

1. **International Conference on Recent Trends in Mathematical Sciences (ICRTMS - 2021)**, presented the paper titled *"Robust Aggregation Operators Under Picture Fuzzy Environment with Application in Decision-Analysis"* held in **Himachal Pradesh University, Shimla,** India, during 11 - 12 December, 2021.

2. **International Conference on Dynamical Systems, Control and Their Applications**, presented the paper titled *"Machine learning Decision making processes using novel picture fuzzy (R-S)-Norm Discriminant Measure"* held in **Indian Institute of Technology, Roorkee,** India, during 1 - 3 July, 2022.

3. **Second International Conference on Emergent Converging Technologies & Biomedical Systems (ETBS- 2022)**, presented the paper titled *"On Some New Similarity Measures for Picture Fuzzy Hypersoft Sets with Application in Medical Diagnosis"* held in **Jaypee University of Information Technology, Solan,** India, during 23 - 24 September, 2022.

Chapter 1

Introduction

Decision-making is an essential component of human behavior, affecting a broad range of sectors from business, economics, health care and engineering. The procedure that involves a selection of alternatives is very important for organizational success, individual growth and social advancement. In today's constantly changing world, making effective decisions has become very difficult due to the plethora of information, the pressing need to make choices, and the existence of information involving a higher amount of vagueness and ambiguity. The significance of making effective decisions cannot be overestimated. Industries as well as individuals face various difficult situations, which include the necessity to strike a balance between conflicting objectives, handle threats, and make decisions based on imprecise and inexact information. Therefore, enhancing strong decision-making structures and algorithms is very crucial for decisive outcomes and accomplishing strategic objectives. For human life survival, business development, promotion advancements, etc. are fully dependable on the ability to make decisions. To select the best possible alternative among the others, the conflicting criteria under the assessment of one or more experts are utilized in multi-criteria decision-making problems. Also, the decision-makers managed a sustainable supply chain management for the utilization of agricultural products with the help of blockchain technology [1], [2]. In addition to this, there is a development of various techniques for the production of hydrogen-based technologies which is beneficial for the environment [3]. The group decision-making models have also been applied for the enhancement of the performance of school students with the utilization of traditional techniques [4].

In recent years, the area of decision-making has changed dramatically over time with the transition from classical decision theory to behavioral decision-making which overcomes the shortcomings of human cognition. Further, advancements in artificial intelligence techniques and computational procedures have played a great role in developing a comprehensive support system for decision-making. In real-world problems, the route towards decision-making is heavily affected by its advantages and dependability on our past knowledge as well as opinions. Also, due to the information deficiencies and the risk of human errors, it is likely believed to have inexact and incomplete knowledge of the systems. As a result of this, it seems to be very difficult to arrive at the best possible choice at a designated time. As the intricacies are growing rapidly, experts face numerous challenges to select promptly by utilizing the given vague and ambiguous information.

1.1 Fundamental Notions and Preliminaries

In this section, some fundamental definitions concerning the picture fuzzy set, picture fuzzy soft set, and picture fuzzy hypersoft set along with their operational laws have been presented as follows.

1.1.1 Picture Fuzzy Set

The idea of fuzzy sets, developed by Zadeh [5] in 1965, drastically changed the way of handling imprecise and vague information. The traditional fuzzy sets give the value of the degree of membership between 0 and 1 for handling uncertain information. Then, Atanassov [6] in 1986, developed one more uncertainty component i.e. degree of non-membership for representing the uncertainty with the inclusion of hesitancy. However, some decision-making situations require a more thorough computational structure. Picture fuzzy sets [7] were introduced to overcome these shortcomings with the incorporation of three uncertainty components: positive membership (ρ), neutral membership (τ) and negative membership (ω). This comprehensive framework is very helpful in situations where the degree of neutrality has a significant role.

Definition 1 *[7]: "A picture fuzzy set (PFS) U in X (universe of discourse) is given by*

$$U = \{< x, \rho_U(x), \tau_U(x), \omega_U(x) >| \ x \in X\} ;$$

where $\rho_U : X \to [0, 1]$, $\tau_U : X \to [0, 1]$ and $\omega_U : X \to [0, 1]$ denotes the degree of positive membership, degree of neutral membership and degree of non-membership respectively and for every $x \in X$ satisfy the condition

$$0 \leq \rho_U(x) + \tau_U(x) + \omega_U(x) \leq 1$$

and the degree of refusal for any picture fuzzy set U and $x \in X$ is given by $\theta_U(x) = 1 - \rho_U(x) - \tau_U(x) - \omega_U(x)$".

The constraint on the degree of membership $\rho_U(x)$, neutral membership $\tau_U(x)$ and non-membership $\omega_U(x)$ is

$$0 \leq \rho_U(x) + \tau_U(x) + \omega_U(x) \leq 1.$$

Definition 2 *[7]: "If $U, V \in PFS(X)$, then the operations can be defined as follows:*

(a) ***Complement:*** $\overline{U} = \{< x, \omega_U(x), \tau_U(x), \rho_U(x) > | \ x \in X\}$*;*

(b) ***Subsethood:*** $U \subseteq V$ *iff* $\forall x \in X, \rho_U(x) \leq \rho_V(x) \ \ \tau_U(x) \geq \tau_V(x)$ *and* $\omega_U(x) \geq \omega_V(x)$*;*

(c) ***Containment:*** $U \supseteq V$ *iff* $\forall x \in X, \rho_U(x) \geq \rho_V(x) \ \ \tau_U(x) \leq \tau_V(x)$ *and* $\omega_U(x) \leq \omega_V(x)$*;*

(d) ***Union:*** $U \cup V = \{< x, \rho_U(x) \vee \rho_V(x), \tau_U(x) \wedge \tau_V(x)$ *and* $\omega_U(x) \wedge \omega_V(x) > | \ x \in X\}$*;*

(e) ***Intersection:*** $U \cap V = \{< x, \rho_U(x) \wedge \rho_V(x), \tau_U(x) \vee \tau_V(x)$ *and* $\omega_U(x) \vee \omega_V(x) > | \ x \in X\}$*."*

1.1.2 Picture Fuzzy Soft Set

For dealing with the parametrization of uncertain information, Molodstov introduced the notion of soft set which offers a flexible structural framework. The combination of picture fuzzy sets and soft sets provides a robust structure capable of handling more complex decision-making situations.

Let $U = \{u_1, u_2, \ldots, u_m\}$ be the universe of discourse and $P = \{p_1, p_2, \ldots, p_n\}$ be the set of parameters. The pair (Φ, P) is called

- "***Soft Set*** [8] over U iff $\Phi : P \to \mathcal{P}(U)$, where $\mathcal{P}(U)$ is the power set of U."
- "***Fuzzy Soft Set*** [9] over $\Phi(U)$, where Φ is a mapping given by $\Phi : P \to (F(U))$ and $F(U)$ denotes the set of all fuzzy sets of U."
- "***Intuitionistic Fuzzy Soft Set*****(IFSS)** [10] over U if $\Phi : P \to IFS(U)$ and can be represented as

  $$(\Phi, P) = \{(p, \Phi(p)) : p \in P, \Phi(p) \in IFS(U)\},$$

  where $IFS(U)$ represents the set of all IFSs of U."
- "***Picture Fuzzy Soft Set*****(PFSS)** [11] over U if $\Phi : P \to PFS(U)$ and can be represented as

  $$(\Phi, P) = \{(p, \Phi(p)) : p \in P, \Phi(p) \in PFS(U)\},$$

  where $PFS(U)$ represents the set of all PFSs of U."

Definition 3 *[12] Let (Φ, Q) and (Ψ, M) be two picture fuzzy soft sets on the same universe of discourse U. Let $Q, M \subseteq P$ be the set of parameters, then*

- **Complement** $(\Phi, Q)^c = (\Phi^c, Q)$ *where $\Phi^c : Q \to TSFS(U)$ is a mapping given by $\Phi^c(p) = (\Phi(p))^c$, for all $p \in Q$.*
- **Subsethood:** $(\Phi, Q) \subseteq (\Psi, M)$, *iff $Q \subseteq M$ and for all $p \in Q$, $\Phi(p) \subseteq \Psi(p)$.*
- **Equality:** $(\Phi, Q) = (\Psi, M)$, *if $(\Phi, Q) \subseteq (\Psi, M)$ and $(\Psi, M) \subseteq (\Phi, Q)$.*
- **Union:** $(\Phi, Q) \cup (\Psi, M) = (H, S)$; *where $S = Q \cup M$ for all $\xi \in S$ and*

$$H(\xi) = \begin{cases} \Phi(\xi) & \xi \in Q - M, \\ \Psi(\xi) & \xi \in M - Q, \\ \Phi(\xi) \cup \Psi(\xi) & \xi \in Q \cap M. \end{cases}$$

In other words, for all $\xi \in Q \cap M$,

$$H(\xi) = \left\{ \left(u, \max(\mu_{\Phi(\xi)}(u), \mu_{\Psi(\xi)}(u)), \min(\eta_{\Phi(\xi)}(u), \eta_{\Psi(\xi)}(u)), \min(\nu_{\Phi(\xi)}(u), \nu_{\Psi(\xi)}(u))\right) \right\}.$$

- **Intersection:** $(\Phi, Q) \cap (\Psi, M) = (H, S)$; *where* $S = Q \cap M$ *for all* $\xi \in S$ *and*

$$H(\xi) = \begin{cases} \Phi(\xi) & \xi \in Q - M, \\ \Psi(\xi) & \xi \in M - Q, \\ \Phi(\xi) \cap \Psi(\xi) & \xi \in Q \cap M. \end{cases}$$

In other words, for all $\xi \in Q \cap M$,

$$H(\xi) = \left\{ \left(u, \min(\mu_{\Phi(\xi)}(u), \mu_{\Psi(\xi)}(u)), \min(\eta_{\Phi(\xi)}(u), \eta_{\Psi(\xi)}(u)), \max(\nu_{\Phi(\xi)}(u), \nu_{\Psi(\xi)}(u))\right) \right\}.$$

Definition 4 *[12] "Suppose* (Φ, Q) *and* (Ψ, M) *are two Picture fuzzy soft sets on the universal set* U*. Let* $Q, M \subseteq P$ *be two subsets of the set of parameters, then as per their definitions, the following properties hold:*

(i) $((\Phi, Q)^c)^c = (\Phi, Q)$.

(ii) $((\Phi, Q) \cap (\Psi, M))^c = (\Phi, Q)^c \cup (\Psi, M)^c$.

(iii) $((\Phi, Q) \cup (\Psi, M))^c = (\Phi, Q)^c \cap (\Psi, M)^c$*."*

Dombi [13] has recommended a particular kind of operation called triangular norm / conorm whose definitions are given below:

Definition 5 *[13] "Let* r *and* s *be any two real numbers. Then, Dombi* $t-norms$ *and* $t-conorms$ *are defined as:*

$$Dom(r, s) = \frac{1}{1 + \left\{ \left(\frac{1-r}{r}\right)^R + \left(\frac{1-s}{s}\right)^R \right\}^{\frac{1}{R}}}$$

$$Dom^*(r, s) = 1 - \frac{1}{1 + \left\{ \left(\frac{r}{1-r}\right)^R + \left(\frac{s}{1-s}\right)^R \right\}^{\frac{1}{R}}}$$

where, $R \geq 1$ *and* $(r, s) \in [0, 1] \times [0, 1]$*."*

1.1.3 Picture Fuzzy HyperSoft Set

The concept of hypersoft sets enhanced the classical soft set structure by allowing for multi sub-attribute computation, by giving a far more structural representation and detailing of data. Also, the blending of hypersoft sets with different extensions of fuzzy sets further allows for the inclusion of uncertainty components which is very crucial while dealing with vague and ambiguous information in real-life decision-making problems.

Definition 6 ***Hypersoft Set (HSS) [14].*** *"Let V be the universal set and P(V) be the power set of V. Consider* $k_1, k_2, \ldots . k_n$ for $n \geq 1$, *be* n *well-defined attributes, whose corresponding attribute values are the sets* $K_1, K_2, \ldots, K_n$ with $K_i \cap K_j = \varphi$ for $i \neq j$ and $i, j \in \{1, 2, \ldots, n\}$. *Let* B_i *be the non-empty subsets of* K_i *for each* $i = 1, 2, \ldots, n$. *Then the pair* $(R, B_1 \times B_2 \times \ldots . B_n)$ *is said to be Hypersoft Set over* V *where* $R : B_1 \times B_2 \times \ldots . \times B_n \rightarrow P(V)$. *In other words, the Hypersoft Set is a multi-parameterized family of subsets of the set V."*

Definition 7 ***Fuzzy Hypersoft Set (FHSS) [14].*** *"Let V be the universal set and* $F(V)$ *be the set all Fuzzy subsets of V. Consider* $k_1, k_2, \ldots . k_n$ for $n \geq 1$, *be* n *well-defined attributes, whose corresponding attribute values are the sets* $K_1, K_2, \ldots, K_n$ *with* $K_i \cap K_j = \varphi$ for $i \neq j$ and $i, j \in \{1, 2, \ldots, n\}$. *Let* B_i *be the non-empty subsets of* K_i *for each* $i = 1, 2, \ldots, n$. *Then the pair* $(R, B_1 \times B_2 \times \ldots . B_n)$ *is said to be Fuzzy Hypersoft Set over* V *where* $R : B_1 \times B_2 \times \ldots . \times B_n \rightarrow F(V)$ *and,* $R(b) = \{v, R(b)(v) | v \in V\}$; $b \in B_1 \times B_2 \times \ldots B_n \subseteq K_1 \times K_2 \times \ldots K_n$."

Definition 8 ***Intuitionistic Fuzzy Hypersoft Set (IFHSS) [14].*** *"Let V be the universal set and IFS(V) be the set of all intuitionistic fuzzy subsets of V. Consider* $k_1, k_2, \ldots . k_n$ for $n \geq 1$, *be* n *well – defined attributes, whose corresponding attribute values are the sets* $K_1, K_2, \ldots, K_n$ with $K_i \cap K_j = \varphi$ for $i \neq j$ and $i, j \in \{1, 2, \ldots, n\}$. *Let* B_i *be the non-empty subsets of* K_i *for each* $i = 1, 2, \ldots, n$. *An intuitionistic Fuzzy Hypersoft Set is defined as the pair,* $(R, B_1 \times B_2 \times \ldots . \times B_n)$ where; $R :$

$B_1 \times B_2 \times \ldots . \times B_n \rightarrow IFS(V)$ *and*

$$R\left(B_1 \times B_2 \times \ldots . \times B_n\right) =$$
$$\left\{< \vartheta, \left(\frac{v}{\rho_{R(\vartheta)}\left(v\right), \omega_{R(\vartheta)}(v)}\right) > \mid v \; \epsilon \; V\right\};$$

where, $\vartheta \in B_1 \times B_2 \times \ldots . \times B_n \subseteq K_1 \times K_2 \times \ldots . K_n$. *It may be noted that* ρ *and* ω *represent membership and non-membership degrees respectively, and satisfies the condition* $0 \leq \rho_{R(\vartheta)}\left(v\right) + \omega_{R(\vartheta)}(v) \leq 1$; *where* $\rho_{R(\vartheta)}\left(v\right), \; \omega_{R(\vartheta)} \in [0,1]$; *and,* $\complement_{R(\vartheta)}\left(v\right) = 1 - \rho_{R(\vartheta)}\left(v\right) - \omega_{R(\vartheta)}(v)$ *is called the degree of indeterminacy."*

Definition 9 ***Pythagorean Fuzzy Hypersoft Set (PyFHSS) [15].*** *"Let* V *be the universal set and PyFS(*V*) be the set of all Pythagorean fuzzy subsets of* V*. Consider* $k_1, k_2, \ldots, k_n$ *for* $n \geq 1$*, be* n *well-defined attributes, whose corresponding attribute values are the sets* $K_1, K_2, \ldots, K_n$ with $K_i \cap K_j = \varphi$ for $i \neq j$ and $i, j \in \{1, 2, \ldots, n\}$. *Let* B_i *be the non-empty subsets of* K_i *for each* $i = 1, 2, \ldots, n$. *A Pythagorean Fuzzy Hypersoft Set is defined as the pair,* $\left(R, B_1 \times B_2 \times \ldots . \times B_n\right)$, *where* $R : B_1 \times B_2 \times \ldots . \times B_n \rightarrow \text{PyFS}\left(V\right)$ *and*

$$R\left(B_1 \times B_2 \times \ldots . \times B_n\right) =$$
$$\left\{< \vartheta, \left(\frac{v}{\rho_{R(\vartheta)}\left(v\right), \omega_{R(\vartheta)}(v)}\right) > \mid v \; \epsilon \; V\right\};$$

where $\vartheta \in B_1 \times B_2 \times \ldots . \times B_n \subseteq K_1 \times K_2 \times \ldots . K_n$. *It may be noted that* ρ and ω *represent membership and non-membership degrees respectively, and satisfies the condition* $0 \leq \rho^2_{R(\vartheta)}\left(v\right) + \omega^2_{R(\vartheta)}(v) \leq 1\}$; *where* $\rho_{R(\vartheta)}\left(v\right), \quad \omega_{R(\vartheta)} \in [0,1]$; *and,* $\complement_{R(\vartheta)}\left(v\right) = \sqrt{1 - \rho^2_{R(\vartheta)}\left(v\right) - \omega^2_{R(\vartheta)}(v)}$ *is called the degree of indeterminacy."*

1.1.4 q- Rung Picture Fuzzy Set

q-rung picture fuzzy information[16] is considered to be a paradigmatic shift in a fuzzy environment as it covers two important additional components of uncertainty, i.e., degree of abstain /refusal and also overcomes the limitation of picture fuzzy set [7], which is very useful in decision-making problems.

Definition 10 *[16]: "A q-rung picture fuzzy set (q-RPFS) U_p in X (universe of discourse) is given by*

$$U_p = \left\{< x, \rho_{U_p}(x), \tau_{U_p}(x), \omega_{U_p}(x) >| \; x \in X\right\};$$

where $\rho_{U_p} : X \to [0,1]$, $\tau_{U_p} : X \to [0,1]$ and $\omega_{U_p} : X \to [0,1]$ denotes the degree of positive membership, degree of neutral membership, and degree of non-membership respectively and for every $x \in X$ satisfy the condition

$$0 \leq \rho^q_{U_p}(x) + \tau^q_{U_p}(x) + \omega^q_{U_p}(x) \leq 1$$

and the degree of refusal for any picture fuzzy set U and $x \in X$ is given by $\theta_{U_p}(x) = \sqrt[q]{1 - (\rho^q_{U_p}(x) + \tau^q_{U_p}(x) + \omega^q_{U_p}(x))}$".

Definition 11 . *"Let U_p and V_p be any two q-rung picture fuzzy sets, then some of the basic operators for these sets are as follows [16]:*

(a) $U_p \oplus \; V_p = \left\langle \sqrt[q]{\rho^q_{U_p} + \rho^q_{V_p} - \rho^q_{U_p}\rho^q_{V_p}}, \tau^q_{U_p}\tau^p_{V_p}, \omega^q_{U_p}\omega^q_{V_p} \right\rangle.$

(b) $U_p \otimes \; V_p = \left\langle \rho^q_{U_p}\rho^q_{V_p}, \sqrt[q]{\tau^q_{U_p} + \tau^q_{V_p} - \tau^q_{U_p}\tau^q_{V_p}}, \sqrt[q]{\omega^q_{U_p} + \omega^q_{V_p} - \omega^q_{U_p}\omega^q_{V_p}} \right\rangle.$

(c) $\kappa U_p = \left\langle \sqrt[q]{\left(1 - \left(1 - \rho^q_{U_p}\right)^{\kappa}\right)}, \left(\tau^q_{U_p}\right)^{\kappa}, \left(\omega^q_{U_p}\right)^{\kappa} \right\rangle.$

(d) $U_p^{\kappa} = \left\langle \left(\rho^q_{U_p}\right)^{\kappa}, \sqrt[q]{1 - \left(1 - \tau^q{U_p}\right)^{\kappa}}, \sqrt[q]{1 - \left(1 - \omega^q{U_p}\right)^{\kappa}} \right\rangle.$"

Definition 12 *[16] "q-rung picture fuzzy weighted geometric (q-RPFWG) and q-rung picture fuzzy weighted arithmetic (q-RPFWA) operators with respect to $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$; $\lambda_j \in [0,1]; \sum_{j=1}^{n} \lambda_j = 1$, defined as follows;*

$$q - RPFWG_{\lambda}(U_1, U_2, U_3, ..., U_n) = \left\{ \prod_{j=1}^{n} (\rho^q_{U_j})^{\lambda_j}, \sqrt[q]{1 - \prod_{j=1}^{n} (1 - \tau^q_{U_j})^{\lambda_j}}, \sqrt[q]{1 - \prod_{j=1}^{n} (1 - \omega^q_{U_j})^{\lambda_j}} \right\} \tag{1.1.1}$$

$$q - RPFWA_{\lambda}(U_1, U_2, U_3, ..., U_n) = \left\{ \sqrt[q]{1 - \prod_{j=1}^{n} (\rho^q_{U_j})^{\lambda_j}}, \prod_{j=1}^{n} (\tau^q_{U_j})^{\lambda_j}, \prod_{j=1}^{n} (\omega^q_{U_j})^{\lambda_j} \right\} \tag{1.1.2}$$

"

1.2 Literature Survey

In this section, there is a brief literature review that is relevant to our presented work as follows:

1.2.1 Discriminant Measures

The notion of discriminant/cross-entropy measure was first developed by Kullback and Leibler [17] which states that this measure gives the difference between the two discrete probability distributions. Then, Bhandari & Pal [18] investigated and expanded the discriminant measure under a fuzzy environment and gave the new notion of fuzzy discriminant measure. Fan and Xie [19] presented a divergence measure based on the exponential function and examined its relationship to the fuzzy exponential entropy. Then, the special classes of divergence measures concerning fuzzy and probabilistic uncertainty were given by Montes et al. [20]. Further, Ghosh et al. [21] have successfully utilized the fuzzy divergence measure in the investigation of automated leukocyte recognition. In addition to this, Vlachos and Sergiadis [22] introduced the discriminant measure for intuitionistic fuzzy setup, which is analogous to the cross-entropy measure of Shang and Jiang [23]. For the validation of the distance and discriminant measures, a new set of axioms were introduced by Wang et al. [24] and Hung et al. [25]. The discriminant measure for the intuitionistic fuzzy setup was introduced by Li [26] and Hung et al. [27] presented the J- divergence measure for the same sets with their utilization in the pattern recognition application. Further, Bajaj et al. [28] presented a R-norm intuitionistic fuzzy cross-entropy measure with its utilization in image thresholding applications. Also, Gandotra et al. [29] introduced parametric entropy under "K-cut" and its distance measure for application in decision-making problems. In addition to this, the bi-parametric discriminant measure for Pythagorean fuzzy sets were given by Guleria & Bajaj [30]

Guiwu Wei [31] introduce the idea of discriminant/cross-entropy measure for picture fuzzy environment and devised *picture fuzzy cross entropy* as $\mathbb{I}_{PFS}(A,B)$ which satisfies two axioms - $\mathbb{I}_{PFS}(A,B) \geq 0$ and $\mathbb{I}_{PFS}(A,B) = 0$ if and only if $A = B$. $\mathbb{I}_{PFS}(A,B)$ can also be called discriminant information measure for PFSs. In general, for fuzzy sets $\mathbb{I}_{FS}(A, B) \neq \mathbb{I}_{FS}(\overline{A}, \overline{B})$. However, for PFSs, $\mathbb{I}_{PFS}(A, B) = \mathbb{I}_{PFS}(\overline{A}, \overline{B})$ holds.

The main goal of multi-criteria decision-making techniques is to attain the best possible optimal alternative assessed under a certain set of criteria. Numerous researchers have worked on the different methodologies for solving the decision-making problems. In literature, Hwang and Yoon [32] proposed the "*Technique for Order Preference by Similarity to Ideal Solutions (TOPSIS)*" decision-making technique, Opricovic [33] devised the "*Vlsekriterijumska Optimizacijai Kompromisno Resenje (VIKOR)*" methods. Wang et al. [34] modified the TOPSIS technique with a fuzzy analytic process for the establishment of radioactive plants in "Vietnam". Also, Pamucar [35] developed the notion of "Geographical Information Systems (GIS)" under the classical technique of "Best-Worst Method (BWM)" and applied it to the assessment of wind turbine locations. Further, Joshi [36] estimated the election results in a polling-bound nation utilizing the VIKOR technique in a picture-fuzzy environment. Yue [37] extended the VIKOR approach to the group decision-making model for assessing the issues related to software reliability and specific experimental studies. Goccer [38] selected a sustainable supply chain approach in an interval-valued picture fuzzy environment by combining the VIKOR method with the analytical hierarchy process in Catastrophic Disruptions. To address the issue with opinion polls, Arya & Kumar [39] expanded the use of VIKOR and TODIM procedures based on picture fuzzy information measures. A picture fuzzy-Choquet integral-based VIKOR approach was proposed by Singh & Kumar [40] to address supplier selection problems. Dutta et al. [41] introduced a new decision-making methodology based on type-2 fuzzy linguistic variables for solving the multi-attribute problem. Also, Tripathi et al. [42] developed a new CRITIC-RS-VIKOR decision-making technique under the intuitionistic fuzzy environment for the assessment of renewable energy source assessment problem.

1.2.2 Soft Set Theory

In the literature, numerous theories have their shortcomings in addressing the impreciseness, vagueness and ambiguity because of the inclusion of multi-attribute tools introduced in the field of engineering, socio-economic situations and other decision-making problems. To cover the multi-attribute feature of the parameters involved in various uncertainty components, Molodtsov [8] proposed a new set called soft set. Further, the new notion of soft set has been utilized by various researchers with many extensions of fuzzy sets as fuzzy soft sets (FSSs) [9], Intuitionistic fuzzy soft sets (IFSSs) [10], Pythagorean fuzzy soft sets (PyFSSs) [43], picture fuzzy soft sets (PFSSs) [11], for various decision-making applications. Further, Das et al. [44] introduced the new concept of neutrosophic soft matrix and successfully applied it to the group decision-making problem of selection of business sectors. Further, a modified decision-making methodology has been developed by Salsabeela et al. [45] with the incorporation of q-rung orthopair fuzzy soft sets.

Various decision-making techniques satisfy the essential requirements for evaluating the best possible alternative. Keshavarz Ghorabaee et al. [46] developed the "*(Evaluation Based On Distance from Average Solution) EDAS*" decision-making technique which is found to be very useful while dealing with conflicting criteria. The EDAS technique computes the "positive distance from the average and the negative distance from the average from the averaging value". The greatest value of positive distance and the lowest value of negative distance will be used to evaluate which option is the most suitable among the available ones. The EDAS technique has the virtue of applying to the average value solution alone, which is particularly useful given the tactile property among decision-makers resulting from an uncertain environment. For solving real-life decision-making problems, the EDAS (Evaluation Based On Distance from Average Solution) technique has been merged with multiple dimensions through distinct fuzzy set extensions. A new modified EDAS-based methodology has been developed by Kahraman et al. [47] for the choice of a proper solid waste disposal site. Then, Peng and Liu [48] developed some new similarity measures for the neutrosophic soft sets and presented a modified EDAS technique with application in the decision-making problem. Further, Feng et al. [49] also gave a new form of EDAS methodology in the hesitant fuzzy environment with the involvement of linguistic parameters. In addition to this, Zhang et al. [50] introduced a picture fuzzy aggregation operators oriented EDAS model for a decision-making problem based on score and accuracy function.

The Dombi aggregation operator was utilized to evaluate renewable energy projects, such as solar, wind, and hydroelectric energy, based on criteria like cost, environmental impact, and efficiency. This method provided more reliable results when handling uncertain or incomplete data, leading to the selection of projects with optimized performance and resource allocation [51].In healthcare, Dombi operators were applied to analyze multi-criteria evaluations of treatment plans for chronic diseases. By aggregating patient data effectively, the method improved the accuracy and reliability of treatment recommendations, resulting in better patient outcomes [52]. Dombi operators have also been implemented in credit risk evaluation, where their ability to handle nonlinear aggregation provided improved decision-making models for credit scoring, enhancing the reliability of risk predictions [53]. Applied in urban planning and sustainability projects, the Dombi aggregation operator enabled the evaluation of ecological impacts under uncertain conditions. This facilitated better policymaking and resource management strategies [54].

1.2.3 Hypersoft Theory

In the different fields of mathematical sciences, like probability, fuzzy information and interval mathematics are regarded as the mathematical tools for solving complex situations involving a variety of uncertainties. It has been observed that it is not easy to handle the inconsistent and imprecise information involving the multi-sub-attribute feature in the parameters of the corresponding alternatives. To overcome such shortcomings in the decision-making processes, Smarandache introduced the notion of a hypersoft set (HSS) that handles multi-sub-attribute features. Hypersoft set has the extra capability to deal with any kind of vague and ambiguous information. Nowadays, there is a great deal of admiration in the field of soft computing for the theory of HSS and its expansions. Fuzzy Hypersoft Sets (FHSS) [14] were also introduced by Smarandache to address uncertainty about the sub-attribute family of parameters. Also, Smarandache [14] has also explored the idea of an Intuitionistic Fuzzy Hypersoft Set (IFHSS) to incorporate the indeterminacy component in the sub-attribute family of parameters. Further, Zulqarnain et al . [15] introduced the notion of a Pythagorean fuzzy hypersoft set (PyFHSS) along with operation laws, algebraic properties and correlation coefficients for the successful utilization in the decision-making application.

The concept of similarity information measures for the assessment of how two or more objects are similar cannot be underestimated. These measures are significant in every field of science and engineering. The similarity information measures are utilized in various areas of “pattern recognition, region extraction, coding theory, image processing, region extraction, coding theory, medical diagnosis, etc.” In literature, researchers developed the similarity measures for the various setups of fuzzy sets “fuzzy sets, vague sets, soft sets, and fuzzy soft sets”. For the hypersoft extensions, Saqlain et al.[55] developed the distance as well as similarity measures and proposed the modified TOPSIS methodology under a neutrosophic hypersoft environment. Further, on similar lines, Jafar et al.[56] presented the trigonometric form of similarity measures for neutrosophic hypersoft sets and utilized them in the renewable energy source selection problem. In addition to this, Rahman et al. [57] also gave the modified decision-making methodology based on the similarity measures for intuitionistic fuzzy hypersoft sets. Kaur and Garg [58] introduced the similarity measures for picture fuzzy hypersoft sets and applied in supplier selection problem. In addition to this, some other kind of similarity measures for picture fuzzy hypersoft sets have been given in the literature for environmental risk assessment [59].

1.2.4 AHP/WASPAS Decision-Making Techniques

The Analytic Hierarchy Process (AHP) [60], was introduced by Thomas L. Saaty in the 1970s which provides a hierarchal framework for observing and solving complex decision-making scenarios. Through this decision-making technique, decision-makers can make the best possible decisions with the reduction of complex situations into a series of pairwise comparison matrices and then analyze the results. While dealing with multiple criteria in difficult real-life circumstances, the AHP has produced effective results in the situations where decision-maker’s knowledge is aggregated through various questionnaire forms. As a result, the AHP has been widely applied in many areas such as “traffic management [61], project risk assessment [62] or decision support systems [63]”. In recent years, numerous researchers have explored the various types of the AHP technique along with the combination of different fuzzy extensions for solving decision-making problems [64]. Laarhoven and Pedrycz [65] modified the AHP technique with a triangular form of fuzzy numbers, and Buckley [66] also developed the hierarchical structure of the AHP technique in its form. Sadiq and Tesfamariam [67] introduced the "intuitionistic fuzzy analytic hierarchy process (IF-AHP)" and applied it to the environmental decision-making problem. Kahraman et al. [68] presented the AHP technique for the interval type-2 fuzzy sets, and Zhu and Xu [69] developed the modified decision-making methodology for the hesitant fuzzy sets.

The weighted aggregated sum product assessment (WASPAS) [70] was developed by Zavadskas et al. in 2012 which combines the principles of the "Weighted Sum Model (WSM) and the Weighted Product Model (WPM)". Turskis et al. [71] apply the fuzzy form of the WASPAS technique for the building site selection problem. Then, Ghorabaee et al. [72] blended the WASPAS technique with interval type-2 fuzzy sets for the assessment of the best possible provider in the distribution systems. Further, Zavadskas et al. [73], [74] modified the WASPAS technique with a mix of single-valued neutrosophic set, interval-valued intuitionistic fuzzy numbers and applied in decision-making applications. Also, Nie et al. [75] developed the all-new WASPAS methodology in an interval-valued neutrosophic environment for the assessment of the best possible solar wind power station.

1.2.5 Motivation & Research Gap

The idea of uncertainty is one of the many paradigm shifts that science and mathematics have experienced in this century. Real-world circumstances are frequently ambiguous and unpredictable, making it difficult to express these situations exactly. For a complete description of these kinds of situations, one would require far more detailed data to recognize, process and comprehend such issues. The inherent ambiguity of human choices as well as the objects being full of uncertainty, the criteria values and/or weights of criteria involved in the multi-criteria decision-making problems are not always expressible in crisp numbers. The best way to deal with such situations is a theory of fuzzy set which is characterized by the membership function and is appropriate to manage such issues. Atanassov extended the FS to an intuitionistic fuzzy set by including the degrees of the dismissal and indeterminacy called non-membership and hesitancy degrees into the investigation. Yager revealed that the existing structures of the fuzzy set and the intuitionistic fuzzy set are not capable enough to depict human opinion in a more practical/broader sense and introduced the notion of Pythagorean fuzzy set which effectively enlarged the span of information by introducing the new conditional constraint. Pythagorean fuzzy set is characterized by a membership value and a non-membership value such that the squared sum of these values is ≤ 1. Cuong and Kreinovich developed the picture fuzzy set which has the involvement of a maximum number of uncertainty components to deal with the ambiguous information in a better way. During our review of the literature, we discovered that:

- Picture fuzzy set presents three uncertainty components: positive membership, neutral membership, and negative membership. This gives a more detailed way of representing the uncertain information as compared to the traditional fuzzy and intuitionistic fuzzy sets.
- With the incorporation of neutral membership, the picture fuzzy set gives a comprehensive framework for modeling real-life problems where the decision-making is not only in the form (true or false) but may have the inclusion of indecisiveness.
- Picture fuzzy sets can be modeled with various other fuzzy systems and methods, strengthening their capacities and giving a more extensive set of tools for handling vague information.
- In the literature survey, we have observed that picture fuzzy discriminant measures help in better handling ambiguous and imprecise information by considering multiple dimensions of membership values. This leads to more reliable and robust decision-making outcomes.
- With the utilization of picture fuzzy soft-hypersoft set up together with the notion of Dombi norms, picture fuzzy hypersoft matrices allows for better handling of conflicting criteria and provide a framework to incorporate different types of uncertainty in the decision-making process.
- Further, by leveraging q-rung picture fuzzy sets into AHP/WASPAS decision-making techniques, a more comprehensive evaluation of alternatives can take place by taking into account multiple criteria and the associated uncertainties.

1.2.6 Objectives of the Study

Based on the above literature survey, motivation and research gap the following objectives have been designed as follows:

- Utilization of bi-parametric picture fuzzy discriminant measure in VIKOR and TOPSIS techniques for the assessment of hydrogen fuel cell technology.
- Construction of picture fuzzy soft Dombi operators along with EDAS technique for the prioritization of agricultural farming.
- Development of picture fuzzy hypersoft matrices for assessing the best possible renewable energy sources.
- Proposition of hybrid q-rung picture fuzzy AHP/WASPAS techniques for the green supply chain management in the energy sector.

Chapter 2

Picture Fuzzy Bi-parametric Discriminant Measure in Decision Making

In this chapter, a new picture fuzzy discriminant/cross-entropy measure involving the parameters R, S has been presented and applied in the VIKOR & TOPSIS multi-criteria decision-making techniques. This discriminant measure provides more flexibility and comprehensiveness in comparison to the existing techniques in the literature. The presented techniques are utilized in the mathematical model for the evaluation of "hydrogen fuel cell (HFC)" technology which gives structural analysis for the expert. While developing the model, the evaluation criteria, criteria weights, and step-by-step performance evaluation of HFC technologies for each criterion in the picture fuzzy framework were taken into consideration. Further, a study of comparison on the helpful observations along with the consistency analysis has been provided to show the effectiveness of the presented work.

2.1 Development of (R,S)-Norm Discriminant Measure Under Picture Fuzzy Framework

This section involves a novel notion of "(R,S)-Norm picture fuzzy discriminant measure" which is analogous to the notion given by Suman and Gandotra [76]. For any two picture fuzzy set U and $V \in PFS(X)$, a bi-parametric picture fuzzy discriminant measure is defined as follows:

$$\mathbb{I}_R^S(U,V) = \frac{R \times S}{n(S-R)} \sum_{i=1}^{n} \begin{bmatrix} \big(\rho_U(x_i)^S \rho_V(x_i)^{(1-S)} + \tau_U(x_i)^S \tau_V(x_i)^{(1-S)} \\ + \omega_U(x_i)^S \omega_V(x_i)^{(1-S)} + \theta_U(x_i)^S \theta_V(x_i)^{(1-S)}\big)^{\frac{1}{S}} \\ - \big(\rho_U(x_i)^R \rho_V(x_i)^{(1-R)} + \tau_U(x_i)^R \tau_V(x_i)^{(1-R)} \\ + \omega_U(x_i)^R \omega_V(x_i)^{(1-R)} + \theta_U(x_i)^R \theta_V(x_i)^{(1-R)}\big)^{\frac{1}{R}} \end{bmatrix}; \tag{2.1.1}$$

where $R, S > 0$; either $0 < S < 1$ and $1 < R < \infty$ or $0 < R < 1$ and $1 < S < \infty$.

Since, the measure $\mathbb{I}_R^S(U,V)$ is not symmetric. Therefore, a symmetrized form of the measure $\mathbb{I}_R^S(U,V)$ may be defined as:

$$\mathbb{J}_R^S(U,V) = \mathbb{I}_R^S(U,V) + \mathbb{I}_R^S(V,U). \tag{2.1.2}$$

Guiwu Wei [31] presented the new concept of cross-entropy/discriminant measure for the picture fuzzy framework and devised *"picture fuzzy cross entropy"* notated as $\mathbb{I}_{PFS}(A,B)$ which satisfies two axioms - "$\mathbb{I}_{PFS}(A,B) \geq 0$ and $\mathbb{I}_{PFS}(A,B) = 0$ iff $A = B$. In general, for fuzzy sets $\mathbb{I}_{FS}(A,B) \neq \mathbb{I}_{FS}(\overline{A},\overline{B})$. However, for $PFSs$, $\mathbb{I}_{PFS}(A,B) = \mathbb{I}_{PFS}(\overline{A},\overline{B})$ holds."

Theorem 1 *The proposed measure $\mathbb{I}_R^S(U,V)$ is a valid picture fuzzy discriminant measure for all U, $V \in PFS(X)$.*

Proof : First, we prove that $\mathbb{I}_R^S(U,V) \geq 0$ with equality if $\rho_U(x_i) = \rho_V(x_i)$, $\tau_U(x_i) = \tau_V(x_i)$ and $\omega_U(x_i) = \omega_V(x_i)$ for all $i = 1, 2, \ldots, n$.

Let $\sum_{i=1}^{n} \rho_U(x_i) = a$, $\sum_{i=1}^{n} \rho_V(x_i) = b$, $\sum_{i=1}^{n} \tau_U(x_i) = c$, $\sum_{i=1}^{n} \tau_V(x_i) = d$, $\sum_{i=1}^{n} \omega_U(x_i) = e$ &

$\sum\limits_{i=1}^{n} \omega_V(x_i) = f$, then

$$\sum_{i=1}^{n} \left(\frac{\rho_U(x_i)}{a}\right)^S \left(\frac{\rho_V(x_i)}{b}\right)^{(1-S)} \geq 1;$$

or

$$\sum_{i=1}^{n} (\rho_U(x_i)^S)(\rho_V(x_i)^{(1-S)}) \geq a^S b^{1-S}. \tag{2.1.3}$$

Similarly, we have

$$\sum_{i=1}^{n} \left(\frac{\tau_U(x_i)}{c}\right)^S \left(\frac{\tau_V(x_i)}{d}\right)^{(1-S)} \geq 1;$$

or

$$\sum_{i=1}^{n} (\tau_U(x_i)^S)\ (\tau_V(x_i)^{(1-S)}) \geq c^S d^{1-S}; \tag{2.1.4}$$

$$\sum_{i=1}^{n} \left(\frac{\omega_U(x_i)}{e}\right)^S \left(\frac{\omega_V(x_i)}{f}\right)^{(1-S)} \geq 1;$$

or

$$\sum_{i=1}^{n} (\omega_U(x_i)^S)(\omega_V(x_i)^{(1-S)}) \geq e^S f^{1-S}; \tag{2.1.5}$$

and

$$\sum_{i=1}^{n} (\theta_U(x_i)^S)(\theta_V(x_i)^{(1-S)}) \ \geq \ (n-a-c-e)^S (n-b-d-f)^{1-S}. \tag{2.1.6}$$

It may be noted that the validity of these inequalities is in accordance with the empirical proof of them in view of the imposed restrictions on the parameters refer [30].

From equations (2.1.3), (2.1.4), (2.1.5) and (2.1.6), we get

$$\sum_{i=1}^{n} \rho_U(x_i)^S \rho_V(x_i)^{(1-S)} \ + \ \tau_U(x_i)^S \tau_V(x_i)^{(1-S)} + \omega_U(x_i)^S \omega_V(x_i)^{(1-S)} + \theta_U(x_i)^S \theta_V(x_i)^{(1-S)}$$
$$\geq a^S b^{1-S} + c^S d^{1-S} + e^S f^{1-S} \ + \ (n-a-c-e)^S (n-b-d-f)^{1-S}. \tag{2.1.7}$$

Case 1: $0 < S < 1$ and $1 < R < \infty$.
Let $\rho_U(x_i)^S \rho_V(x_i)^{(1-S)} + \tau_U(x_i)^S \tau_V(x_i)^{(1-S)} + \omega_U(x_i)^S \omega_V(x_i)^{(1-S)} + \theta_U(x_i)^S \theta_V(x_i)^{(1-S)} = z_i$.
Since $z_i < 1$ and $\frac{1}{S} > 1$, therefore, $z_i > (z_i)^{\frac{1}{S}}$.

As $\frac{R\times S}{n(S-R)} < 0$, then

$$\frac{R\times S}{n(S-R)}\sum_{i=1}^{n}\left[(z_i)^{\frac{1}{S}}\right] \;>\; \frac{R\times S}{n(S-R)}\sum_{i=1}^{n}(z_i) \tag{2.1.8}$$

and for $R>1$,

$$\frac{R\times S}{n(S-R)}\sum_{i=1}^{n}\left[(z_i)^{\frac{1}{R}}\right] \;<\; \frac{R\times S}{n(S-R)}\sum_{i=1}^{n}(z_i). \tag{2.1.9}$$

Therefore, from (2.1.8) and (2.1.9), we have $\mathbb{I}_R^S(U,V) > 0$ and if $\rho_U(x_i) = \rho_V(x_i)$, $\tau_U(x_i)=\tau_V(x_i)$, $\omega_U(x_i)=\omega_V(x_i)$ in (2.1.1), have $\mathbb{I}_R^S(U,V)=0$. Hence, conclude that $\mathbb{I}_R^S(U,V)\geq 0$.

Next we prove the convexity of $\mathbb{I}_R^S(U,V)$ in this case.

For $0<S<1$, equation (2.1.7) may be written as

$$\left(\sum_{i=1}^{n}\left(\rho_U(x_i)^S\rho_V(x_i)^{(1-S)}+\tau_U(x_i)^S\tau_V(x_i)^{(1-S)} + \omega_U(x_i)^S\omega_V(x_i)^{(1-S)}+\theta_U(x_i)^S\theta_V(x_i)^{(1-S)}\right)\right)^{\frac{1}{S}}$$
$$\leq \left(a^Sb^{1-S}+c^Sd^{1-S}+e^Sf^{1-S} + (n-a-c-e)^S(n-b-d-f)^{1-S}\right)^{\frac{1}{S}}.$$

Also, we can write the above equation as

$$\sum_{i=1}^{n}\left[\left(\rho_U(x_i)^S\rho_V(x_i)^{(1-S)} + \tau_U(x_i)^S\tau_V(x_i)^{(1-S)} + \omega_U(x_i)^S\omega_V(x_i)^{(1-S)}+\theta_U(x_i)^S\theta_V(x_i)^{(1-S)}\right)^{\frac{1}{S}}\right]$$
$$\leq \left[\sum_{i=1}^{n}\left(\rho_U(x_i)^S\rho_V(x_i)^{(1-S)} + \tau_U(x_i)^S\tau_V(x_i)^{(1-S)} + \omega_U(x_i)^S\omega_V(x_i)^{(1-S)}+\theta_U(x_i)^S\theta_V(x_i)^{(1-S)}\right)\right]^{\frac{1}{S}}. \tag{2.1.10}$$

Next, for $R>1$, from equation (2.1.6), we have

$$\left(\sum_{i=1}^{n}\left(\rho_U(x_i)^R\rho_V(x_i)^{(1-R)} + \tau_U(x_i)^R\tau_V(x_i)^{(1-R)}+\omega_U(x_i)^R\omega_V(x_i)^{(1-R)}+\theta_U(x_i)^R\theta_V(x_i)^{(1-R)}\right)\right)^{\frac{1}{R}}$$
$$\geq \left(a^Rb^{1-R}+c^Rd^{1-R}+e^Rf^{1-R} + (n-a-c-e)^R(n-b-d-f)^{1-R}\right)^{\frac{1}{R}};$$

and the above equation can be written as

$$\sum_{i=1}^{n}\left[\left(\rho_U(x_i)^R\rho_V(x_i)^{(1-R)} + \tau_U(x_i)^R\tau_V(x_i)^{(1-R)}+\omega_U(x_i)^R\omega_V(x_i)^{(1-R)}+\theta_U(x_i)^R\theta_V(x_i)^{(1-R)}\right)^{\frac{1}{R}}\right]$$
$$\geq \left[\sum_{i=1}^{n}\left(\rho_U(x_i)^R\rho_V(x_i)^{(1-R)} + \tau_U(x_i)^R\tau_V(x_i)^{(1-R)}+\omega_U(x_i)^R\omega_V(x_i)^{(1-R)}+\theta_U(x_i)^R\theta_V(x_i)^{(1-R)}\right)\right]^{\frac{1}{R}}. \tag{2.1.11}$$

Since $\frac{R\times S}{n(S-R)} < 0$, therefore, from (2.1.10) and (2.1.11), we get

$$\mathbb{I}_R^S(U,V)\geq \frac{R\times S}{n(S-R)}\left[\begin{array}{c}\left\{a^Sb^{1-S}+c^Sd^{1-S}+e^Sf^{1-S} + (n-a-c-e)^S(n-b-d-f)^{1-S}\right\}^{\frac{1}{S}}\\ -\left\{a^Rb^{1-R}+c^Rd^{1-R}+e^Rf^{1-R} + (n-a-c-e)^R(n-b-d-f)^{1-R}\right\}^{\frac{1}{R}}\end{array}\right]. \tag{2.1.12}$$

Further, if we take

$$\psi(a,b) = \frac{R\times S}{n(S-R)}\left[\begin{array}{c}\{a^S b^{1-S} + c^S d^{1-S} + e^S f^{1-S} \ + \ (n-a-c-e)^S (n-b-d-f)^{1-S}\} \\ -\{a^R b^{1-R} + c^R d^{1-R} + e^R f^{1-R} \ + \ (n-a-c-e)^R (n-b-d-f)^{1-R}\}\end{array}\right],$$

then

$$\frac{\partial\psi(a,\ b)}{\partial a} = \frac{R\ \times\ S}{n(S-R)}\left[\begin{array}{c}\left\{S\left(\frac{a}{b}\right)^{S-1} - \ S\left(\frac{n-a-c-e}{n-b-d-f}\right)^{S-1}\right\} \\ - \ \left\{R\left(\frac{a}{b}\right)^{R-1} - R\left(\frac{n-a-c-e}{n-b-d-f}\right)^{R-1}\right\}\end{array}\right], \tag{2.1.13}$$

and

$$\frac{\partial^2\psi(a,\ b)}{\partial a^2} = \frac{R\ \times\ S}{n(S-R)}\left[\begin{array}{c}\left\{\frac{S(S-1)}{b}\left(\frac{a}{b}\right)^{S-2} + \ \frac{S(S-1)}{n-b-d-f}\left(\frac{n-a-c-e}{n-b-d-f}\right)^{S-2}\right\} \\ -\left\{\frac{R(R-1)}{b}\left(\frac{a}{b}\right)^{R-2} + \ \frac{R(R-1)}{n-b-d-f}\left(\frac{n-a-c-e}{n-b-d-f}\right)^{R-2}\right\}\end{array}\right] > 0. \tag{2.1.14}$$

This proves the convexity of $\psi(a,b)$ in a. Further, the minimum value of the measure will be 0 for $\frac{a}{b} = \frac{n-a-c-e}{n-b-d-f}$. For $a = b$, $c = d$ and $e = f$, value of the measure becomes 0.

Case 2: $S > 1$ and $0 < R < 1$.
Let $\rho_U(x_i)^S\rho_V(x_i)^{(1-S)} + \tau_U(x_i)^S\tau_V(x_i)^{(1-S)} + \omega_U(x_i)^S\omega_V(x_i)^{(1-S)} + \theta_U(x_i)^S\theta_V(x_i)^{(1-S)} = z_i$. Since $z_i < 1$ and $\frac{1}{S} < 1$, therefore, $z_i < (z_i)^{\frac{1}{S}}$.

As $\frac{R\times S}{n(S-R)} > 0$, therefore,

$$\frac{R\times S}{n(S-R)}\sum_{i=1}^{n}\left[(z_i)^{\frac{1}{S}}\right] > \frac{R\times S}{n(S-R)}\sum_{i=1}^{n}(z_i); \tag{2.1.15}$$

and for $0 < R < 1$,

$$\frac{R\ \times\ S}{n(S-R)}\sum_{i=1}^{n}\left[(z_i)^{\frac{1}{R}}\right] \ > \ \frac{R\times S}{n(S-R)}\sum_{i=1}^{n}(z_i). \tag{2.1.16}$$

Therefore, from (2.1.15) and (2.1.16), we have $\mathbb{I}_R^S(U,V) > 0$.
If $\rho_U(x_i) = \rho_V(x_i)$, $\tau_U(x_i) = \tau_V(x_i)$ and $\omega_U(x_i) = \omega_V(x_i)$ in (2.1.1), we have $\mathbb{I}_R^S(U,V) = 0$.

which concludes $\mathbb{I}_R^S(U,V) \geq 0$.

Similarly, the convexity of $\mathbb{I}_R^S(U,V)$ can be done same as Case 1.

In the last,

$\mathbb{I}_R^S(U,V) \geq 0$, where equality holds only when

$$\rho_U(x_i) = \rho_V(x_i), \tau_U(x_i) = \tau_V(x_i) \text{ for every } i$$

and $a = b$, $c = d$ and $e = f$, i.e., $U = V$. Therefore, $\mathbb{I}_R^S(U,V)$ is a well-defined "bi-parametric picture fuzzy discriminant measure".

2.2 MCDM Methodology Based on (R,S)-Norm Picture Fuzzy Information

In this section, we introduce two revised decision-making methodologies which are based on VIKOR and TOPSIS techniques that take into account the concepts of bi-parametric picture fuzzy information measures.

Consider a problem of multi-criteria decision-making, where $\mathbb{A} = \{A_1, A_2, \ldots, A_m\}$ be the set of available alternatives and $\mathbb{E} = \{E_1, E_2, \ldots, E_n\}$ be the set of criteria. For opinions on the available alternatives w.r.t.each criterion, let $\mathbb{D} = \{D_1, D_2, \ldots, D_l\}$ be the set of experts/decision-makers who assesses the alternatives and give their decisions in terms of qualitative variables. Let $\mathbb{R}_k = (v_{ij}^k)$, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ be the qualitative matrix which is given concerning individual expert, say k^{th} expert, where v_{ij}^k gives the evaluation of alternative A_i in reference with criterion C_j, in the form of qualitative variables.

Further, to find the appropriate and the most promising alternative out of the m alternatives, we developed a revised methodology based on VIKOR/TOPSIS techniques that take into account the (R,S)-Norm picture fuzzy information measures. To illustrate the suggested method, we describe the procedural steps by dividing them into two stages as shown in Figure 2.1.

The various mathematical procedural steps involved in the proposed methodology is shown as under:

- **Step 1: Evaluation of the Criteria by Experts**

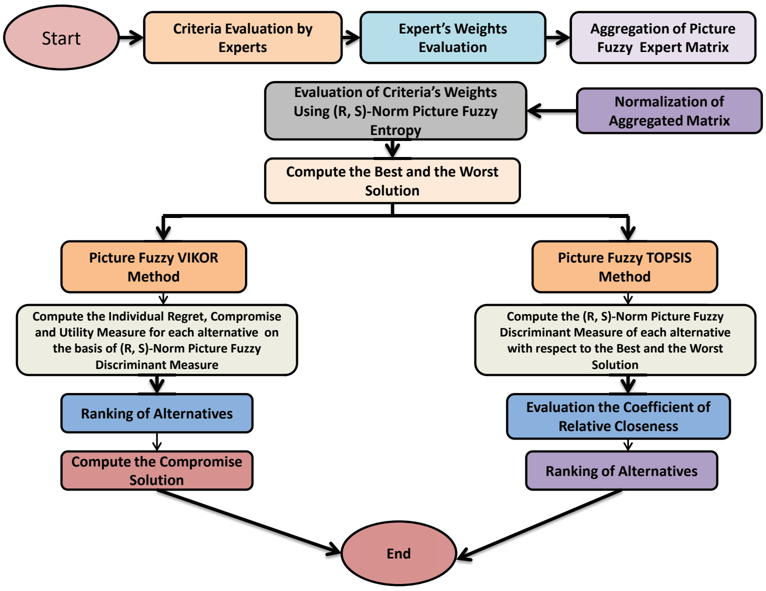

Figure 2.1: VIKOR & TOPSIS - (R, S)-Norm Picture Fuzzy Information Measures

In connection with the picture fuzzy number, selected experts give their valuable opinions for individual criteria with the help of the defined set of qualitative variables.

- **Step 2: Assessment of the Expert's Weights**

  In a decision-making situation, it is noted that figuring out the decision maker's weight is crucial. We suppose the main component of expert is computed by the pre-defined qualitative terms and expressed in the terms of picture fuzzy numbers. Analogous to the formula given in [78], the weight of k^{th} expert is computed as:

$$\phi_k = \frac{\rho_k + \theta_k\left[\frac{\rho_k}{\rho_k+\tau_k+\omega_k}\right]}{\sum\limits_{k=1}^{l} \rho_k + \theta_k\left[\frac{\rho_k}{\rho_k+\tau_k+\omega_k}\right]}; \tag{2.2.1}$$

  where $\sum\limits_{k=1}^{l} \phi_k = 1$ and $\phi_k \geq 0 \ \forall \ k$.

- **Step 3: Computing the Aggregated Picture Fuzzy Expert Matrix**

  For aggregating each of the expert matrices into one cluster based on the expert's perception, we shall make use of an average aggregation operator to formulate the aggregated expert matrix. Further, we apply the following picture fuzzy operator given by Wei [77]:

$\tilde{\mathbb{R}} = [(\tilde{r}_{ij})]_{m\times n}$, where $\tilde{r}_{ij}$ is

$$\tilde{r}_{ij} = PFWA_{\phi}(v_{ij}^{(1)}, v_{ij}^{(2)}, \ldots, v_{ij}^{(l)}) = \left(1 - \prod_{k=1}^{l}(1-\rho_{ij})^{\phi_k}, \prod_{k=1}^{l}(\tau_{ij})^{\phi_k}, \prod_{k=1}^{l}(\omega_{ij})^{\phi_k}\right) \tag{2.2.2}$$

- **Step 4: Normalization of Picture Fuzzy Expert Matrix**

  To evaluate all the criteria on a equal footing, it becomes utmost important to normalize them before their application in the methodology. Therefore, the expert matrix $\tilde{\mathbb{R}} = [\tilde{r}_{ij}]_{m\times n}$ is transformed to another expert matrix, say, $\mathbb{R} = [r_{ij}]_{m\times n}$ where r_{ij} is given by

$$r_{ij} = (\rho_{ij}, \tau_{ij}, \omega_{ij}) = \begin{cases} \tilde{r}_{ij}, & \text{for benefits criteria ;} \\ \tilde{r}_{ij}^{c}, & \text{for cost criteria.} \end{cases} \tag{2.2.3}$$

- **Step 5: Computing the Weights of the Criterions**

  It may be observed that the order of ranking of the alternatives will be affected by considering distinct criteria weights. To avoid such shortcomings, we compute the criteria weights by utilizing the (R, S)-Norm picture fuzzy information measure as follows:

$$\vartheta_j = \frac{1 - e_j}{n - \sum\limits_{j=1}^{n} e_j}, \quad j = 1, 2, \ldots, n; \tag{2.2.4}$$

  where $e_j = \frac{1}{m}\sum\limits_{i=1}^{m} \mathbb{H}_R^S(z_{ij})$, and

$$\mathbb{H}_R^S(z_{ij}) = \frac{R\times S}{(R-S)}\sum_{i=1}^{n}\frac{1}{n}\left[\begin{array}{c}\left(\rho_U(x_i)^S + \tau_U(x_i)^S + \omega_U(x_i)^S + \theta_U(x_i)^S\right)^{\frac{1}{S}} \\ -\left(\rho_U(x_i)^R + \tau_U(x_i)^R + \omega_U(x_i)^R + \theta_U(x_i)^R\right)^{\frac{1}{R}}\end{array}\right]$$

  is the (R, S) picture fuzzy entropy for $z_{ij} = (p_{ij}, q_{ij})$.

- **Step 6: Evaluation of "Best/Worst Solution"**

  In this step, "the best and the worst" solution are computed in terms of picture fuzzy ideal solution r_j^+ and picture fuzzy negative ideal solution r_j^-, as follows:

$$r_j^+ = \begin{cases} \max\limits_i \rho_{ij}, & \text{for benefit criterion } C_j, \\ \min\limits_i \tau_{ij}, & \text{for cost criterion } C_j, \\ \min\limits_i \omega_{ij}, & \text{for cost criterion } C_j; \end{cases} \tag{2.2.5}$$

and

$$r_j^- = \begin{cases} \min\limits_i \rho_{ij}, & \text{for benefit criterion } C_j, \\ \max\limits_i \tau_{ij}, & \text{for cost criterion } C_j, \\ \max\limits_i \omega_{ij}, & \text{for cost criterion } C_j. \end{cases} \tag{2.2.6}$$

Remarks: In stage 1, the above-mentioned six steps remain the same for both techniques. Now, based on the choice of the individual or any organization, we may choose to go either for the picture fuzzy VIKOR or picture fuzzy TOPSIS methods. The procedural steps of the two methods have been given in two parts as follows:

— Picture Fuzzy VIKOR Method

With the presumption that a compromise is acceptable, Opricovic [33] created one of the key MCDM methodologies, the VIKOR (VIseKriterijumska Optimizacija I Kompromisno Resenje), to address decision making problems with opposing criteria. This approach is one of the MCDM techniques that is frequently used in literature to compromise solutions that simultaneously satisfy all the incompatible criteria. In addition to the calculations for the six steps mentioned above, perform the following additional computations to complete the decision-making process below:

- **Step 7: Necessary Measures for all the alternatives**

  Here, we compute **Group Utility** S_i, **Individual Regret** R_i & **Compromise Measure** Q_i of each A_i by making use of the concept of (R, S)-Norm picture fuzzy discriminant measure. Further, we use the following formula to obtain these measures for choices A_i $(i = 1, 2, \ldots, m)$ as under:

$$S_i = \sum_{j=1}^{n} \vartheta_j \frac{\mathbb{I}_R^S\left(r_j^+, r_{ij}\right)}{I_R^S\left(r_j^+, r_j^-\right)}; \tag{2.2.7}$$

$$R_i = \max_{1 \le j \le n} \vartheta_j \frac{\mathbb{I}_R^S\left(r_j^+, r_{ij}\right)}{I_R^S\left(r_j^+, r_j^-\right)}; \tag{2.2.8}$$

and

$$Q_i = y\frac{\left(S_i - \min_i S_i\right)}{\left(\max_i S_i - \min_i S_i\right)} + (1-y)\frac{\left(R_i - \min_i R_i\right)}{\left(\max_i R_i - \min_i R_i\right)}. \tag{2.2.9}$$

Here, y and $1-y$ represents the weights of the maximum group utility approach and the weight of individual regret respectively.

- **Step 8: Ordering of the Alternatives**
  Next, the ranking of the corresponding alternatives can be done based on the values of S_i, R_i, Q_i in the decreasing order.

- **Step 9: Computing the Compromise Solution**
  To have the optimal and unique solution, the available alternatives must satisfy the following constraints:

  - C_1- **Advantage within the Range of Acceptability**

    $$Q(A_{(2)}) - Q(A_{(1)}) \geq \frac{1}{m-1}, \tag{2.2.10}$$

    where $A_{(1)}$ and $A_{(2)}$ are the first two optimal alternatives which are computed from the measure of Q.

  - C_2- **Stability which is Acceptable in Decision-Making**
    It may be noted that $A_{(1)}$ must be the top ranked by S_i or/and R_i. For a decision problem, the compromise solution is stable and may be the fixed highest utility (for $y > 0.5$) or unanimous (for $y > 0.5$) or with prohibition (for $y < 0.5$).

Additionally, if the constraint C_1 is not fulfilled, then the highest value of B should be determined and obtained from the expression "$Q(A_{(B)}) - Q(A_{(1)}) \leq \frac{1}{m-1}$;" where B denotes the random position in the ranking of the alternatives to the most suitable one. As a result, for some $i = 1, 2, ..., m$, the alternative $A(i)$ is the acceptable choice.

— Picture Fuzzy TOPSIS Method

The "TOPSIS(Technique for Order of Preference by Similarity to Ideal Solution)" method for MCDM was developed by Hwang and Yoon [32] which has been extensively applied. The basic underlying strategy is to pick an alternative with the "minimum geometric distance" from the "positive ideal solution (PIS)" and the "maximum geometric distance" from the "negative ideal solution (NIS)".

In continuation of the above-mentioned six steps of stage 1, we carry out some more computations to complete the decision process as below:

- **Step 7: Calculation of "(R, S)-Norm Discriminant Measure"**
  We compute the discriminant measures of the alternatives A_i's $\forall\ i = 1, 2, \ldots, m$ from r_j^+ and r_j^- respectively using the discriminant measure (2.1.1).

- **Step 8: Computation of "Coefficient of Relative Closeness"**
  We calculate the "coefficient of relative closeness", i.e, CRC_i's , $(i = 1, 2, \ldots m)$ as

$$CRC_i = \frac{\mathbb{I}_R^S(A_i, r_j^-)}{\mathbb{I}_R^S(A_i, r_j^+)\ +\ \mathbb{I}_R^S(A_i, r_j^-)}. \tag{2.2.11}$$

- **Step 9. Ordering of Alternatives**
  Thus, the ordering of the alternatives can be done by listing the obtained score of the coefficient of relative closeness in the increasing order i.e., the maximum score would represent the optimal alternative.

Hence, we have systematically provided novel VIKOR and TOPSIS decision-making approaches with the incorporation of the proposed (R, S)-Norm picture fuzzy information measures.

2.3 Methodologies for Hydrogen Fuel Cell Technology Assessment

In this section, we incorporate the (R, S)-Norm picture fuzzy information measures in VIKOR/TOPSIS algorithms. The HFCs under consideration must have been chosen after careful consideration and expert consultation. Based on the expert's judgment and the body of relevant literature, all factors influencing the cell assessment have been established. The alternatives which are to be assessed include household electric-heat composite systems [79], portable fuel cell power facilities [80], distributive fuel cell power generation systems [79], [81] and fuel cell backup power systems [82], [83], [79]. These are the five available HFCs, say, H_1, H_2, H_3, H_4 & H_5, which are to be assessed in solving this problem. These HFCs have been thoroughly assessed referring to the four major criteria and 19 sub-criteria (Refer Table 2.1) and shown in

Table 2.1: Study on Classification of Assessment Criteria in HFC Technology

"Major Criteria"	**"Sub-criteria"**	**"Existing Literature"**	**"Classification"**
Economic	Acquisition cost (E_1)	[84]	Cost
	Cost of use (E_2)	[85]	Cost
	Logistics costs (E_3)	[86]	Cost
	Quantity discount (E_4)	[87][88]	Benefit
	Global market demand (E_5)	[89]	Benefit
Environment	Energy efficiency (E_6)	[90]	Benefit
	Carbon-dioxide emission (E_7)	[91]	Benefit
	Geographical location (E_8)	[92][93]	Benefit
	Environment-friendlily (E_9)		Benefit
Society	Safeguards (E_{10})	[94]	Benefit
	Use environment maturity (E_{11})	[94]	Benefit
	Social acceptability (E_{12})	[95]	Benefit
	Fulfilling the urgent requirements (E_{13})	[96] [97]	Benefit
	Information disclosure (E_{14})	[98] [99]	Benefit
Technology Capability	Reliability (E_{15})	[100]	Benefit
	System Performance (E_{16})	[89]	Benefit
	Product maturity (E_{17})	[95]	Benefit
	Product development potential (E_{18})	[95]	Benefit
	Domestic technological ability (E_{19})	[80]	Benefit

Figure 2.2. Now, this model under the expert's opinion and criterion weights using the picture fuzzy orientation to VIKOR/TOPSIS technique, problem of choosing the best potential hydrogen fuel cell from the set of possibilities is being analytically solved.

Steps to Solve the Selection Problem Procedurally:

- **Step 1.** The decision-makers provided qualitative computations of the 19 criteria that are under consideration (Table 2.5) and converted them into picture fuzzy information by making use of pre-defined numerical ranges on the picture fuzzy number scale provided in Table 2.2. Given Table 2.3 and the five available hydrogen fuel cells $H_1, H_2, H_3, H_4 \,\&\, H_5$, the experts define the linguistic information for each of the 19 criteria (Table 2.4) and has been converted into picture fuzzy information.

- **Step 2.** Further, we discuss the significance of the expert's opinion by making use of qualitative variables which are converted into picture fuzzy information using the defined quantitative ranking in terms of picture fuzzy numbers. These

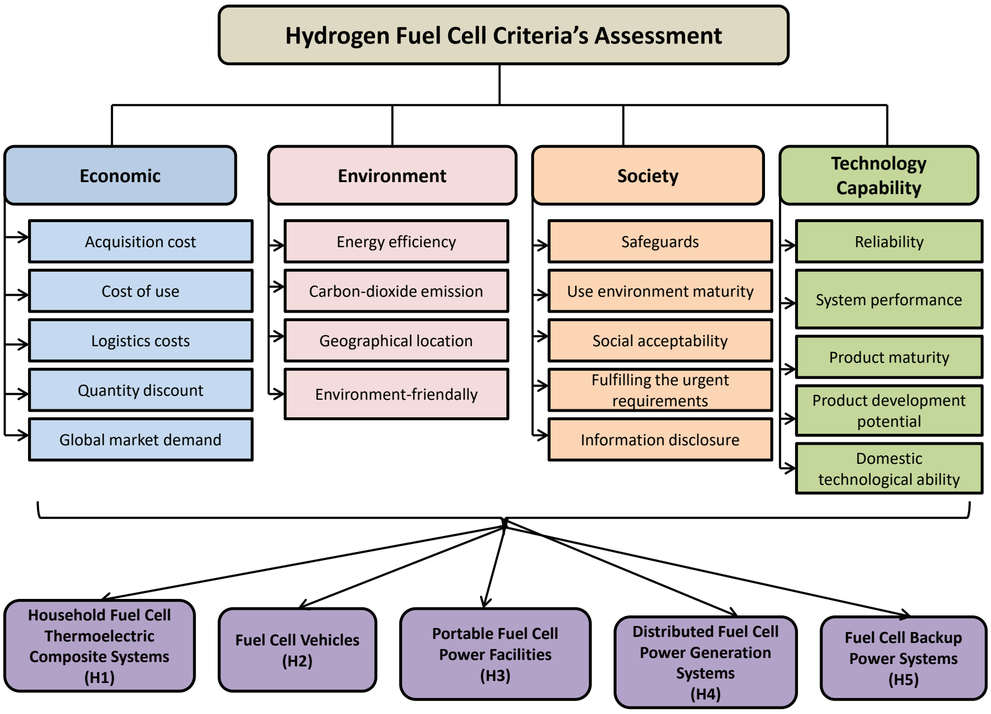

Figure 2.2: Hydrogen Fuel Cell Criteria's Indicators

Table 2.2: Qualitative variables for Ranking the Weightage of Criteria and Decision Makers

Qualitative Term	PFNs
"Extremely Important (EI)"	(0.83, 0.04, 0.11)
"Very Important (VI)"	(0.60, 0.05, 0.21)
"Important (I)"	(0.53, 0.12, 0.25)
"Less Important(LI)"	(0.45, 0.15, 0.30)
"Very Less Important (VLI)"	(0.30, 0.25, 0.35)

Table 2.3: Qualitative variables for computing the performance of alternatives

Qualitative Term	PFNs
"Absolutely High (AH)"	(0.83, 0.04, 0.11)
"Very Very High (VVH)"	(0.75, 0.05, 0.15)
"Very High (VH)"	(0.62, 0.1, 0.2)
"High (H)"	(0.55, 0.11, 0.25)
"Medium High (MH)"	(0.50, 0.15, 0.30)
"Medium (M)"	(0.45, 0.20, 0.35)
"Medium Low (ML)"	(0.40, 0.22, 0.37)
"Low (L)"	(0.35, 0.25, 0.40)
"Very Low (VL)"	(0.25, 0.30, 0.43)
"Very Very Low (VVL)"	(0.15, 0.35, 0.48)

Table 2.4: Qualitative Computation for Ranking of the Alternatives by Experts

	H_1	H_2	H_3	H_4	H_5
E_1	(H, M H, M)	(H, M H, M L)	(MH, H, VH)	(V V H, V V H, H)	(M, VH, H)
E_2	(M H, M, V L)	(H, V H, V H)	(V H, M H, M H)	(VH, VH, VH)	(V H, M, M H)
E_3	(V L, L, M L)	(M L, L, ML)	(H, M, V L)	(V H, V L, M L)	(V V L, V L, L)
E_4	(L, M, M)	(V L, ML, M L)	(M H, V L, M L)	(V H, H, M L)	(M L, V L, V V L)
E_5	(V L, V L, V L)	(M L, L, L)	(V H, M, M L)	(V H, V V H, V V H)	(L, V L, M L)
E_6	(V V H, M H, V H)	(H, L, M L)	(M L, M L, MH)	(V L, H, L)	(V V H, H, L)
E_7	(H, M, M H)	(M H, M L, V L)	(M L, L, V L)	(MH, L, M)	(V H, M, L)
E_8	(M H, V H, V L)	(M L, L, L)	(V H, M L, L)	(V H, M H, V L)	(V H, L, ML)
E_9	(H, M L, L)	(L, V V L, ML)	(L, M L, V V L)	(VH, MH, L)	(V H, M L, L)
E_{10}	(V V H, M H, H)	(H, M L, L)	(H, L, M L)	(V H, H, L)	(V V H, M H, MH)
E_{11}	(M L, L, V V L)	(L, V V L, L)	(V H, M L, L)	(V H, L, M L)	(H, V L, M H)
E_{12}	(V V H, M H, M L)	(M H, M L, V L)	(M H, V L, M L)	(M H, L, M L)	(V V H, H, H)
E_{13}	(H, M H, L)	(L, M L, V L)	(H, V H, M L)	(H, ML, V H)	(V H, M H, V H)
E_{14}	(V H, H, M L)	(M H, M, V L)	(M L, L, V V L)	(H, V L, M L)	(V V H, M L, L)
E_{15}	(M H, M L, L)	(M H, L, M L)	(H, M L, L)	(V H, H, L)	(V H, M H, L)
E_{16}	(M H, H, L)	(M L, M L, L)	(M H, L, H)	(L, L, V L)	(H, M H, M)
E_{17}	(V L, L, L)	(M L, L, V L)	(V H, H, M)	(H, V H, M L)	(V L, M L, MH)
E_{18}	(V H, M L, L)	(H, H, V L)	(V H, M L, H)	(V V H, H, M H)	(V V H, H, M H)
E_{19}	(H, V L, L)	(M, V L, M L)	(H, M H, L)	(V L, V V L, M L)	(H, H, M H)

Table 2.5: Qualitative Computation for Ranking the Criterions

	DM_1	DM_2	DM_3
E_1	V H	M H	H
E_2	M H	M L	L
E_3	V V H	M	M L
E_4	H	V H	L
E_5	M H	M H	V H
E_6	M L	L	V L
E_7	V V L	H	M L
E_8	H	V L	V V L
E_9	V H	H	H
E_{10}	M	V L	V L
E_{11}	V L	M H	V H
E_{12}	M H	M	H
E_{13}	M L	L	H
E_{14}	V L	L	L
E_{15}	M H	M	L
E_{16}	M	H	V V H
E_{17}	M	M H	V V L
E_{18}	M H	M	M H
E_{19}	L	V L	L

values are being tabulated in Table 2.2. Then, the expert's weights are computed using equation (2.2.1), and the results are summarized in Table 2.6.

Table 2.6: Experts Weights

	D M1	**D M2**	**D M3**
"Qualitative Variable"	V I	I	V V I
"Weight"	0.329621	0.279867	0.390512

- **Step 3.** In this step, we all expert matrices acquired from the various experts into a single expert matrix (shown in Table 2.7) using the picture fuzzy weighted averaging aggregation operator.

Table 2.7: Aggregated Picture Fuzzy Expert Matrix

	H_1	H_2	H_3	H_4	H_5
E_1	(0.499, 0.152, 0.300)	(0.481, 0.157, 0.307)	(0.564, 0.117, 0.243)	(0.685, 0.068, 0.183,)	(0.541, 0.130, 0.262)
E_2	(0.398, 0.213, 0.361)	(0.598, 0.103, 0.215)	(0.543, 0.131, 0.262)	(0.620, 0.100, 0.200)	(0.531, 0.142, 0.274)
E_3	(0.339, 0.252, 0.397)	(0.386, 0.228, 0.378)	(0.419, 0.192, 0.339)	(0.451, 0.185, 0.315)	(0.261, 0.294, 0.433)
E_4	(0.419, 0.215, 0.366)	(0.354, 0.244, 0.389)	(0.399, 0.211, 0.360)	(0.524, 0.140, 0.271)	(0.268, 0.288, 0.427)
E_5	(0.250, 0.300, 0.430)	(0.367, 0.240, 0.390)	(0.496, 0.165, 0.297)	(0.713, 0.063, 0.165)	(0.344, 0.250, 0.396)
E_6	(0.643, 0.089, 0.204)	(0.442, 0.181, 0.332)	(0.441, 0.189, 0.341)	(0.385, 0.211, 0.359)	(0.572, 0.117, 0.254)
E_7	(0.504, 0.147, 0.295)	(0.383, 0.219, 0.382)	(0.330, 0.257, 0.401)	(0.442, 0.194, 0.345)	(0.480, 0.174, 0.307)
E_8	(0.458, 0.176, 0.308)	(0.341, 0.252, 0.400)	(0.467, 0.178, 0.311)	(0.465, 0.172, 0.302)	(0.472, 0.176, 0.309)
E_9	(0.437, 0.184, 0.335)	(0.321, 0.261, 0.408)	(0.294, 0.275, 0.420)	(0.494, 0.160, 0.294)	(0.467, 0.178, 0.311)
E_{10}	(0.573, 0.121, 0.260)	(0.437, 0.184, 0.335)	(0.442, 0.181, 0.332)	(0.509, 0.147, 0.279)	(0.602, 0.104, 0.239)
E_{11}	(0.297, 0.273, 0.419)	(0.299, 0.275, 0.421)	(0.467, 0.178, 0.311)	(0.472, 0.176, 0.309)	(0.457, 0.164, 0.312)
E_{12}	(0.573, 0.121, 0.260)	(0.383, 0.219, 0.366)	(0.399, 0.211, 0.360)	(0.422, 0.201, 0.353)	(0.629, 0.181, 0.211)
E_{13}	(0.465, 0.165, 0.316)	(0.328, 0.259, 0.402)	(0.520, 0.140, 0.274)	(0.543, 0.129, 0.256)	(0.590, 0.112, 0.224)
E_{14}	(0.524, 0.140, 0.271)	(0.398, 0.213, 0.360)	(0.297, 0.273, 0.402)	(0.419, 0.191, 0.340)	(0.536, 0.141, 0.283)
E_{15}	(0.417, 0.204, 0.356)	(0.422, 0.201, 0.353)	(0.437, 0.184, 0.335)	(0.509, 0.147, 0.279)	(0.494, 0.160, 0.294)
E_{16}	(0.462, 0.168, 0.319)	(0.381, 0.231, 0.381)	(0.483, 0.153, 0.303)	(0.313, 0.268, 0.411)	(0.499, 0.151, 0.300)
E_{17}	(0.319, 0.265, 0.410)	(0.330, 0.257, 0.401)	(0.540, 0.135, 0.265)	(0.520, 0.140, 0.274)	(0.399, 0.210, 0.358)
E_{18}	(0.467, 0.178, 0.311)	(0.451, 0.163, 0.309)	(0.539, 0.129, 0.259)	(0.614, 0.096, 0.227)	(0.614, 0.096, 0.227)
E_{19}	(0.366, 0.215, 0.360)	(0.379, 0.232, 0.379)	(0.437, 0.184, 0.335)	(0.288, 0.277, 0.418)	(0.531, 0.124, 0.268)

- **Step 4.** Since the first three attributes are cost type, therefore the normalization of the above-aggregated picture fuzzy expert matrix can be done by using equation (3.2.2), and the resulting normalized matrix is shown in Table 2.8.

- **Step 5.** The criteria's weights have been computed by making use of (R, S)-Norm picture fuzzy entropy measure which is given by [76] (here are 3 experts, i.e., $n = 3$) and the resulting values are shown in Table 2.9.

Table 2.8: Normalized Aggregated Picture Fuzzy Expert Matrix

	H_1	H_2	H_3	H_4	H_5
E_1	(0.300, 0.152, 0.499)	(0.307, 0.157, 0.481)	(0.243, 0.117, 0.564)	(0.183, 0.068, 0.685,)	(0.262, 0.130, 0.541)
E_2	(0.361, 0.213, 0.398)	(0.215, 0.103, 0.598)	(0.262, 0.131, 0.543)	(0.200, 0.100, 0.620)	(0.274, 0.142, 0.531)
E_3	(0.397, 0.252, 0.339)	(0.378, 0.228, 0.386)	(0.339, 0.192, 0.419)	(0.315, 0.185, 0.451)	(0.433, 0.294, 0.261)
E_4	(0.419, 0.215, 0.366)	(0.354, 0.244, 0.389)	(0.399, 0.211, 0.360)	(0.524, 0.140, 0.271)	(0.268, 0.288, 0.427)
E_5	(0.250, 0.300, 0.430)	(0.367, 0.240, 0.390)	(0.496, 0.165, 0.297)	(0.713, 0.063, 0.165)	(0.344, 0.250, 0.396)
E_6	(0.643, 0.089, 0.204)	(0.442, 0.181, 0.332)	(0.441, 0.189, 0.341)	(0.385, 0.211, 0.359)	(0.572, 0.117, 0.254)
E_7	(0.504, 0.147, 0.295)	(0.383, 0.219, 0.382)	(0.330, 0.257, 0.401)	(0.442, 0.194, 0.345)	(0.480, 0.174, 0.307)
E_8	(0.458, 0.176, 0.308)	(0.341, 0.252, 0.400)	(0.467, 0.178, 0.311)	(0.465, 0.172, 0.302)	(0.472, 0.176, 0.309)
E_9	(0.437, 0.184, 0.335)	(0.321, 0.261, 0.408)	(0.294, 0.275, 0.420)	(0.494, 0.160, 0.294)	(0.467, 0.178, 0.311)
E_{10}	(0.573, 0.121, 0.260)	(0.437, 0.184, 0.335)	(0.442, 0.181, 0.332)	(0.509, 0.147, 0.279)	(0.602, 0.104, 0.239)
E_{11}	(0.297, 0.273, 0.419)	(0.299, 0.275, 0.421)	(0.467, 0.178, 0.311)	(0.472, 0.176, 0.309)	(0.457, 0.164, 0.312)
E_{12}	(0.573, 0.121, 0.260)	(0.383, 0.219, 0.366)	(0.399, 0.211, 0.360)	(0.422, 0.201, 0.353)	(0.629, 0.181, 0.211)
E_{13}	(0.465, 0.165 0.316)	(0.328, 0.259, 0.402)	(0.520, 0.140, 0.274)	(0.543, 0.129, 0.256)	(0.590, 0.112, 0.224)
E_{14}	(0.524, 0.140, 0.271)	(0.398, 0.213, 0.360)	(0.297, 0.273, 0.402)	(0.419, 0.191 0.340)	(0.536, 0.141, 0.283)
E_{15}	(0.417, 0.204, 0.356)	(0.422, 0.201, 0.353)	(0.437, 0.184, 0.335)	(0.509, 0.147, 0.279)	(0.494, 0.160, 0.294)
E_{16}	(0.462, 0.168, 0.319)	(0.381, 0.231, 0.381)	(0.483, 0.153, 0.303)	(0.313, 0.268, 0.411)	(0.499, 0.151, 0.300)
E_{17}	(0.319, 0.265, 0.410)	(0.330, 0.257, 0.401)	(0.540, 0.135, 0.265)	(0.520, 0.140, 0.274)	(0.399, 0.210, 0.358)
E_{18}	(0.467, 0.178, 0.311)	(0.451, 0.163, 0.309)	(0.539, 0.129, 0.259)	(0.614, 0.096, 0.227)	(0.614, 0.096, 0.227)
E_{19}	(0.366, 0.215, 0.360)	(0.379, 0.232, 0.379)	(0.437, 0.184, 0.335)	(0.288, 0.277, 0.418)	(0.531, 0.124, 0.268)

Table 2.9: Computation of the Criteria's Weights

Criteria	**Weights** (ϑ_j)
E_1	0.0213
E_2	0.0344
E_3	0.0432
E_4	0.0309
E_5	0.0215
E_6	0.0353
E_7	0.0243
E_8	0.0209
E_9	0.0166
E_{10}	0.0264
E_{11}	0.0166
E_{12}	0.0294
E_{13}	0.0224
E_{14}	0.0274
E_{15}	0.0257
E_{16}	0.0221
E_{17}	0.0193
E_{18}	0.0171
E_{19}	0.0226

Table 2.10: Determination of Compromise Measure for Each HFC

	S_i	R_i	Q_i
H_1	0.181	0.087	0.164
H_2	0.550	0.211	0.948
H_3	0.484	0.233	0.914
H_4	0.502	0.187	0.827
H_5	0.165	0.028	0.000
Ranking Order	$S_5 > S_1 > S_3 > S_4 > S_2$	$R_5 > R_1 > R_4 > R_2 > R_3$	$Q_5 > Q_1 > Q_4 > Q_3 > Q_2$

- **Step 6.** In this step, the calculated values of "picture fuzzy positive ideal solution" r_j^+ and "picture fuzzy negative ideal solution" r_j^- are as follows:

$$\begin{aligned} r_j^+ = \{&(0.481, 0.157, 0.307), (0.398, 0.213, 0.361), (0.261, 0.294, 0.433), (0.524, 0.140, 0.271),\\ &(0.713, 0.063, 0.165), (0.642, 0.089, 0.204), (0.504, 0.148, 0.295), (0.472, 0.172, 0.302),\\ &(0.494, 0.160, 0.294), (0.602, 0.104, 0.239), (0.472, 0.164, 0.309), (0.630, 0.121, 0.211),\\ &(0.590, 0.112, 0.224), (0.536, 0.140, 0.271), (0.509, 0.147, 0.280), (0.499, 0.151, 0.300),\\ &(0.540, 0.135, 0.265), (0.614, 0.096, 0.227), (0.531, 0.124, 0.268)\}; \end{aligned} \tag{2.3.1}$$

and

$$\begin{aligned} r_j^- = \{&(0.685, 0.068, 0.183), (0.620, 0.100, 0.200), (0.451, 0.185, 0.315), (0.268, 0.288, 0.427),\\ &(0.250, 0.300, 0.397), (0.385, 0.211, 0.366), (0.330, 0.257, 0.430), (0.341, 0.252, 0.398),\\ &(0.294, 0.275, 0.420), (0.437, 0.184, 0.335), (0.297, 0.275, 0.421), (0.383, 0.219, 0.366),\\ &(0.328, 0.259, 0.419), (0.297, 0.303, 0.419), (0.417, 0.204, 0.353), (0.313, 0.268, 0.411),\\ &(0.319, 0.265, 0.401), (0.451, 0.178, 0.319), (0.288, 0.277, 0.418)\}; \end{aligned} \tag{2.3.2}$$

Remark: The above-mentioned steps cover the first 6 common stages of the proposed technique. Now, we go for the computations in two stages - the picture fuzzy VIKOR method and the picture fuzzy TOPSIS method.

— Picture Fuzzy "VIKOR" Method

- **Step 7.** Now by making use of equations (2.2.7), (2.2.8) and (2.2.9), we compute the values of S_i, R_i and Q_i respectively. To compute the values of the compromise measure, we take $y = 0.5$. The calculated values are shown in the Table 2.10.

- **Step 8.** Based on the values obtained from S_i, R_i and Q_i in the above step, the rating results are found to be as follows:

$S_5 > S_1 > S_3 > S_4 > S_2$; $R_5 > R_1 > R_4 > R_2 > R_3$; $Q_5 > Q_1 > Q_4 > Q_3 > Q_2$.

Table 2.11: Analysis of Sensitivity for Different Values of y

	H_1	H_2	H_3	H_4	H_5
S_i	0.181	0.550	0.484	0.502	0.165
R_i	0.087	0.211	0.233	0.187	0.028
Q_i $(y = 0.0)$	0.28746	0.89589	1.0000	0.77753	0.0
Q_i $(y = 0.1)$	0.26282	0.90630	0.98284	0.78747	0.0
Q_i $(y = 0.2)$	0.23818	0.91671	0.96569	0.79742	0.0
Q_i $(y = 0.3)$	0.21354	0.92712	0.94853	0.80736	0.0
Q_i $(y = 0.4)$	0.18889	0.93753	0.93137	0.8173	0.0
Q_i $(y = 0.5)$	0.16425	0.94794	0.91421	0.82724	0.0
Q_i $(y = 0.6)$	0.13961	0.95836	0.89706	0.83718	0.0
Q_i $(y = 0.7)$	0.11497	0.96877	0.8799	0.84712	0.0
Q_i $(y = 0.8)$	0.09033	0.97918	0.86274	0.85706	0.0
Q_i $(y = 0.9)$	0.06569	0.98959	0.84558	0.8670	0.0
Q_i $(y = 1.0)$	0.04105	1.0000	82843	0.87694	0.0

- **Step 9.** Based on the values obtained in the descending order of the Q_i's, the HFC H_5 is considered to be the optimal alternative. Since

$$Q(A_{(2)}) - Q(A_{(1)}) = 0.164 < \frac{1}{5-1} = 0.25,$$

hence, the HFC H_5 satisfies the condition C_2. Therefore, the HFC H_5 is the optimal choice.

Discussion Over Compromise Solution's Sensitivity:

Based on the proposed methodology and computational procedure, we present the following analysis in context with the sensitivity:

- Here, we take different possible values of weights y ranging from 0 to 1 to maximize the group utility and prepare the sensitivity analysis chart concerning the compromise solution.
- It may also be noted that the values of the parameters are also varying to understand the sensitivity issue.
- The computed values have been tabulated in Table 2.11 and presented with the help of Figure 2.3 and Figure 2.4.

Based on the results obtained in Table 2.11, it is to be concluded that the fuel cell H_1 is the most suitable HFC which can also be viewed from Figure 2.5.

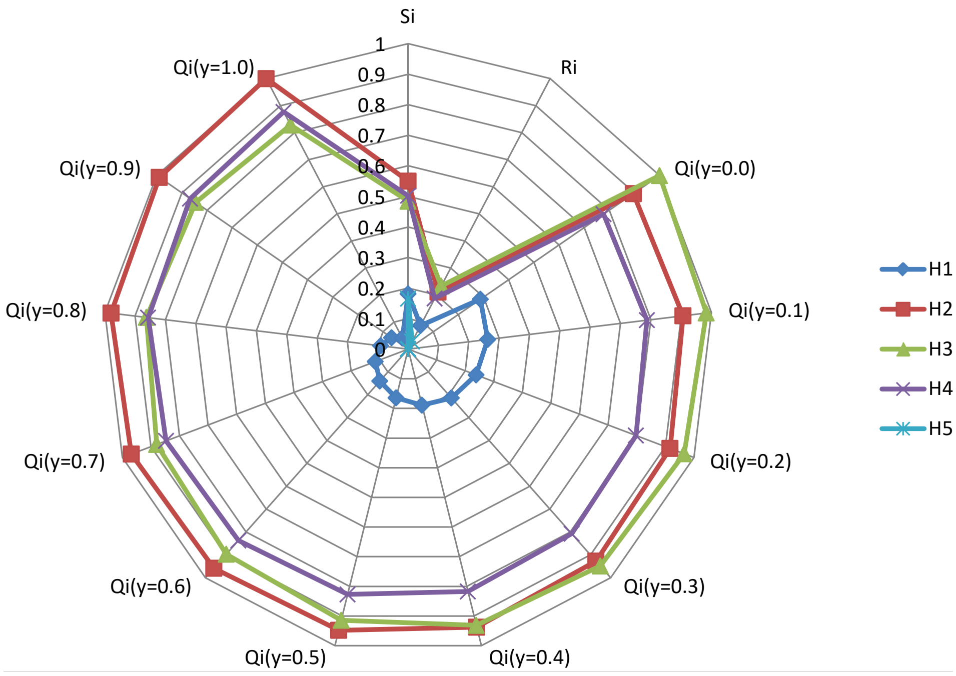

Figure 2.3: Sensitivity Analysis of Alternatives w.r.t. Measures

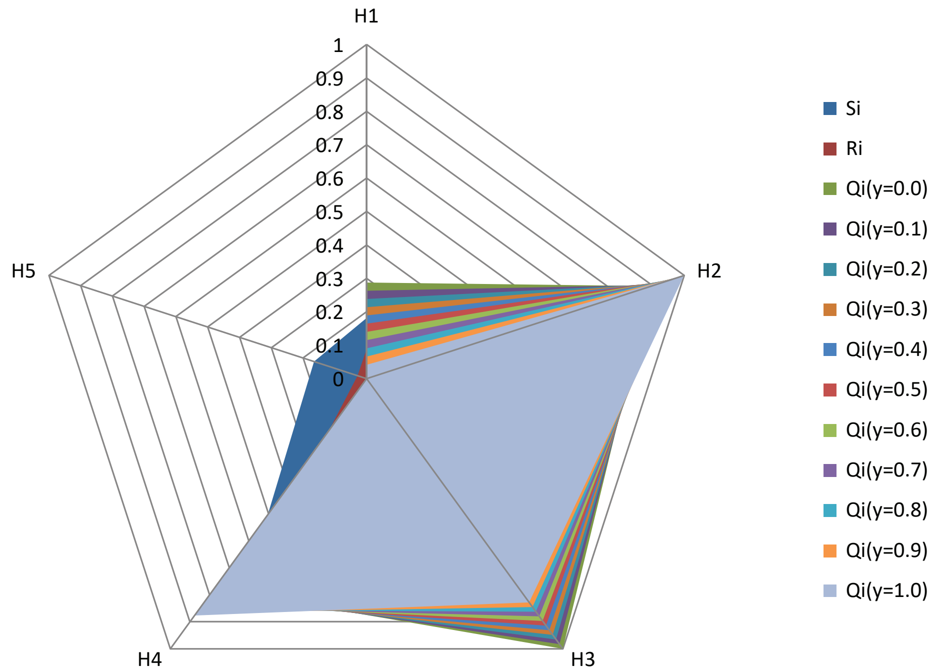

Figure 2.4: Sensitivity Analysis of Compromise Measure

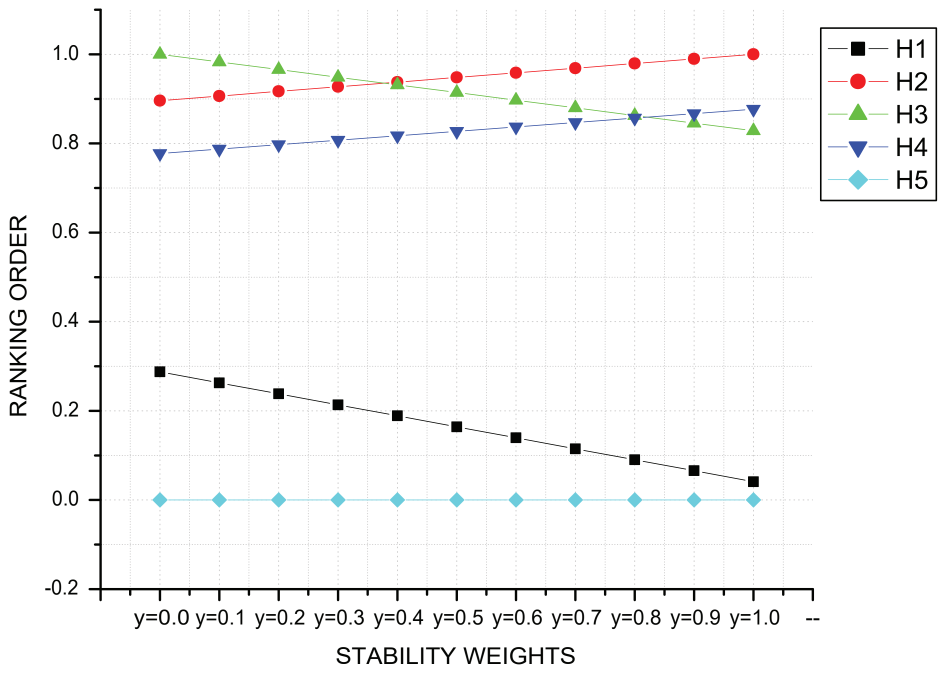

Figure 2.5: Order of Ranking w.r.t. Weights of Stability (y)

— **Picture Fuzzy "TOPSIS" Method**

- **Step 7.** The different values of the discriminant measures of H_i's $\forall$ i = 1, 2, 3, 4, 5 from r_j^+ and r_j^- respectively have been computed by making use of the equation (2.1.1) and tabulated in Table 2.12.

Table 2.12: Computation of Discriminant Measure between $H_i's$ and r_j^+/r_j^-

	$\mathbb{I}_R^S(H_i, r_j^+)$	$\mathbb{I}_R^S(H_i, r_j^-)$
H_1	0.3785	0.4531
H_2	0.4108	0.3891
H_3	0.3878	0.4370
H_4	0.3908	0.4409
H_5	0.3742	0.4639

- **Step 8.** Finally, we evaluate the values of the "coefficient of relative closeness" with the help of equation (2.2.11) and shown in Table 2.13.

- **Step 9.** Now, based on the evaluated values of the "coefficient of relative closeness", the order of prioritization for HFCs is listed as follows:

$$H_5 > H_1 > H_4 > H_3 > H_2.$$

Table 2.13: Relative Closeness's Coefficient

Sites	Closeness Index
H_1	0.5448
H_2	0.4864
H_3	0.5298
H_4	0.5301
H_5	0.5535

Therefore, based on the "coefficient of relative closeness", the HFC H_5 is optimal for use.

Remark: It may be noted that under the systematic process of prioritization with the help of the proposed methodologies of (R, S)-Norm picture fuzzy information measures synced with VIKOR and TOPSIS techniques, both the methods work consistently and appropriately. Based on the numerical values under consideration, the fuel cell power backup system H_5 was found to be the best solution by both methods.

The (R, S)-Norm discriminant measure is designed to address issues of uncertainty and imprecision more effectively than TOPSIS and VIKOR. Empirical results from simulations using benchmark datasets in supply chain management indicate higher consistency in ranking when handling data with significant overlaps or missing values.

Compared to TOPSIS and VIKOR, the (R, S)-Norm reduces the number of iterative calculations. For instance, in a dataset with many alternatives and criteria, the (R, S)-Norm achieved decision results faster due to its simplified normalization and aggregation mechanisms.

2.4 Comparative Analysis and Advantages

Here, the advantageous features of the presented picture fuzzy VIKOR/TOPSIS techniques are listed along with comparative remarks in contrast with the existing techniques. Based on the obtained results and motivation behind the proposed methodologies, the following remarks are being listed:

- The overall computational analysis incorporating the (R, S)-Norm picture information measures (entropy/discriminant measure) provides a wider/broader coverage of the imprecise information in the fulfillment of the requirements, as well as the issue of information loss, has been greatly reduced in the suggested approaches with the involvement of the two important uncertainty components, i.e., degree of abstain and degree of refusal. Also, with the parameters R and S, we gain flexibility in the computations for better and optimal results.

- The proposed bi-parametric information measure includes the idea of picture fuzzy sets while many researchers have applied FSs/IFSs/PyFSs in which the degree of abstain and refusal are missing, which may have a risk of information loss. As mentioned in the section of the introduction, picture fuzzy sets have a greater range of coverage for inaccurate and missing information.

- To have complete and precise decisions for the multi-criteria problems under consideration, we have properly assigned the expert's weights in devising the proposed techniques for the optimal solution, while Juli et al. [101] implemented the picture fuzzy TOPSIS in risk management problems without considering the expert's weights. Also, Boran et al. [78] applied the TOPSIS in a straight way in group decision-making problems.

- The ability to produce a compromise solution that maximizes the group utility while minimizing individual regret is one of the key advantages of the picture fuzzy VIKOR method. In comparison to the ideal solution, the compromise solution has been found by using the modified VIKOR approach which is the optimal one.

- The idea of picture fuzzy numbers can used to handle the ambiguous and insufficient information that can arise in MCDM problems. The use of the picture fuzzy number is determined to be more suited since the input parameters, such as the expert's weights, the weights assigned to the criterion, selection of alternatives may have some degree of extra uncertainty which is easily covered by the degree of abstain and refusal.

- It may be observed that the main components of an MCDM technique are the criterion's weights, the expert's weights, and the computation of the available alternatives using the established criteria. Hence, any novel MCDM technique

Table 2.14: Comparison with the Existing Techniques

"Research Articles"	**"Expert's Weightage"**	**"Criterion's Weight"**	**"Qualitative Variables"**	**"Entropy and Discriminant Measure"**	**"Assessment Information of Alternatives"**
Kaya & Kahraman [102]	Taken into account	Partly Known	✓	×	"Fuzzy Set"
Kahraman & Kaya [103]	Taken into account	Partly Known	✓	×	"Fuzzy Set"
Mousavi et al. [104]	Calculated	Totally Unknown	✓	×	"Hesitant Fuzzy Set"
Mishra et al. [105]	Taken into account	Partly Known	✓	Discriminant Measure	"Intuitionistic Fuzzy Set"
Schitea et al. [106]	Calculated	Totally Unknown	✓	×	"Intuitionistic Fuzzy Set"
Alipour et al. [86]	Calculated	Totally Unknown	✓	Entropy Measure	"Pythagorean Fuzzy Set"
Proposed Methods	Calculated	Totally Unknown	✓	Entropy and Discriminant Measure	"Picture Fuzzy Set"

Table 2.15: Comparative Analysis with the Various Existing MCDM Methods

	"WASPAS" [107]	"SWARA-COPRAS" [86]	"Fuzzy MCDM" [80]	"Proposed VIKOR"	"Proposed TPOSIS"
H_1	2	2	2	2	2
H_2	5	5	5	5	5
H_3	4	4	4	4	4
H_4	3	3	3	3	3
H_5	1	1	1	1	1

stresses these mentioned characteristics. Here, we compare our proposed technique along with its advantages based on the above-mentioned characteristics with various existing techniques in the literature as shown in Table 2.14.

- The final order of rankings for the potential HFCs that have been the subject of recent studies by a variety of scholars are compiled and presented in Table 2.15, which demonstrates the sharp consistency of the suggested methodology. The results are statistically comparable, the proposed techniques are distinct from the other techniques described in the literature.

- It may also be noted that the qualitative data used in the present work is identical to the data in the study [80] and [86]. The qualitative data has been incorporated with the simple fuzzy/Pythagorean fuzzy information and conjunction with some other MCDM methodologies. The framed model is following [107] to have a valid comparison. On the other hand, the proposed study utilizes the picture fuzzy set with the modified MCDM techniques of VIKOR/TOPSIS. To have a valid comparison we should take the same informative data otherwise comparing distinct data problems has no meaning, and the results are exists not consistent.

- The application of the proposed technique is under the necessity of a MCDM problem. VIKOR can enhance the collective utility and reduce individual regret whereas TOPSIS's compensating strategy enables the barter between the criteria. Generally speaking, our proposed techniques accommodate the maximum impreciseness and vagueness of the information having the components of abstain and refusal, which is appropriate for complex situations.

2.5 Conclusions

The evaluation techniques discussed in this chapter give policymakers valuable information to help them with the selection of the best possible HFC technology. The criteria under assessment were deemed to be highly diversified by integrating the technological, environmental, economic, and social factors and were managed through the application of TOPSIS and VIKOR decision-making methodologies within the framework of "bi-parametric fuzzy picture discriminant measure". The evaluation results for the alternative "Fuel Cell Backup Power Systems" after utilizing the suggested approaches are found to be consistent and acceptable.

Chapter 3

Picture Fuzzy Soft Dombi Aggregation Operators

In this chapter, the notion of the score/accuracy function of picture fuzzy soft numbers and picture fuzzy soft Dombi aggregation operators (weighted/ordered weighted average, hybrid/weighted geometric) have been introduced along with various operational laws and properties. Further, for the sake of providing a larger space to the decision makers and including the parametrization feature of the imprecise information, the traditional "EDAS (Evaluation Based On Distance from Average Solution)" method has been modified and presented in the light of the proposed score/accuracy function and the introduced Dombi aggregation operators. In addition to this, an illustrative example related to digital farming has been studied in detail showing that the proposed methodology is highly helpful in finding the best alternative to have sustainable farming among various types of agrifarming. To understand the feasibility, loftiness and dependability of the proposed modified EDAS methodology, the comparative remarks and advantages have been listed for better understanding and readability with some existing MCDM approaches.

3.1 Picture Fuzzy Soft Dombi Arithmetic Aggregation Operators

In this section, for the sake of computations, a picture fuzzy soft set (PFSS) is regarded as $I_u = (\rho_{I_u}, \tau_{I_u}, \omega_{I_u})$ and called as picture fuzzy soft number (PFSN), where u is referential subscript which is used for building a relationship between alternatives and attributes in the required examples. For applications, we have to prioritize these numbers, for which we propose the score and accuracy functions for the picture fuzzy soft numbers as follows:

Definition 13 *Let $I_u = (\rho_{I_u}, \tau_{I_u}, \omega_{I_u})$ be the picture fuzzy soft number, then*

- *the score function is given as $S(I_u) = \rho^n_{I_u} - \omega^n_{I_u}$; $S(I_u) \in [-1, 1]$.*
- *the accuracy function is given as $H(I_u) = \rho^n_{I_u} + \tau^n_{I_u} + \omega^n_{I_u}$; $H(I_u) \in [0, 1]$.*

Next, the order-relation between two picture fuzzy soft numbers are given as follows: Let $I_u = (\rho_{I_u}, \tau_{I_u}, \omega_{I_u})$ and $I_v = (\rho_{I_v}, \tau_{I_v}, \omega_{I_v})$ be two picture fuzzy soft numbers then

- $I_u \geq I_v$ if $S(I_u) \geq S(I_v)$.
- $I_u \leq I_v$ if $S(I_u) \leq S(I_v)$.

  In case, if $S(I_u) = S(I_v)$ for any two PFSN, then

- $I_u \geq I_v$ if $H(I_u) \geq H(I_v)$.
- $I_u \leq I_v$ if $H(I_u) \leq H(I_v)$.
- $I_u \sim I_v$ if $H(I_u) = H(I_v)$.

Remark: From, the above definition, the score function is monotonically increasing with respect to its variables.

Definition 14 *Let $I_u = (\rho_{I_u} \tau_{I_u}, \omega_{I_u})$ and $I_v = (\rho_{I_v}, \tau_{I_v}, \omega_{I_v})$ two picture fuzzy soft numbers and $\lambda > 0$ be any real number. Then the following operations are defined over the two picture fuzzy soft numbers:*

(a) $I_u \oplus I_v = (\rho_{I_u} + \rho_{I_v} - \rho_{I_u}\rho_{I_v},\ \tau_{I_u}\tau_{I_v},\ \omega_{I_u}\omega_{I_v})$.

(b) $I_u \otimes I_v = (\rho_{I_u}\rho_{I_v},\ \tau_{I_u} + \tau_{I_v} - \tau_{I_u}\tau_{I_v},\ \omega_{I_u} + \omega_{I_v} - \omega_{I_u}\omega_{I_v})$.

(c) $\lambda I_u = (1 - (1 - \rho_{I_u})^\lambda, \tau_{I_u}^\lambda, \omega_{I_u}^\lambda)$.

(d) $I_u^\lambda = \big((\rho_{I_u})^\lambda,\ 1 - (1 - \tau_{I_u})^\lambda,\ 1 - (1 - \omega_{I_u})^\lambda\big)$.

(e) $I_u^c = (\omega_{I_u},\ \tau_{I_u},\ \rho_{I_u})$.

Definition 15 *Suppose* $I_u = (\rho_{I_u}, \tau_{I_u}, \omega_{I_u})$ *and* $I_v = (\rho_v, \tau_v, \omega_v)$ *be two picture fuzzy soft numbers and* $\lambda, \lambda_1, \lambda_2 > 0$ *be the real numbers. Then the following operational laws hold:*

(i) $I_u \oplus I_v = I_v \oplus I_u$

(ii) $I_u \otimes I_v = I_v \otimes I_u$

(iii) $\lambda(I_u \oplus I_v) = \lambda I_v \oplus \lambda I_u$

(iv) $(I_u \otimes I_v)^\lambda = I_v^\lambda \otimes I_u^\lambda$

(v) $\lambda_1 I_u \oplus \lambda_2 I_u = (\lambda_1 + \lambda_2) I_u$

(vi) $I_u^{\lambda_1} \otimes I_u^{\lambda_2} = I_u^{(\lambda_1 + \lambda_2)}$

(vii) $(I_u^{\lambda_1})^{\lambda_2} = I_u^{\lambda_1 \lambda_2}$.

Definition 16 *Suppose* $\mathcal{T}$ *is a collection of all picture fuzzy soft numbers.*
Let $(I_{u_1},\ I_{u_2},\ \ldots,\ I_{u_m}) \in \mathcal{T}^m$. *A mapping* $PFSWA_\psi : \mathcal{T}^m \longrightarrow \mathcal{T}$ *is said to be picture fuzzy soft weighted averaging operator*($PFSWA$)*, if*

$$PFSWA_\psi(I_{u_1}, I_{u_2}, \ldots, I_{u_m}) = \oplus_{j=1}^m (\psi_j I_{u_j}) = \left(1 - \prod_{j=1}^m (1 - \rho_{I_{u_j}})^{\psi_j},\ \prod_{j=1}^m (\tau_{I_{u_j}})^{\psi_j},\ \prod_{j=1}^m (\omega_{I_{u_j}})^{\psi_j}\right); \tag{3.1.1}$$

where $\psi = (\psi_1, \psi_2, \ldots, \psi_m)^T$ *is the weight vector corresponding to* $(I_{u_j})_{j=1}^m$ *such that* $\psi_j \geq 0,$ *for all* j; $\sum_{j=1}^m \psi_j = 1$.

Definition 17 *Suppose* $\mathcal{T}$ *is a collection of all picture fuzzy soft numbers.*
Let $(I_{u_1},\ I_{u_2},\ \ldots,\ I_{u_m}) \in \mathcal{T}^m$. *A mapping* $PFSOWA_\psi : \mathcal{T}^m \longrightarrow \mathcal{T}$ *is called picture fuzzy soft ordered weighted averaging operator*($PFSOWA$)*, if*

$$PFSOWA_\psi(I_{u_1}, I_{u_2}, \ldots, I_{u_m}) = \left(1 - \prod_{j=1}^m (1 - \rho_{I_{u_{\sigma(j)}}})^{\psi_j},\ \prod_{j=1}^m (\tau_{I_{u_{\sigma(j)}}})^{\psi_j},\ \prod_{j=1}^m (\omega_{I_{u_{\sigma(j)}}})^{\psi_j}\right); \tag{3.1.2}$$

where $\psi = (\psi_1, \psi_2, \ldots, \psi_m)^T$ *is the weight vector corresponding to* $(I_{u_j})_{j=1}^m$ *such that* $\psi_j \geq 0$, *for all* j; $\sum_{j=1}^{m} \psi_j = 1$ *and* $(\sigma(1), \sigma(2), \ldots, \sigma(m))$ *is a possible permutation of* $(1, 2, \ldots, m)$, *s.t.* $I_{u_{\sigma(j+1)}} \leq I_{u_{\sigma(j)}}$ *for all* $j = 1, 2, \ldots, m-1$.

Definition 18 *Suppose* $\mathcal{P}$ *is a collection of all picture fuzzy soft numbers. Let* $(T_{u_1}, T_{u_2}, \ldots, T_{u_m}) \in \mathcal{P}^m$. *A mapping* $PFSHA_\omega : \mathcal{P}^m \longrightarrow \mathcal{P}$ *is called picture fuzzy soft hybrid averaging operator, if*

$$PFSHA_{\psi,\gamma}(I_{u_1}, I_{u_2}, \ldots, I_{u_m}) = \left(1 - \prod_{j=1}^{m}(1-\rho_{\widetilde{I}_{u_{\sigma(j)}}})^{\gamma_j}, \prod_{j=1}^{m}(\tau_{\widetilde{I}_{u_{\sigma(j)}}})^{\gamma_j}, \prod_{j=1}^{m}(\omega_{\widetilde{I}_{u_{\sigma(j)}}})^{\gamma_j}\right) \tag{3.1.3}$$

where $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m)^T$ *is the weight vector corresponding to* $(\widetilde{I}_{u_{\sigma(j)}})_{j=1}^m$ *such that*
$\gamma_j \geq 0$, *for all* j; $\sum_{j=1}^{m} \gamma_j = 1$.
$\widetilde{I}_{u_{\sigma(j)}}$ *is the* j^{th} *largest of the weighted PFSNs* $\widetilde{I}_{u_j}$; *where* $\widetilde{I}_{u_j} = (m\psi_j)I_{u_j}$ *and* m *is the balancing coefficient with* $\psi = (\psi_1, \psi_2, \ldots, \psi_n)^T$ *being the weight vector of* I_{u_j} *with* $\psi_j \geq 0$, *for all* j; $\sum_{j=1}^{m} \psi_j = 1$.

Remarks:

- In case, we take uniform distribution of weights as $\gamma = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$ then the picture fuzzy soft hybrid averaging operator gives picture fuzzy soft weighted averaging operator.
- However, if we take $\psi = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$ then the picture fuzzy soft hybrid averaging operator gives picture fuzzy soft ordered weighted averaging operator.

Definition 19 *Let* $I_u = (\rho_{I_u}, \tau_{I_u}, \omega_{I_u})$ *and* $I_v = (\rho_{I_v}, \tau_{I_v}, \omega_{I_v})$ *be two PFSNs,* $R \geq 1$ *and* $\lambda > 0$. *Then Dombi* $t-$*norm and* $t-$*conorm operations of PFSNs are as follows:*

$$(i)\quad I_u \oplus I_v = \left\langle 1 - \frac{1}{1+\left\{\left(\frac{\rho_{I_u}}{1-\rho_{I_u}}\right)^R + \left(\frac{\rho_{I_v}}{1-\rho_{I_v}}\right)^R\right\}^{\frac{1}{R}}}, \frac{1}{1+\left\{\left(\frac{1-\tau_{I_u}}{\tau_{I_u}}\right)^R + \left(\frac{1-\tau_{I_v}}{\tau_{I_v}}\right)^R\right\}^{\frac{1}{R}}}, \frac{1}{1+\left\{\left(\frac{1-\omega_{I_u}}{\omega_{I_u}}\right)^R + \left(\frac{1-\omega_{I_v}}{\omega_{I_v}}\right)^R\right\}^{\frac{1}{R}}} \right\rangle$$

$$(ii)\quad I_u \otimes I_v = \left\langle \frac{1}{1+\left\{\left(\frac{1-\rho_{I_u}}{\rho_{I_u}}\right)^R + \left(\frac{1-\rho_{I_v}}{\rho_{I_v}}\right)^R\right\}^{\frac{1}{R}}}, 1 - \frac{1}{1+\left\{\left(\frac{\tau_{I_u}}{1-\tau_{I_u}}\right)^R + \left(\frac{\tau_{I_v}}{1-\tau_{I_v}}\right)^R\right\}^{\frac{1}{R}}}, 1 - \frac{1}{1+\left\{\left(\frac{\omega_{I_u}}{1-\omega_{I_u}}\right)^R + \left(\frac{\omega_{I_v}}{1-\omega_{I_v}}\right)^R\right\}^{\frac{1}{R}}} \right\rangle$$

(iii) $\lambda I_u = \left\langle 1 - \frac{1}{1+\left\{\lambda\left(\frac{\rho_{I_u}}{1-\rho_{I_u}}\right)^R\right\}^{\frac{1}{R}}}, \frac{1}{1+\left\{\lambda\left(\frac{1-\tau_{I_u}}{\tau_{I_u}}\right)^R\right\}^{\frac{1}{R}}}, \frac{1}{1+\left\{\lambda\left(\frac{1-\omega_{I_u}}{\omega_{I_u}}\right)^R\right\}^{\frac{1}{R}}} \right\rangle$

(iv) $I_u^\lambda = \left\langle \frac{1}{1+\left\{\lambda\left(\frac{1-\rho_{I_u}}{\rho_{I_u}}\right)^R\right\}^{\frac{1}{R}}}, 1 - \frac{1}{1+\left\{\lambda\left(\frac{\tau_{I_u}}{1-\tau_{I_u}}\right)^R\right\}^{\frac{1}{R}}}, 1 - \frac{1}{1+\left\{\lambda\left(\frac{\omega_{I_u}}{1-\omega_{I_u}}\right)^R\right\}^{\frac{1}{R}}} \right\rangle$

Theorem 2 *Let* $I_u = (\rho_{I_u},\ \tau_{I_u},\ \omega_{I_u})$, $I_v = (\rho_{I_v},\ \tau_{I_v},\ \omega_{I_v})$ *and* $I_w = (\rho_{I_w},\ \tau_{I_w},\ \omega_{I_w})$ *be three PFSNs and* $\lambda,\ \lambda_1,\ \lambda_2 > 0$ *be the real numbers. Then the following operational laws (in terms of Dombi soft operations) hold:*

(i) $I_u \oplus I_v = I_v \oplus I_u$

(ii) $I_u \otimes I_v = I_v \otimes I_u$

(iii) $\lambda(I_u \oplus I_v) = \lambda I_v \oplus \lambda I_u$

(iv) $(I_u \otimes I_v)^\lambda = I_v^\lambda \otimes I_u^\lambda$

(v) $\lambda_1 I_w \oplus \lambda_2 I_w = (\lambda_1 + \lambda_2) I_w$

(vi) $I_w^{\lambda_1} \otimes I_w^{\lambda_2} = I_w^{(\lambda_1 + \lambda_2)}$

(vii) $(I_w^{\lambda_1})^{\lambda_2} = I_w^{\lambda_1 \lambda_2}$.

Proof: For the proof of this theorem we shall make use of the definition 19. As per the definition 19, we get

(i)
$$I_u \oplus I_v = \left\langle 1 - \frac{1}{1+\left\{\left(\frac{\rho_{I_u}}{1-\rho_{I_u}}\right)^R+\left(\frac{\rho_{I_v}}{1-\rho_{I_v}}\right)^R\right\}^{\frac{1}{R}}}, \frac{1}{1+\left\{\left(\frac{1-\tau_{I_u}}{\tau_{I_u}}\right)^R+\left(\frac{1-\tau_{I_v}}{\tau_{I_v}}\right)^R\right\}^{\frac{1}{R}}}, \frac{1}{1+\left\{\left(\frac{1-\omega_{I_u}}{\omega_{I_u}}\right)^R+\left(\frac{1-\omega_{I_v}}{\omega_{I_v}}\right)^R\right\}^{\frac{1}{R}}} \right\rangle$$
$$= \left\langle 1 - \frac{1}{1+\left\{\left(\frac{\rho_{I_v}}{1-\rho_{I_v}}\right)^R+\left(\frac{\rho_{I_u}}{1-\rho_{I_u}}\right)^R\right\}^{\frac{1}{R}}}, \frac{1}{1+\left\{\left(\frac{1-\tau_{I_v}}{\tau_{I_v}}\right)^R+\left(\frac{1-\tau_{I_u}}{\tau_{I_u}}\right)^R\right\}^{\frac{1}{R}}}, \frac{1}{1+\left\{\left(\frac{1-\omega_{I_v}}{\omega_{I_v}}\right)^R+\left(\frac{1-\omega_{I_u}}{\omega_{I_u}}\right)^R\right\}^{\frac{1}{R}}} \right\rangle$$
$$= I_v \oplus I_u$$

(ii) Proof of (ii) will be the same as proof of (i).

(iii) Let $x = 1 - \frac{1}{1+\left\{\left(\frac{\rho_{I_u}}{1-\rho_{I_u}}\right)^R+\left(\frac{\rho_{I_v}}{1-\rho_{I_v}}\right)^R\right\}^{\frac{1}{R}}}$.

This implies that $\frac{x}{1-x} = \left\{\left(\frac{\rho_{I_u}}{1-\rho_{I_u}}\right)^R + \left(\frac{\rho_{I_v}}{1-\rho_{I_v}}\right)^R\right\}^{\frac{1}{R}} \implies \left(\frac{x}{1-x}\right)^R = \left\{\left(\frac{\rho_{I_u}}{1-\rho_{I_u}}\right)^R + \left(\frac{\rho_{I_v}}{1-\rho_{I_v}}\right)^R\right\}$.

Now, using the above argument, we get

$$\lambda\,(I_u \oplus I_v)$$
$$= \lambda \left\langle 1 - \frac{1}{1+\left\{\left(\frac{\rho_{I_u}}{1-\rho_{I_u}}\right)^R+\left(\frac{\rho_{I_v}}{1-\rho_{I_v}}\right)^R\right\}^{\frac{1}{R}}}, \frac{1}{1+\left\{\left(\frac{1-\tau_{I_u}}{\tau_{I_u}}\right)^R+\left(\frac{1-\tau_{I_v}}{\tau_{I_v}}\right)^R\right\}^{\frac{1}{R}}}, \frac{1}{1+\left\{\left(\frac{1-\omega_{I_u}}{\omega_{I_u}}\right)^R+\left(\frac{1-\omega_{I_v}}{\omega_{I_v}}\right)^R\right\}^{\frac{1}{R}}} \right\rangle$$
$$= \left\langle 1 - \frac{1}{1+\left\{\lambda\left(\frac{\rho_{I_u}}{1-\rho_{I_u}}\right)^R+\lambda\left(\frac{\rho_{I_v}}{1-\rho_{I_v}}\right)^R\right\}^{\frac{1}{R}}}, \frac{1}{1+\left\{\lambda\left(\frac{1-\tau_{I_u}}{\tau_{I_u}}\right)^R+\lambda\left(\frac{1-\tau_{I_v}}{\tau_{I_v}}\right)^R\right\}^{\frac{1}{R}}}, \frac{1}{1+\left\{\lambda\left(\frac{1-\omega_{I_u}}{\omega_{I_u}}\right)^R+\lambda\left(\frac{1-\omega_{I_v}}{\omega_{I_v}}\right)^R\right\}^{\frac{1}{R}}} \right\rangle.$$

Further,

$\lambda I_u \oplus \lambda I_v$

$$= \left\langle 1-\frac{1}{1+\left\{\lambda\left(\frac{\rho_{I_u}}{1-\rho_{I_u}}\right)^R\right\}^{\frac{1}{R}}}, \frac{1}{1+\left\{\lambda\left(\frac{1-\tau_{I_u}}{\tau_{I_u}}\right)^R\right\}^{\frac{1}{R}}}, \frac{1}{1+\left\{\lambda\left(\frac{1-\omega_{I_u}}{\omega_{I_u}}\right)^R\right\}^{\frac{1}{R}}} \right\rangle$$

$$\oplus \left\langle 1-\frac{1}{1+\left\{\lambda\left(\frac{\rho_{I_v}}{1-\rho_{I_v}}\right)^R\right\}^{\frac{1}{R}}}, \frac{1}{1+\left\{\lambda\left(\frac{1-\tau_{I_v}}{\tau_{I_v}}\right)^R\right\}^{\frac{1}{R}}}, \frac{1}{1+\left\{\lambda\left(\frac{1-\omega_{I_v}}{\omega_{I_v}}\right)^R\right\}^{\frac{1}{R}}} \right\rangle$$

$$= \left\langle 1-\frac{1}{1+\left\{\lambda\left(\frac{\rho_{I_u}}{1-\rho_{I_u}}\right)^R+\lambda\left(\frac{\rho_{I_v}}{1-\rho_{I_v}}\right)^R\right\}^{\frac{1}{R}}}, \frac{1}{1+\left\{\lambda\left(\frac{1-\tau_{I_u}}{\tau_{I_u}}\right)^R+\lambda\left(\frac{1-\tau_{I_v}}{\tau_{I_v}}\right)^R\right\}^{\frac{1}{R}}}, \frac{1}{1+\left\{\lambda\left(\frac{1-\omega_{I_u}}{\omega_{I_u}}\right)^R+\lambda\left(\frac{1-\omega_{v}}{\omega_{I_v}}\right)^R\right\}^{\frac{1}{R}}} \right\rangle.$$

$= \lambda\left(I_u \oplus I_v\right).$

(iv) $(I_u \otimes I_v)^\lambda$

$$= \left\langle \frac{1}{1+\left\{\left(\frac{1-\rho_{I_u}}{\rho_{I_u}}\right)^R+\left(\frac{1-\rho_{I_v}}{\rho_{I_v}}\right)^R\right\}^{\frac{1}{R}}}, 1-\frac{1}{1+\left\{\left(\frac{\tau_{I_u}}{1-\tau_{I_u}}\right)^R+\left(\frac{\tau_{I_v}}{1-\tau_{I_v}}\right)^R\right\}^{\frac{1}{R}}}, 1-\frac{1}{1+\left\{\left(\frac{\omega_{I_u}}{1-\omega_{I_u}}\right)^R+\left(\frac{\omega_{I_v}}{1-\omega_{I_v}}\right)^R\right\}^{\frac{1}{R}}} \right\rangle^\lambda$$

$$= \left\langle \frac{1}{1+\left\{\lambda\left(\frac{1-\rho_{I_u}}{\rho_{I_u}}\right)^R+\lambda\left(\frac{1-\rho_{I_v}}{\rho_{I_v}}\right)^R\right\}^{\frac{1}{R}}}, 1-\frac{1}{1+\left\{\lambda\left(\frac{\tau_{I_u}}{1-\tau_{I_u}}\right)^R+\lambda\left(\frac{\tau_{I_v}}{1-\tau_{I_v}}\right)^R\right\}^{\frac{1}{R}}}, 1-\frac{1}{1+\left\{\lambda\left(\frac{\omega_{I_u}}{1-\omega_{I_u}}\right)^R+\lambda\left(\frac{\omega_{I_v}}{1-\omega_{I_v}}\right)^R\right\}^{\frac{1}{R}}} \right\rangle$$

$$= \left\langle \frac{1}{1+\left\{\lambda\left(\frac{1-\rho_{I_u}}{\rho_{I_u}}\right)^R\right\}^{\frac{1}{R}}}, 1-\frac{1}{1+\left\{\lambda\left(\frac{\tau_{I_u}}{1-\tau_{I_u}}\right)^R\right\}^{\frac{1}{R}}}, 1-\frac{1}{1+\left\{\lambda\left(\frac{\omega_{I_u}}{1-\omega_{I_u}}\right)^R\right\}^{\frac{1}{R}}} \right\rangle$$

$$\otimes \left\langle \frac{1}{1+\left\{\lambda\left(\frac{1-\rho_{I_v}}{\rho_{I_v}}\right)^R\right\}^{\frac{1}{R}}}, 1-\frac{1}{1+\left\{\lambda\left(\frac{\tau_{I_v}}{1-\tau_{I_v}}\right)^R\right\}^{\frac{1}{R}}}, 1-\frac{1}{1+\left\{\lambda\left(\frac{\omega_{I_v}}{1-\omega_{I_v}}\right)^R\right\}^{\frac{1}{R}}} \right\rangle$$

$= I_v^\lambda \otimes I_u^\lambda.$

(v) $$\lambda_1 I_w \oplus \lambda_2 I_w = \left\langle 1-\frac{1}{1+\left\{\lambda_1\left(\frac{\rho_{I_w}}{1-\rho_{I_w}}\right)^R\right\}^{\frac{1}{R}}}, \frac{1}{1+\left\{\lambda_1\left(\frac{1-\tau_{I_w}}{\tau_{I_w}}\right)^R\right\}^{\frac{1}{R}}}, \frac{1}{1+\left\{\lambda_1\left(\frac{1-\omega_{I_w}}{\omega_{I_w}}\right)^R\right\}^{\frac{1}{R}}} \right\rangle$$

$$\oplus \left\langle 1-\frac{1}{1+\left\{\lambda_2\left(\frac{\rho_{I_w}}{1-\rho_{I_w}}\right)^R\right\}^{\frac{1}{R}}}, \frac{1}{1+\left\{\lambda_2\left(\frac{1-\tau_{I_w}}{\tau_{I_w}}\right)^R\right\}^{\frac{1}{R}}}, \frac{1}{1+\left\{\lambda_2\left(\frac{1-\omega_{I_w}}{\omega_{I_w}}\right)^R\right\}^{\frac{1}{R}}} \right\rangle$$

$$= \left\langle 1-\frac{1}{1+\left\{(\lambda_1+\lambda_2)\left(\frac{\rho_{I_w}}{1-\rho_{I_w}}\right)^R\right\}^{\frac{1}{R}}}, \frac{1}{1+\left\{(\lambda_1+\lambda_2)\left(\frac{1-\tau_{I_w}}{\tau_{I_w}}\right)^R\right\}^{\frac{1}{R}}}, \frac{1}{1+\left\{(\lambda_1+\lambda_2)\left(\frac{1-\omega_{I_w}}{\omega_{I_w}}\right)^R\right\}^{\frac{1}{R}}} \right\rangle$$

$= (\lambda_1+\lambda_2)\, I_w.$

(vi) $$I_w^{\lambda_1} \otimes I_w^{\lambda_2} = \left\langle \frac{1}{1+\left\{\lambda_1\left(\frac{1-\rho_{I_w}}{\rho_{I_w}}\right)^R\right\}^{\frac{1}{R}}}, 1-\frac{1}{1+\left\{\lambda_1\left(\frac{\tau_{I_w}}{1-\tau_{I_w}}\right)^R\right\}^{\frac{1}{R}}}, 1-\frac{1}{1+\left\{\lambda_1\left(\frac{\omega_{I_w}}{1-\omega_{I_w}}\right)^R\right\}^{\frac{1}{R}}} \right\rangle$$

$$\otimes \left\langle \frac{1}{1+\left\{(\lambda_1+\lambda_2)\left(\frac{1-\rho_{I_w}}{\rho_{I_w}}\right)^R\right\}^{\frac{1}{R}}}, \frac{1}{1+\left\{(\lambda_1+\lambda_2)\left(\frac{\tau_{I_w}}{1-\tau_{I_w}}\right)^R\right\}^{\frac{1}{R}}}, \frac{1}{1+\left\{(\lambda_1+\lambda_2)\left(\frac{\omega_{I_w}}{1-\omega_{I_w}}\right)^R\right\}^{\frac{1}{R}}} \right\rangle$$

$$= \left\langle \frac{1}{1+\left\{(\lambda_1+\lambda_2)\left(\frac{1-\rho_{I_w}}{\rho_{I_w}}\right)^R\right\}^{\frac{1}{R}}}, \frac{1}{1+\left\{(\lambda_1+\lambda_2)\left(\frac{\tau_{I_w}}{1-\tau_{I_w}}\right)^R\right\}^{\frac{1}{R}}}, \frac{1}{1+\left\{(\lambda_1+\lambda_2)\left(\frac{\omega_{I_w}}{1-\omega_{I_w}}\right)^R\right\}^{\frac{1}{R}}} \right\rangle$$

$= I_w^{(\lambda_1+\lambda_2)}.$

(vii) The proof of this can be outlined on similar lines.

Definition 20 *Suppose $\mathcal{T}$ is a collection of all picture fuzzy soft numbers. Let $(I_{u_1},\ I_{u_2},\ \ldots,\ I_{u_m}) \in \mathcal{T}^m$. A mapping $PFSDWA_{\psi} : \mathcal{T}^m \longrightarrow \mathcal{T}$ is said to be a picture fuzzy soft Dombi weighted averaging operator, if*

$$PFSDWA_{\psi}(I_{u_1},\ I_{u_2},\ \ldots,\ I_{u_m}) = \oplus_{j=1}^{m} (\psi_j I_{u_j}); \tag{3.1.4}$$

where $\psi = (\psi_1,\ \psi_2,\ \ldots,\ \psi_m)^T$ is the weight vector corresponding to $(I_{u_j})_{j=1}^m$ s.t. $\psi_j \geq 0$, for all j; $\sum\limits_{j=1}^{m} \psi_j = 1$.

Theorem 3 *The picture fuzzy soft Dombi weighted averaging operator $PFSDWA_{\psi}$ aggregates all the input values and yields a PFSN given by*

$$\begin{aligned} & PFSDWA_{\psi}(I_{u_1},\ I_{u_2},\ \ldots,\ I_{u_m}) = \oplus_{j=1}^{m} (\psi_j I_{u_j}) \\ = & \left\langle 1 - \frac{1}{1+\left\{\sum_{j=1}^{m} \psi_j \left(\frac{\rho_{I_{u_j}}}{1-\rho_{I_{u_j}}}\right)^R\right\}^{\frac{1}{R}}},\ \frac{1}{1+\left\{\sum_{j=1}^{m} \psi_j \left(\frac{1-\tau_{I_{u_j}}}{\tau_{I_{u_j}}}\right)^R\right\}^{\frac{1}{R}}}, \right. \\ & \left. \frac{1}{1+\left\{\sum_{j=1}^{m} \psi_j \left(\frac{1-\omega_{I_{u_j}}}{\omega_{I_{u_j}}}\right)^R\right\}^{\frac{1}{R}}} \right\rangle. \end{aligned}$$

where $\psi = (\psi_1,\ \psi_2,\ \ldots,\ \psi_m)^T$ is the weight vector corresponding to $(I_{u_j})_{j=1}^m$ s.t. $\omega_j \geq 0$, for all j; $\sum\limits_{j=1}^{m} \omega_j = 1$.

Proof: We will prove this theorem by making use of the principle of mathematical induction.

(i) For $m = 2$, we get

$$\begin{aligned} & PFSDWA_{\psi}(I_{u_1}, I_{u_2}) = I_{u_1} \oplus I_{u_2} = \left(\rho_{I_{u_1}}, \tau_{I_{u_1}}, \omega_{I_{u_1}}\right) \oplus \left(\rho_{I_{u_2}}, \tau_{I_{u_2}}, \omega_{I_{u_2}}\right) \\ = & \left\langle 1 - \frac{1}{1+\left\{\psi_1 \left(\frac{\rho_{I_{u_1}}}{1-\rho_{I_{u_1}}}\right)^R + \psi_2 \left(\frac{\rho_{I_{u_2}}}{1-\rho_{I_{u_2}}}\right)^R\right\}^{\frac{1}{R}}},\ \frac{1}{1+\left\{\psi_1 \left(\frac{1-\tau_{I_{u_1}}}{\tau_{I_{u_1}}}\right)^R + \psi_2 \left(\frac{1-\tau_{I_{u_2}}}{\tau_{I_{u_2}}}\right)^R\right\}^{\frac{1}{R}}}, \right. \\ & \left. \frac{1}{1+\left\{\psi_1 \left(\frac{1-\omega_{I_{u_1}}}{\omega_{I_{u_1}}}\right)^R + \psi_2 \left(\frac{1-\omega_{I_{u_2}}}{\omega_{I_{u_2}}}\right)^R\right\}^{\frac{1}{R}}} \right\rangle; \end{aligned}$$

$$= \left\langle 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{2} \psi_j \left(\frac{\rho_{I_{u_j}}}{1-\rho_{I_{u_j}}} \right)^R \right\}^{\frac{1}{R}}}, \frac{1}{1 + \left\{ \sum_{j=1}^{2} \psi_j \left(\frac{1-\tau_{I_{u_j}}}{\tau_{I_{u_j}}} \right)^R \right\}^{\frac{1}{R}}}, \frac{1}{1 + \left\{ \sum_{j=1}^{2} \psi_j \left(\frac{1-\omega_{I_{u_j}}}{\omega_{I_{u_j}}} \right)^R \right\}^{\frac{1}{R}}} \right\rangle.$$

Hence, the result holds for $m = 2$.

(ii) Suppose that the result holds for $m = k$, then by using definition 5, we get

$$PFSDWA_{\psi}(I_{u_1}, I_{u_2}, \ldots, I_{u_k}) = \left\langle 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{k} \psi_j \left(\frac{\rho_{I_{u_j}}}{1-\rho_{I_{u_j}}} \right)^R \right\}^{\frac{1}{R}}}, \frac{1}{1 + \left\{ \sum_{j=1}^{k} \psi_j \left(\frac{1-\tau_{I_{u_j}}}{\tau_{I_{u_j}}} \right)^R \right\}^{\frac{1}{R}}}, \frac{1}{1 + \left\{ \sum_{j=1}^{k} \psi_j \left(\frac{1-\omega_{I_{u_j}}}{\omega_{I_{u_j}}} \right)^R \right\}^{\frac{1}{R}}} \right\rangle.$$

Now, for $m = k + 1$, we get

$$PFSDWA_{\psi}(I_{u_1}, I_{u_2}, \ldots, I_{u_k}, I_{u_{k+1}}) = \oplus_{j=1}^{k} (\psi_j I_{u_j}) \oplus (\psi_{k+1} I_{u_{k+1}})$$

$$= \left\langle 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{k} \psi_j \left(\frac{\rho_{I_{u_j}}}{1-\rho_{I_{u_j}}} \right)^R \right\}^{\frac{1}{R}}}, \frac{1}{1 + \left\{ \sum_{j=1}^{k} \psi_j \left(\frac{1-\tau_{I_{u_j}}}{\tau_{I_{u_j}}} \right)^R \right\}^{\frac{1}{R}}}, \frac{1}{1 + \left\{ \sum_{j=1}^{k} \psi_j \left(\frac{1-\omega_{I_{u_j}}}{\omega_{I_{u_j}}} \right)^R \right\}^{\frac{1}{R}}} \right\rangle \oplus \left\langle 1 - \frac{1}{1 + \left\{ \psi_{k+1} \left(\frac{\rho_{I_{u_{k+1}}}}{1-\rho_{I_{u_{k+1}}}} \right)^R \right\}^{\frac{1}{R}}}, \frac{1}{1 + \left\{ \psi_{k+1} \left(\frac{1-\tau_{I_{u_{k+1}}}}{\tau_{I_{u_{k+1}}}} \right)^R \right\}^{\frac{1}{R}}}, \frac{1}{1 + \left\{ \psi_{k+1} \left(\frac{1-\omega_{I_{u_{k+1}}}}{\omega_{I_{u_{k+1}}}} \right)^R \right\}^{\frac{1}{R}}} \right\rangle$$

$$= \left\langle 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{k+1} \psi_j \left(\frac{\rho_{I_{u_j}}}{1-\rho_{I_{u_j}}} \right)^R \right\}^{\frac{1}{R}}}, \frac{1}{1 + \left\{ \sum_{j=1}^{k+1} \psi_j \left(\frac{1-\tau_{I_{u_j}}}{\tau_{I_{u_j}}} \right)^R \right\}^{\frac{1}{R}}}, \frac{1}{1 + \left\{ \sum_{j=1}^{k+1} \psi_j \left(\frac{1-\omega_{I_{u_j}}}{\omega_{I_{u_j}}} \right)^R \right\}^{\frac{1}{R}}} \right\rangle.$$

Therefore, the result holds for $m = k + 1$. Hence, the proof of the theorem.
Next, we define and prove some properties that are related to the picture fuzzy soft Dombi weighted averaging operators as follows:

(i) ***Idempotency :*** If $I_{u_j} = I_u$; for all $j = 1, 2, \ldots, m$, then

$$PFSDWA_{\psi}(I_{u_1},\ I_{u_2},\ \ldots I_{u_m}) = I_u.$$

Proof: For the proof of this property, we shall make use of the theorem 3. As per the theorem,

$$PFSDWA_{\psi}(I_{u_1},\ I_{u_2},\ \ldots,\ I_{u_m}) = \oplus_{j=1}^{m}\left(\psi_j I_{u_j}\right)$$

$$= \left\langle 1 - \frac{1}{1+\left\{\sum_{j=1}^{m}\psi_j\left(\frac{\rho_{I_{u_j}}}{1-\rho_{I_{u_j}}}\right)^R\right\}^{\frac{1}{R}}},\ \frac{1}{1+\left\{\sum_{j=1}^{m}\psi_j\left(\frac{1-\tau_{I_{u_j}}}{\tau_{I_{u_j}}}\right)^R\right\}^{\frac{1}{R}}},\ \frac{1}{1+\left\{\sum_{j=1}^{m}\psi_j\left(\frac{1-\omega_{I_{u_j}}}{\omega_{I_{u_j}}}\right)^R\right\}^{\frac{1}{R}}}\right\rangle$$

$$= \left\langle 1 - \frac{1}{1+\left\{\left(\frac{\rho_{I_u}}{1-\rho_{I_u}}\right)^R\right\}^{\frac{1}{R}}},\ \frac{1}{1+\left\{\left(\frac{1-\tau_{I_u}}{\tau_{I_u}}\right)^R\right\}^{\frac{1}{R}}},\ \frac{1}{1+\left\{\left(\frac{1-\omega_{I_u}}{\omega_{I_u}}\right)^R\right\}^{\frac{1}{R}}}\right\rangle$$

$$= \left\langle 1 - \frac{1}{1+\left\{\left(\frac{\rho_{I_u}}{1-\rho_{I_u}}\right)\right\}},\ \frac{1}{1+\left\{\left(\frac{1-\tau_{I_u}}{\tau_{I_u}}\right)\right\}},\ \frac{1}{1+\left\{\left(\frac{1-\omega_{I_u}}{\omega_{I_u}}\right)\right\}}\right\rangle$$

$$= (\rho_{I_u},\ \tau_{I_u},\ \omega_{I_u}) = I_u.$$

Hence, the proof.

(ii) ***Boundedness*:** If I_{u_j} $(j = 1, 2, \ldots, m)$ be the collection of PFSNs.
Let $I^- = \min(I_{u_1},\ I_{u_2},\ \ldots\ I_{u_m})$ and $I^+ = \max(I_{u_1},\ I_{u_2},\ \ldots\ I_{u_m})$. Then

$$I^- \leq PFSDWA_{\psi}(I_{u_1}, I_{u_2}, \ldots I_{u_m}) \leq I^+.$$

Proof: Let $I^- = \min(I_{u_1}, I_{u_2}, \ldots I_{u_m}) = \left(\rho^-_{I_u}, \tau^-_{I_u}, \omega^-_{I_u}\right)$ and $I^+ = \max(I_{u_1}, I_{u_2}, \ldots I_{u_m}) = \left(\rho^+_{I_u}, \tau^+_{I_u}, \omega^+_{I_u}\right)$. Then, we have $\rho^-_{I_u} = \min\limits_j \{\rho_{I_j}\}, \tau^-_{I_u} = \max\limits_j \{\tau_{I_j}\}, \omega^-_{I_u} = \max\limits_j \{\omega_{I_j}\}, \rho^+_{I_u} =$

$\max_j \{\rho_{I_j}\}, \tau^+_{I_u} = \min_j \{\tau_j\}, \omega^+_u = \min_{I_j} \{\omega_{I_j}\}$.

Now, we have the following three inequalities,

$$1-\frac{1}{1+\left\{\sum_{j=1}^m \psi_j\left(\frac{\rho^-_{I_u}}{1-\rho_{I^-_u}}\right)^R\right\}^{\frac{1}{R}}} \leq 1-\frac{1}{1+\left\{\sum_{j=1}^m \psi_j\left(\frac{\rho_{I_u}}{1-\rho_{I_u}}\right)^R\right\}^{\frac{1}{R}}} \leq 1-\frac{1}{1+\left\{\sum_{j=1}^m \psi_j\left(\frac{\rho^+_{I_u}}{1-\rho_{I^+_u}}\right)^R\right\}^{\frac{1}{R}}};$$

$$\frac{1}{1+\left\{\sum_{j=1}^m \psi_j\left(\frac{1-\tau^+_{I_u}}{\tau_{I^+_u}}\right)^R\right\}^{\frac{1}{R}}} \leq \frac{1}{1+\left\{\sum_{j=1}^m \psi_j\left(\frac{1-\tau_{I_u}}{\tau_{I_u}}\right)^R\right\}^{\frac{1}{R}}} \leq \frac{1}{1+\left\{\sum_{j=1}^m \psi_j\left(\frac{1-\tau^-_{I_u}}{\tau_{I^-_u}}\right)^R\right\}^{\frac{1}{R}}};$$

and

$$\frac{1}{1+\left\{\sum_{j=1}^m \psi_j\left(\frac{1-\omega^+_{I_u}}{\omega_{I^+_u}}\right)^R\right\}^{\frac{1}{R}}} \leq \frac{1}{1+\left\{\sum_{j=1}^m \psi_j\left(\frac{1-\omega_{I_u}}{\omega_{I_u}}\right)^R\right\}^{\frac{1}{R}}} \leq \frac{1}{1+\left\{\sum_{j=1}^m \psi_j\left(\frac{1-\omega^-_{I_u}}{\omega_{I^-_u}}\right)^R\right\}^{\frac{1}{R}}}.$$

Therefore,

$$I^- \leq PFSDWA_\psi(I_{u_1}, I_{u_2}, \ldots I_{u_m}) \leq I^+.$$

(iii) ***Monotonicity*:** Let I_{u_j} and $I_{u'_j}$ $(j = 1, 2, \ldots, m)$ be the two collections of picture fuzzy soft numbers. If $I_{u_j} \leq I_{u'_j}$ then

$$PFSDWA_\psi(I_{u_1},\ I_{u_2},\ \ \ldots\ I_{u_m}) \ \leq\ PFSDWA_\psi(I_{u'_1}, I_{u'_2},\ \ \ldots\ I_{u'_m}).$$

Proof: The proof of this property can similarly be done.

Next, we would introduce the notion of picture fuzzy soft Dombi ordered weighted average ($PFSDOWA$) aggregation operators and their properties.

Definition 21 *Suppose $\mathcal{T}$ is a collection of all picture fuzzy soft numbers. Let $(I_{u_1},\ I_{u_2},\ \ldots,\ I_{u_m}) \in \mathcal{T}^m$. A mapping $PFSDOWA_\psi : \mathcal{T}^m \longrightarrow \mathcal{T}$ is said to be picture fuzzy soft Dombi ordered weighted averaging operator, if*

$$PFSDOWA_\psi(I_{u_1},\ I_{u_2},\ \ldots,\ I_{u_m}) \ =\ \oplus_{j=1}^m (\psi_j I_{u_{\sigma(j)}}); \qquad (3.1.5)$$

where $\psi = (\psi_1,\ \psi_2,\ \ldots,\ \psi_m)^T$ is the weight vector corresponding to $(I_{u_j})_{j=1}^m$ s.t. $\psi_j \geq 0$, for all j; $\sum_{j=1}^m \psi_j = 1$ and $(\sigma(1),\ \sigma(2),\ \ldots,\ \sigma(m))$ is a possible permutation of $(1,\ 2,\ \ldots,\ m)$, s.t. $I_{u_{\sigma(j+1)}} \leq I_{u_{\sigma(j)}}$ for all $j = 1,\ 2,\ \ldots,\ m-1$.

Theorem 4 *The picture fuzzy soft Dombi ordered weighted averaging operator* $PFSDOWA_\psi$ *aggregates all the input values and yields a PFSN given by*

$$PFSDOWA_\psi(I_{u_1},\ I_{u_2},\ \ldots,\ I_{u_m})\ =\ \oplus_{j=1}^{m}(\psi_j I_{u_{\sigma(j)}})$$

$$= \left\langle 1-\frac{1}{1+\left\{\sum_{j=1}^{m}\psi_j\left(\frac{\rho_{I_{u_{\sigma(j)}}}}{1-\rho_{I_{u_{\sigma(j)}}}}\right)^R\right\}^{\frac{1}{R}}},\ \frac{1}{1+\left\{\sum_{j=1}^{m}\psi_j\left(\frac{1-\tau_{I_{u_{\sigma(j)}}}}{\tau_{I_{u_{\sigma(j)}}}}\right)^R\right\}^{\frac{1}{R}}},\ \frac{1}{1+\left\{\sum_{j=1}^{m}\psi_j\left(\frac{1-\omega_{I_{u_{\sigma(j)}}}}{\omega_{I_{u_{\sigma(j)}}}}\right)^R\right\}^{\frac{1}{R}}}\right\rangle.$$

where $\psi\ =\ (\psi_1,\ \psi_2,\ \ldots,\ \psi_m)^T$ *is the weight vector corresponding to* $(I_{u_j})_{j=1}^m$ *s.t.* $\omega_j\ \geq\ 0,$ *for all* j; $\sum_{j=1}^{m}\psi_j = 1$ *and* $(\sigma(1),\ \sigma(2),\ \ldots,\ \sigma(m))$ *is a possible permutation of* $(1,\ 2,\ \ldots,\ m)$, *s.t.* $I_{u_{\sigma(j+1)}}\ \leq\ I_{u_{\sigma(j)}}$ *for all* $j\ =\ 1,\ 2,\ \ldots,\ m-1.$

Proof: The proof of this theorem is the same as the proof of Theorem 3.

Remark: On similar lines, we can define and prove the properties of **Idempotency**, **Monotonicity** and **Boundedness** for $PFSDOWA$ aggregation operators by making use of the above definitions.

Note: The picture fuzzy soft Dombi weighted averaging operator takes weights of PFSN into account, while the picture fuzzy soft Dombi ordered weighted averaging operator takes only the weights of the ordered positions of PFSNs into account. This means that both operators are dealing with the same factor. For the better incorporation of both aspects together, we introduce the picture fuzzy soft Dombi hybrid averaging (PFSDHA) operator and define it as follows:

Definition 22 *Suppose* $\mathcal{T}$ *is a collection of all picture fuzzy soft numbers. Let* $(I_{u_1},\ I_{u_2},\ \ldots,\ I_{u_m})\ \in\ \mathcal{T}^m$. *A mapping* $PFSDHA_\psi : \mathcal{T}^m \longrightarrow \mathcal{T}$ *is called a picture fuzzy soft Dombi hybrid averaging operator, if*

$$PFSDHA_{\psi,\gamma}(I_{u_1},\ I_{u_2},\ \ldots,\ I_{u_m})\ =\ \oplus_{j=1}^{m}(\gamma_j\widetilde{I}_{u_{\sigma(j)}});$$

where $\gamma\ =\ (\gamma_1,\ \gamma_2,\ \ldots,\ \gamma_m)^T$ *is the weight vector corresponding to* $(\widetilde{I}_{u_{\sigma(j)}})_{j=1}^m$ *s.t.* $\gamma_j\ \geq\ 0,$ *for all* j; $\sum_{j=1}^{m}\gamma_j\ =\ 1.$ $\widetilde{I}_{u_{\sigma(j)}}$ *is the* j^{th} *largest of the weighted PFSNs* $\widetilde{I}_{u_j}$;

where $\widetilde{I}_{u_j} = (m\psi_j)I_{u_j}$ *and* m *is the balancing coefficient with* $\psi = (\psi_1,\ \psi_2,\ \ldots,\ \psi_n)^T$ *being the weight vector of* I_{u_j} *with* $\psi_j \geq 0,$ *for all* $j;$ $\sum_{j=1}^{m} \psi_j = 1.$

Remarks:

- If we take weights as $\gamma = (\frac{1}{n},\ \frac{1}{n},\ \ldots,\ \frac{1}{n})^T$ then picture fuzzy soft Dombi hybrid averaging operator gives picture fuzzy Dombi soft weighted averaging operator.

- However, if we take $\psi = (\frac{1}{n},\ \frac{1}{n},\ \ldots, \frac{1}{n})^T$ then picture fuzzy soft Dombi hybrid averaging operator gives picture fuzzy soft ordered weighted averaging operator.

Theorem 5 *The picture fuzzy soft Dombi hybrid averaging operator* $PFSDHA_{\psi,\gamma}$ *aggregates all the input values and yields a PFSN is given by*

$$PFSDHA_{\psi,\gamma}(I_{u_1}, I_{u_2}, \ldots, I_{u_m}) = \left\langle 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{m} \psi_j \left(\frac{\rho_{I_{u_{\sigma(j)}}}}{1-\rho_{I_{u_{\sigma(j)}}}} \right)^R \right\}^{\frac{1}{R}}}, \frac{1}{1 + \left\{ \sum_{j=1}^{m} \psi_j \left(\frac{1-\tau_{I_{u_{\sigma(j)}}}}{\tau_{I_{u_{\sigma(j)}}}} \right)^R \right\}^{\frac{1}{R}}}, \frac{1}{1 + \left\{ \sum_{j=1}^{m} \psi_j \left(\frac{1-\omega_{I_{u_{\sigma(j)}}}}{\omega_{I_{u_{\sigma(j)}}}} \right)^R \right\}^{\frac{1}{R}}} \right\rangle.$$

Proof: The proof is the same as that of the proof of Theorem 3 and can easily be done.

Also, on similar lines, the other properties like Idempotency, Boundedness and Monotonicity related to the picture fuzzy Dombi soft hybrid averaging operator can easily be defined and proved with the help of the definitions.

3.2 Picture Fuzzy Soft Dombi Geometric Aggregation Operators

In this section, we present the notion of picture fuzzy soft Dombi geometric aggregation operator and discuss their properties and results.

Definition 23 *Suppose $\mathcal{T}$ is a collection of all picture fuzzy soft numbers. Let $(I_{u_1},\ I_{u_2},\ \ldots,\ I_{u_m}) \in \mathcal{T}^m$. A mapping $PFSDWG_\psi : \mathcal{T}^m \longrightarrow \mathcal{T}$ is said to be a picture fuzzy sof Dombi weighted geometric operator, if*

$$PFSDWG_\psi(I_{u_1},\ I_{u_2},\ \ldots,\ I_{u_m}) = \otimes_{j=1}^m (I_{u_j})^{\psi_j}; \tag{3.2.1}$$

where $\psi = (\psi_1,\ \psi_2,\ \ldots,\ \psi_m)^T$ is the weight vector corresponding to $(I_{u_j})_{j=1}^m$ s.t. $\psi_j \geq 0,$ for all j; $\sum\limits_{j=1}^m \psi_j = 1$.

Theorem 6 *The picture fuzzy soft Dombi weighted geometric operator $PFSDWG_\psi$ aggregates all the input values and yields a PFSN given by*

$$PFSDWG_\psi(I_{u_1},\ I_{u_2},\ \ldots,\ I_{u_m}) = \otimes_{j=1}^m (I_{u_j})^{\psi_j}$$

$$= \left\langle \frac{1}{1+\left\{\sum_{j=1}^m \psi_j \left(\frac{1-\rho_{I_{u_j}}}{\rho_{I_{u_j}}}\right)^R\right\}^{\frac{1}{R}}},\ 1-\frac{1}{1+\left\{\sum_{j=1}^m \psi_j \left(\frac{\tau_{I_{u_j}}}{1-\tau_{I_{u_j}}}\right)^R\right\}^{\frac{1}{R}}},\ 1-\frac{1}{1+\left\{\sum_{j=1}^m \psi_j \left(\frac{\omega_{I_{u_j}}}{1-\omega_{I_{u_j}}}\right)^R\right\}^{\frac{1}{R}}} \right\rangle.$$

where $\psi = (\psi_1,\ \psi_2,\ \ldots,\ \psi_m)^T$ is the weight vector corresponding to $(I_{u_j})_{j=1}^m$ s.t. $\omega_j \geq 0,$ for all j; $\sum\limits_{j=1}^m \omega_j = 1$.

Proof: The proof of the theorem can be done as that of Theorem 3.
Here, are some properties analogous to the weighted averaging operators.

(i) ***Idempotency*:** If $I_{u_j} = I_u$; for all $j = 1, 2, \ldots, m$, then

$$PFSDWG_\psi(I_{u_1}, I_{u_2}, \ldots I_{u_m}) = I_u.$$

(ii) ***Boundedness*:** If I_{u_j} $(j = 1, 2, \ldots, m)$ be the collection of PFSNs. Let $I^- = \min(I_{u_1}, I_{u_2}, \ldots I_{u_m})$ and $I^+ = \max(I_{u_1}, I_{u_2}, \ldots I_{u_m})$. Then

$$I^- \leq PFSDWG_\psi(I_{u_1}, I_{u_2}, \ldots I_{u_m}) \leq I^+.$$

(iii) ***Monotonicity*:** Let I_{u_j} and $I_{u'_j}$ $(j = 1, 2, \ldots, m)$ be the two collections of picture fuzzy soft numbers. If $I_{u_j} \leq I_{u'_j}$ then

$$PFSDWG_\psi(I_{u_1}, I_{u_2}, \ldots I_{u_m}) \leq PFSDWG_\psi(I_{u'_1}, I_{u'_2}, \ldots I_{u'_m}).$$

Note: The proof of all these properties can similarly be given by making use of the definitions.

Next, we would introduce the notion of picture fuzzy soft Dombi-ordered weighted geometric ($PFSDOWG$) aggregation operators an their properties.

Definition 24 *Suppose $\mathcal{T}$ is a collection of all picture fuzzy soft numbers. Let $(I_{u_1},\ I_{u_2},\ \ldots,\ I_{u_m}) \in \mathcal{T}^m$. A mapping $PFSDOWG_\psi : \mathcal{T}^m \longrightarrow \mathcal{T}$ is said to be picture fuzzy soft Dombi ordered weighted geometric operator, if*

$$PFSDOWG_\psi(I_{u_1},\ I_{u_2},\ \ldots,\ I_{u_m}) = \otimes_{j=1}^{m} (I_{u_{\sigma(j)}})^{\psi_j}; \qquad (3.2.2)$$

where $\psi = (\psi_1,\ \psi_2,\ \ldots,\ \psi_m)^T$ is the weight vector corresponding to $(I_{u_j})_{j=1}^m$ s.t. $\psi_j \geq 0$, for all j; $\sum_{j=1}^{m} \psi_j = 1$ and $(\sigma(1),\ \sigma(2),\ \ldots,\ \sigma(m))$ is a possible permutation of $(1,\ 2,\ \ldots,\ m)$, s.t. $I_{u_{\sigma(j+1)}} \leq I_{u_{\sigma(j)}}$ for all $j = 1,\ 2,\ \ldots,\ m-1$.

Theorem 7 *The picture fuzzy soft Dombi ordered weighted geometric operator $PFSDOWG_\psi$ aggregates all the input values and yields a PFSN given by*

$$PFSDOWG_\psi(I_{u_1},\ I_{u_2},\ \ldots,\ I_{u_m}) = \otimes_{j=1}^{m} (I_{u_{\sigma(j)}})^{\psi_j}$$

$$= \left\langle \frac{1}{1 + \left\{ \sum_{j=1}^{m} \psi_j \left(\frac{1-\rho_{I_{u_{\sigma(j)}}}}{\rho_{I_{u_{\sigma(j)}}}} \right)^R \right\}^{\frac{1}{R}}},\ 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{m} \psi_j \left(\frac{\tau_{I_{u_{\sigma(j)}}}}{1-\tau_{I_{u_{\sigma(j)}}}} \right)^R \right\}^{\frac{1}{R}}}, \right.$$
$$\left. 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{m} \psi_j \left(\frac{\omega_{I_{u_{\sigma(j)}}}}{1-\omega_{I_{u_{\sigma(j)}}}} \right)^R \right\}^{\frac{1}{R}}} \right\rangle.$$

where $\psi = (\psi_1,\ \psi_2,\ \ldots,\ \psi_m)^T$ is the weight vector corresponding to $(I_{u_j})_{j=1}^m$ s.t.$\omega_j \geq 0$, for all j; $\sum_{j=1}^{m} \psi_j = 1$ and $(\sigma(1), \sigma(2),\ \ldots,\ \sigma(m))$ is a possible permutation of $(1, 2, \ldots, m)$, s.t. $I_{u_{\sigma(j+1)}} \leq I_{u_{\sigma(j)}}$ for all $j = 1, 2,\ \ldots,\ m-1$.

Proof: The proof of this theorem is same as of Theorem 3.
Note: Further, the hybrid operators for geometric can similarly be defined as in averaging case.

3.3 EDAS Methodology Based on Picture Fuzzy Soft Dombi Aggregation Operators

The classical "EDAS (Evaluation based on Distance from Average Solution)" method Keshavarz Ghorabaee et al.[46] which involves the different attributes, has been studied in many MCDM problems. This method involves the computation of the average solution (AV), this also describes the distinction between the average solution and all the alternatives which are based on the two different distance measures, i.e., "Positive Distance from Average (PDA)" and "Negative Distance from Average (NDA)". For the best choice of alternatives, higher values of PDA and lower values of NDA are preferred.
We have combined the existing EDAS model with picture fuzzy Dombi soft aggregation operators and PFSNs, which give rise to ***picture fuzzy soft Dombi EDAS model (PFSD-EDAS Methodology)***. The proposed methodology involves the various steps of computations as follows:

Suppose that there are n alternatives $\{A_1,\ A_2,\ ...,\ A_n\}$, m criterions $\{C_1,\ C_2,\ ...,\ C_m\}$ & k experts $\{E_1,\ E_2,\ ...,\ E_k\}$. Let $\{\psi_1,\ \psi_2,\ ...,\ \psi_m\}$ & $\{\emptyset_1,\ \emptyset_2,\ ...,\ \emptyset_k\}$ be the weighting vectors of criterion's and experts respectively, where $\psi_j \in [0,1], \emptyset_j \in [0,1], \forall j;$ s.t. $\sum\limits_{j=1}^{m} \psi_j = 1,\ \sum\limits_{r=1}^{k} \emptyset_r = 1.$

The various steps of the proposed algorithm are illustrated with the help of the following chart:

The detailed procedural steps of the proposed ***(PFSD-EDAS Methodology)*** have been explained below:

Step 1. The information given by the experts collected in the form of picture fuzzy soft numbers of all the alternatives w.r.t. different criteria and construct the picture fuzzy soft evaluation matrix $Z = \left[(\rho_{ij}^r, \tau_{ij}^r, \omega_{ij}^r)\right]_{n\times m} i = 1,2,...,n, j = 1,2,...,m, r =$

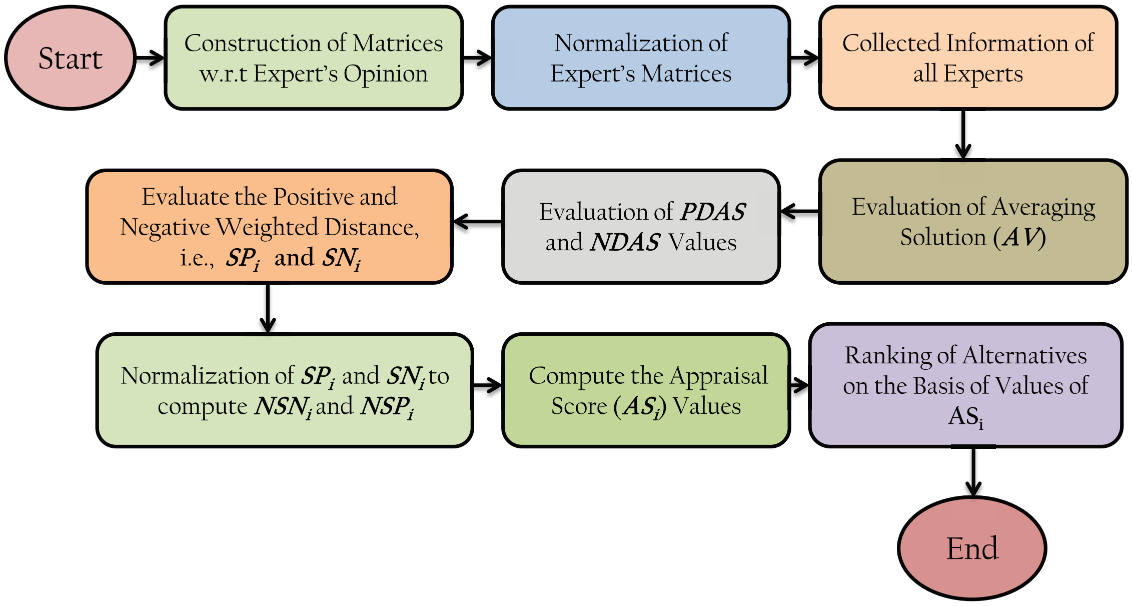

Figure 3.1: Procedural Steps of Picture Fuzzy Soft Dombi EDAS Model

$1, 2, ..., k$, given by

$$[Z]_{n\times m} = \begin{array}{c} \\ A^1 \\ A^2 \\ \vdots \\ A^n \end{array} \begin{array}{c} \begin{array}{cccc} C^1 & C^2 & \cdots & C^m \end{array} \\ \begin{pmatrix} (\rho^r_{11},\ \tau^r_{11},\ \omega^r_{11}) & (\rho^r_{12},\ \tau^r_{12},\ \omega^r_{12}) & \cdots & (\rho^r_{1m},\ \tau^r_{1m},\ \omega^r_{1m}) \\ (\rho^r_{21},\ \tau^r_{21},\ \omega^r_{21}) & (\rho^r_{22},\ \tau^r_{22},\ \omega^r_{22}) & \cdots & (\rho^r_{2m},\ \tau^r_{2m},\ \omega^r_{2m}) \\ \vdots & \vdots & \ddots & \vdots \\ (\rho^r_{n1},\ \tau^r_{n1},\ \omega^r_{n1}) & (\rho^r_{n2},\ \tau^r_{n2},\ \omega^r_{n2}) & \cdots & (\rho^r_{nm},\ \tau^r_{nm},\ \omega^r_{nm}) \end{pmatrix} \end{array}$$

Step 2. We normalize the evaluation matrices obtained from Step 1, i.e., from Z to Z';

(i) For beneficial criterion
$$\left[(\rho'^r_{ij},\ \tau'^r_{ij},\ \omega'^r_{ij})\right] = \left[(\rho^r_{ij},\ \tau^r_{ij},\ \omega^r_{ij})\right];\ i = 1, 2, .., n,\ j = 1, 2, ..., m,\ r = 1, 2, ..., k.$$

(ii) For non-beneficial criterion
$$\left[(\rho'^r_{ij},\ \tau'^r_{ij},\ \omega'^r_{ij})\right] = \left[(\omega^r_{ij},\ \tau^r_{ij},\ \rho^r_{ij})\right];\ i = 1, 2, .., n,\ j = 1, 2, ..., m,\ r = 1, 2, ..., k.$$

Step 3. Next, we compute the aggregated matrix $[Z]_{n\,\times\,m}$ using "$PFSDWA\,(PFSDWG/PFSDOWA/PFSDOWG)$" operator and convert $\left[(\rho'^r_{ij},\ \tau'^r_{ij},\ \omega'^r_{ij})\right]$ to $\left[(\rho'_{ij},\ \tau'_{ij},\ \omega'_{ij})\right]$ with the help of the expert's weighting vector $\{\emptyset_1,\ \emptyset_2,\ ...,\ \emptyset_k\}$. The obtained results can be shown as:

$$[Z]_{n\,\times\,m} \;=\; \begin{array}{c} \\ A^1 \\ A^2 \\ \vdots \\ A^n \end{array} \begin{array}{c} \begin{array}{cccc} C^1 & C^2 & \cdots & C^m \end{array} \\ \begin{pmatrix} (\rho^{'}_{11},\ \tau^{'}_{11},\ \omega^{'}_{11}) & (\rho^{'}_{12},\ \tau^{'}_{12},\ \omega^{'}_{12}) & \cdots & (\rho^{'}_{1m},\ \tau^{'}_{1m},\ \omega^{'}_{1m}) \\ (\rho^{'}_{21},\ \tau^{'}_{21},\ \omega^{'}_{21}) & (\rho^{'}_{22},\ \tau^{'}_{22},\ \omega^{'}_{22}) & \cdots & (\rho^{'}_{2m},\ \tau^{'}_{2m},\ \omega^{'}_{2m}) \\ \vdots & \vdots & \ddots & \vdots \\ (\rho^{'}_{n1},\ \tau^{'}_{n1},\ \omega^{'}_{n1}) & (\rho^{'}_{n2},\ \tau^{'}_{n2},\ \omega^{'}_{n2}) & \cdots & (\rho^{'}_{nm},\ \tau^{'}_{nm},\ \omega^{'}_{nm}) \end{pmatrix} \end{array}.$$

Step 4. Now, we compute the average solution (AV) by using all the proposed criterions as follows:

$$AV = [AV_j]_{1\times m} = \left[\frac{\sum_{i=1}^{n}\left[(\rho^{'}_{ij},\tau^{'}_{ij},\omega^{'}_{ij})\right]}{n}\right]_{1\times m}.$$

On the basis of the definition 14, we get

$$\sum_{i=1}^{n}\left[(\rho^{'}_{ij},\tau^{'}_{ij},\omega^{'}_{ij})\right] = \left(1-\prod_{j=1}^{m}(1-\rho^{'}_{ij}),\ \prod_{j=1}^{m}(\tau^{'}_{ij}),\prod_{j=1}^{m}(\omega^{'}_{ij})\right)_{1\times m};$$

and $\quad AV = [AV_j]_{1\times m} = \left[\frac{\sum_{i=1}^{n}\left[(\rho^{'}_{ij},\tau^{'}_{ij},\omega^{'}_{ij})\right]}{n}\right]_{1\times m}$

$= \left(1-\prod_{j=1}^{m}(1-\rho^{'}_{ij})^{\frac{1}{n}},\ \prod_{j=1}^{m}(\tau^{'}_{ij})^{\frac{1}{n}},\prod_{j=1}^{m}(\omega^{'}_{ij})^{\frac{1}{n}}\right)_{1\times m}.$

Similarly, for $(PFSDWG/PFSDOWA/PFSDOWG)$ operators, we can evaluate AV using the other parts of definition 14.

Step 5. Now, from the results of average solution (AV), we can find the PDA and by making use of th following expressions:

$$PDA_{ij} \;=\; [PDA_{ij}]_{n\,\times\,m} \;=\; \frac{\max\left(0,((\rho^{'}_{ij},\ \tau^{'}_{ij},\ \omega^{'}_{ij})\ -\ AV_j)\right)}{AV_j};$$

$$NDA_{ij} \;=\; [NDA_{ij}]_{n\,\times\,m} \;=\; \frac{\max\left(0,(AV_j\ -\ (\rho^{'}_{ij},\ \tau^{'}_{ij},\ \omega^{'}_{ij}))\right)}{AV_j}.$$

For simplifications, we can make use of the score function as defined in Definition 13 to compute PDA and NDA as follows:

$$PDA_{ij} \;=\; [PDA_{ij}]_{n\,\times\,m} \;=\; \frac{\max\left(0,(S((\rho^{'}_{ij},\ \tau^{'}_{ij},\ \omega^{'}_{ij}))\ -\ S(AV_j))\right)}{S(AV_j)};$$

$$NDA_{ij} \;=\; [NDA_{ij}]_{n\,\times\,m} \;=\; \frac{\max\left(0,(S(AV_j)\ -\ S((\rho^{'}_{ij},\ \tau^{'}_{ij},\ \omega^{'}_{ij})))\right)}{S(AV_j)}.$$

Step 6. Next, we compute the weighted sum values of PDA and NDA, i.e., the values of SP_i and SN_i which can be calculated as follows:

$$SP_i \;=\; \sum_{j=1}^{m} \psi_j \; PDA_{ij}, \;\; SN_i \;=\; \sum_{j=1}^{m} \psi_j \; NDA_{ij}.$$

Step 7. The normalized form of SP_i and SN_i can be computed as:

$$NSP_i \;=\; \frac{SP_i}{\max\limits_i (SP_i)}, \;\; NSN_i \;=\; 1 \;-\; \frac{SN_i}{\max\limits_i (SN_i)}.$$

Step 8. Now, for NSP_i and NSN_i w.r.t. each alternative compute the values of appraisal score (AS_i) as:

$$AS_i \;=\; \frac{1}{2} \left(NSP_i \;+\; NSN_i\right).$$

Step 9. In the final step, ranking of alternativ can be done with supreme values of appraisal score (AS_i).

3.4 PFSD-EDAS Methodology in Sustainable Agrifarming

In the moder era, there is the development of a new industry known as "Agriculture Farming" which ensures the farmer's financial benefits and the long-term viability of their production. There is one more important application of sustainable farming termed "Green Agriculture" which minimizes the usage of pesticides to avoid harmful effects on the health of consumers and farmers. In addition to this, it may be noted that the irregular and destructive kind of farming may result in some kind of crisis. Therefore, the concept of precision agriculture and digital farming will play an important role in the growth of humankind for their development. The four essential alternatives about sustainable and smart agrifarming may be considered as follows:

(1) **Good Farm Production** (A_1): Devising high-yielding techniques, self-reliant and economical production which gives a good earning. Also, it would generate job opportunities for poor people and is very good for society. Further, there will be the development of rural areas which would help the good connections between rural and urban areas.

(2) **Environmental Control** (A_2): The first benefit of agriculture is environmental control, as it reduces deforestation, the depletion of natural resources, the progressing bio-diversity and the reduction of CO_2 emissions. Also, it controls the quality of water, air and soil by preserving and protecting the regions.

(3) **Availability of Ecological Resources** (A_3): To improve the quality of resources to be used in farming. The other main difficulty while facing green agriculture is torrent depletion and loss of these resources. The availability of ecological resources improves agriculture which is very beneficial for the ecosystem.

(4) **Food Safety and Efficiency** (A_4): To increase the efficiency of food and energy in sales and production. As the world population is rapidly increasing, food security issues can be solved with sustainable agriculture practices by producing more in minimal time.

Now, these four alternatives are being evaluated based on five attributes. Suppose that the following are the five attributes of robotic agrifarming for the above alternatives:

(1) **Superior Quality Production (**C_1): There are specific agriculture factors, such as soil and time for a crop to ripe, that play a role in the product quality. For crops ike rice, wheat, maize, barley, cereals and other pulses, ripeness and degree of waterlessness matter.

(2) **Confining the Requirement of Man Power (**C_2): In farming, the cost of labor is very expensive and therefore the suitability of employees and the manpower are in great demand.

(3) **Minimal Production Cost (**C_3): In agriculture, the cost of production can be reduced with the usage of agriculture robots. In addition, we need to deal with some of the variables like environmental conditions, purchasing different brands of seeds and using numerous chemicals.

(4) **Accomplishment of Time-Constraint Project (**C_4): To complete the sophisticated projects quickly and within time, one should make use of scientists, technicians, farmers, scholars and robotization.

(5) **Role of Consistency in Project** (C_5): For maintaining a consistent site, the use of artificial intelligence in agricultural farming from the process of seeding to the process of harvesting.

Further, to assess the given alternatives w.r.t. the above stated criterions, a panel of three experts $\{E_1, E_2, E_3\}$ has been appointed. The proposed methodology involves the various steps of computations as follows:

Step 1. The information given by the experts collected in the form of picture fuzzy soft numbers of all the alternatives w.r.t. different criteria and construct the picture fuzzy soft evaluation matrix given in Table 3.1, Table 3.2 and Table 3.3.

Step 2. Normalized evaluation matrices obtained from Step 1 are given in Table 3.4, Table 3.5 and Table 3.6.

Step 3. The collected information of all the Experts for "*PFSDWA, PFSDWG, PFSDOWA, PFSDOWG*" operators are given in Table 3.7, Table 3.8, Table 3.9 and Table 3.10.

Step 4. The averaging solutions (AV) of the proposed picture fuzzy Dombi soft aggregation operations are given in Table 3.11 and Table 3.12.

Step 5. Now, PDA_{ij} and NDA_{ij} are computed as per the following cases:

Case 1 - "PF Soft Dombi WA aggregation operator" given in Table 3.13 and Table 3.14.

Case 2 - "PF Soft Dombi WG aggregation operator" given in Table 3.15 and Table 3.16.

Case 3 - "PF Soft Dombi OWA aggregation operator" given in Table 3.17 and Table 3.18.

Case 4 - "PF Soft Dombi OWG aggregation operator" given in Table 3.19 and Table 3.20.

Step 6. On the basis of proposed operators, the positive and negative weighted distances, i.e., SP_i and SN_i have been tabulated in Table 3.21 and Table 3.22.

Step 7. Normalized form of SP_i and SN_i, i.e., NSP_i and NSN_i are being provided in Table 3.23 and Table 3.24.

Step 8. The appraisal score values (AS_i) are being tabulated in Table 3.25.

Step 9. Now, on the basis of the values of (AS_i), the ranking of the alternatives is being done and given in Table 3.26.

On the basis of the above steps of methodology and the necessary computations carried out, we find that the alternative A_4 is mostly preferred by the experts.

Hence, in the process of agriculture farming, it is suggested that food safety and efficiency have to be given the highest priority. All the tables for ready reference have been provided.

Table 3.1: Picture Fuzzy Soft Information by Expert E_1

	C_1	C_2	C_3	C_4	C_5
A_1	(0.20, 0.12, 0.56)	(0.23, 0.15, 0.62)	(0.10, 0.33, 0.47)	(0.15, 0.34, 0.51)	(0.20, 0.37, 0.43)
A_2	(0.18, 0.25, 0.43)	(0.22, 0.28, 0.50)	(0.17, 0.29, 0.54)	(0.19, 0.17, 0.64)	(0.37, 0.13, 0.50)
A_3	(0.08, 0.32, 0.60)	(0.30, 0.12, 0.58)	(0.28, 0.11, 0.62)	(0.05, 0.15, 0.80)	(0.25, 0.10, 0.65)
A_4	(0.10, 0.22, 0.58)	(0.26, 0.13, 0.61)	(0.18, 0.27, 0.55)	(0.07, 0.26, 0.67)	(0.23, 0.01, 0.76)

Table 3.2: Picture Fuzzy Soft Information by Expert E_2

	C_1	C_2	C_3	C_4	C_5
A_1	(0.17, 0.35, 0.48)	(0.10, 0.27, 0.53)	(0.19, 0.28, 0.61)	(0.05, 0.15, 0.80)	(0.64, 0.29, 0.07)
A_2	(0.10, 0.27, 0.53)	(0.19, 0.17, 0.64)	(0.20, 0.37, 0.43)	(0.65, 0.22, 0.23)	(0.45, 0.28, 0.27)
A_3	(0.14, 0.20, 0.66)	(0.20, 0.21, 0.59)	(0.77, 0.11, 0.18)	(0.10, 0.17, 0.73)	(0.08, 0.27, 0.55)
A_4	(0.31, 0.28, 0.41)	(0.50, 0.32, 0.18)	(0.39, 0.32, 0.29)	(0.17, 0.34, 0.49)	(0.18, 0.14, 0.68)

Table 3.3: Picture Fuzzy Soft Information by Expert E_3

	C_1	C_2	C_3	C_4	C_5
A_1	(0.25, 0.24, 0.51)	(0.18, 0.12, 0.70)	(0.23, 0.25, 0.52)	(0.20, 0.12, 0.56)	(0.56, 0.18, 0.26)
A_2	(0.23, 0.08, 0.69)	(0.10, 0.21, 0.59)	(0.63, 0.02, 0.35)	(0.15, 0.09, 0.76)	(0.19, 0.34, 0.47)
A_3	(0.42, 0.21, 0.37)	(0.32, 0.23, 0.45)	(0.13, 0.14, 0.73)	(0.08, 0.27, 0.55)	(0.19, 0.17, 0.64)
A_4	(0.33, 0.25, 0.42)	(0.15, 0.19, 0.66)	(0.17, 0.26, 0.57)	(0.50, 0.32, 0.18)	(0.08, 0.10, 0.82)

3.5 Advantageous Remarks

The following important advantageous remarks are listed as follows:

- The formal inclusion of parametrization through the picture fuzzy soft sets proves to be more consistent in the orientation of the obtained results and the involved parameters give rise to robustness in the proposed methodology.
- Also, in literature, it has been found that the implementation of Dombi norms[108] leads to certain superiorities in the results in terms of variability concerning the functioning of parameters.

Table 3.4: Normalized Picture Fuzzy Soft Information by Expert E_1

	C_1	C_2	C_3	C_4	C_5
A_1	(0.56, 0.12, 0.20)	(0.62, 0.15, 0.23)	(0.47, 0.33, 0.10)	(0.51, 0.34, 0.15)	(0.43, 0.37, 0.20)
A_2	(0.43, 0.25, 0.18)	(0.50, 0.28, 0.22)	(0.54, 0.29, 0.17)	(0.64, 0.17, 0.19)	(0.50, 0.13, 0.37)
A_3	(0.60, 0.32, 0.08)	(0.58, 0.12, 0.30)	(0.62, 0.11, 0.28)	(0.80, 0.15, 0.05)	(0.65, 0.10, 0.25)
A_4	(0.58, 0.22, 0.10)	(0.61, 0.13, 0.26)	(0.55, 0.27, 0.18)	(0.67, 0.26, 0.07)	(0.76, 0.01, 0.23)

Table 3.5: Normalized Picture Fuzzy Soft Information by Expert E_2

	C_1	C_2	C_3	C_4	C_5
A_1	(0.48, 0.35, 0.17)	(0.53, 0.27, 0.10)	(0.61, 0.28, 0.19)	(0.80, 0.15, 0.05)	(0.07, 0.29, 0.64)
A_2	(0.53, 0.27, 0.10)	(0.64, 0.17, 0.19)	(0.43, 0.37, 0.20)	(0.23, 0.22, 0.65)	(0.27, 0.28, 0.45)
A_3	(0.66, 0.20, 0.14)	(0.59, 0.21, 0.20)	(0.18, 0.11, 0.77)	(0.73, 0.17, 0.10)	(0.55, 0.27, 0.08)
A_4	(0.41, 0.28, 0.31)	(0.18, 0.32, 0.50)	(0.29, 0.32, 0.39)	(0.49, 0.34, 0.17)	(0.68, 0.14, 0.18)

Table 3.6: Normalized Picture Fuzzy Soft Information by Expert E_3

	C_1	C_2	C_3	C_4	C_5
A_1	(0.51, 0.24, 0.25)	(0.70, 0.12, 0.18)	(0.52, 0.25, 0.23)	(0.56, 0.12, 0.20)	(0.26, 0.18, 0.56)
A_2	(0.69, 0.08, 0.23)	(0.59, 0.21, 0.10)	(0.35, 0.02, 0.63)	(0.76, 0.09, 0.15)	(0.47, 0.34, 0.19)
A_3	(0.37, 0.21, 0.42)	(0.45, 0.23, 0.32)	(0.73, 0.14, 0.13)	(0.55, 0.27, 0.08)	(0.64, 0.17, 0.19)
A_4	(0.42, 0.25, 0.33)	(0.66, 0.19, 0.15)	(0.57, 0.26, 0.17)	(0.18, 0.32, 0.50)	(0.82, 0.10, 0.08)

Table 3.7: Collected information of all the Experts for $PFSDWA$ Operators

	A_1	A_2	A_3	A_4
C_1	(0.5234, 0.1864, 0.2116)	(0.5986, 0.1288, 0.1695)	(0.5408, 0.2360, 0.1455)	(0.4867, 0.2436, 0.1815)
C_2	(0.6488, 0.1465, 0.1661)	(0.5750, 0.2188, 0.1399)	(0.5943, 0.1716, 0.2107)	(0.5943, 0.1759, 0.2107)
C_3	(0.5262, 0.2793, 0.1536)	(0.4457, 0.0412, 0.2658)	(0.6468, 0.1217, 0.2011)	(0.5252, 0.2738, 0.1959)
C_4	(0.6346, 0.1636, 0.1165)	(0.6781, 0.1255, 0.1941)	(0.7135, 0.1932, 0.068)	(0.5003, 0.2993, 0.14113)
C_5	(0.3042, 0.2418, 0.3489)	(0.4515, 0.2114, 0.2390)	(0.6289, 0.1452, 0.1595)	(0.7818, 0.0244 0.1211)

Table 3.8: Collected information of all the Experts for $PFSDWG$ Operators

	A_1	A_2	A_3	A_4
C_1	(0.5197, 0.2293, 0.2178)	(0.5424, 0.1868, 0.1893)	(0.4756, 0.2506, 0.2799)	(0.4624, 0.2461, 0.2595)
C_2	(0.6310, 0.1846, 0.1840)	(0.5633, 0.2288, 0.1636)	(0.5148, 0.1904, 0.2917)	(0.4225, 0.2012, 0.2869)
C_3	(0.5160, 0.2858, 0.1805)	(0.4168, 0.2123, 0.4796)	(0.4363, 0.1238, 0.4661)	(0.4727, 0.2762, 0.2289)
C_4	(0.5748, 0.2169, 0.1559)	(0.4979, 0.1472, 0.3477)	(0.6537, 0.2121, 0.0738)	(0.2915, 0.3045, 0.3409)
C_5	(0.1851, 0.2785, 0.5039)	(0.4169, 0.2657, 0.2705)	(0.6229, 0.1701, 0.1933)	(0.7672, 0.0793, 0.1579)

Table 3.9: Collected information of all the Experts for $PFSDOWA$ Operators

	A_1	A_2	A_3	A_4
C_1	(0.5167, 0.0823, 0.0914)	(0.5734, 0.0608, 0.0636)	(0.5543, 0.1052, 0.0523)	(0.4848, 0.1184, 0.0773)
C_2	(0.6227, 0.0669, 0.0636)	(0.5763, 0.0948, 0.0645)	(0.5344, 0.0753, 0.1238)	(0.5218, 0.0787, 0.1014)
C_3	(0.5377, 0.1373, 0.0648)	(0.4457, 0.0188, 0.0991)	(0.5783, 0.0517, 0.0964)	(0.4715, 0.1371, 0.0910)
C_4	(0.6798, 0.0681, 0.0369)	(0.6151, 0.058, 0.0899)	(0.7135, 0.0800, 0.0289)	(0.5004, 0.1504, 0.0512)
C_5	(0.2691, 0.1163, 0.1672)	(0.4096, 0.0914, 0.1086)	(0.6087, 0.0632, 0.05614)	(0.7609, 0.0089, 0.0549)

Table 3.10: Collected information of all th Experts for $PFSDOWG$ Operators

	A_1	A_2	A_3	A_4
C_1	(0.5116, 0.2384, 0.2056)	(0.5178, 0.1904, 0.1656)	(0.4808, 0.2483, 0.2454)	(0.4593, 0.2479, 0.2375)
C_2	(0.5982, 0.1844, 0.1656)	(0.5599, 0.2196, 0.1636)	(0.5159, 0.1797, 0.2663)	(0.2978, 0.2181, 0.3332)
C_3	(0.5219, 0.2858, 0.1677)	(0.4168, 0.2286, 0.4247)	(0.3028, 0.1207, 0.5684)	(0.3954, 0.2841, 0.2646)
C_4	(0.5963, 0.2169, 0.1278)	(0.3664, 0.1555, 0.437)	(0.6537, 0.1999, 0.874)	(0.2915, 0.3019, 0.2982)
C_5	(0.1248, 0.2785, 0.4973)	(0.3581, 0.2455, 0.2705)	(0.5991, 0.1806, 0.1685)	(0.7398, 0.0773, 0.1579)

Table 3.11: Averaging Aggregation Operators (AV)

	WA	OWA
C_1	(0.5391, 0.1928, 0.1754)	(0.5335, 0.0888, 0.0696)
C_2	(0.4499, 0.1764, 0.1924)	(0.5655, 0.0783, 0.0847)
C_3	(0.5419, 0.1399, 0.2002)	(0.5112, 0.0654, 0.0866)
C_4	(0.6398, 0.1856, 0.1216)	(0.6356, 0.083, 0.0471)
C_5	(0.5807, 0.1160, 0.2003)	(0.5518, 0.0496, 0.0865)

Table 3.12: Geometric Aggregation Operators(AV)

	WG	OWG
C_1	(0.4992, 0.2286, 0.2374)	(0.4918, 0.2316, 0.2141)
C_2	(0.5273, 0.2015, 0.2338)	(0.4763, 0.2007, 0.2356)
C_3	(0.4589, 0.2271, 0.3527)	(0.4017, 0.2326 0.3756)
C_4	(0.4832, 0.2222, 0.2386)	(0.4517, 0.2204, 0.2484)
C_5	(0.4383, 0.2023, 0.2959)	(0.3752, 0.199, 0.2882)

Table 3.13: PDA_{ij} (Soft Dombi WA)

	C_1	C_2	C_3	C_4	C_5
A_1	(0.8249)	(4.9502)	(0.8479)	(0.5578)	(0.0000)
A_2	(1.5109)	(4.3635)	(0.0000)	(0.4551)	(0.0000)
A_3	(1.3130)	(2.1152)	(1.2104)	(0.9397)	(0.7756)
A_4	(0.7645)	(3.7293)	(0.6329)	(0.0796)	(1 .4996)

Table 3.14: NDA_{ij}(Soft Dombi WA)

	C_1	C_2	C_3	C_4	C_5
A_1	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.1691)
A_2	(0.0000)	(0.0000)	(0.1040)	(0.0000)	(0.2985)
A_3	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
A_4	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)

Table 3.15: PDA_{ij}(Soft Dombi WG)

	C_1	C_2	C_3	C_4	C_5
A_1	(0.1533)	(0.1430)	(2.9416)	(0.0000)	(2.1389)
A_2	(0.5267)	(0.0000)	(0.2762)	(0.4437)	(0.0000)
A_3	(0.0000)	(0.0000)	(0.8412)	(0.0000)	(3.0721)
A_4	(0.6412)	(0.0000)	(1.2945)	(0.0000)	(3.2785)

Table 3.16: NDA_{ij} (Soft Dombi WG)

	$\boldsymbol{C_1}$	$\boldsymbol{C_2}$	$\boldsymbol{C_3}$	$\boldsymbol{C_4}$	$\boldsymbol{C_5}$
$\boldsymbol{A_1}$	(0.0000)	(0.0000)	(0.0000)	(2.3032)	(0.0000)
$\boldsymbol{A_2}$	(0.0000)	(0.2398)	(0.0000)	(0.0000)	(1.4409)
$\boldsymbol{A_3}$	(0.4262)	(0.5011)	(0.0000)	(1.2178)	(0.0000)
$\boldsymbol{A_4}$	(0.0000)	(0.3087)	(0.0000)	(1.2021)	(0.0000)

Table 3.17: PDA_{ij}(Soft Dombi OWA)

	$\boldsymbol{C_1}$	$\boldsymbol{C_2}$	$\boldsymbol{C_3}$	$\boldsymbol{C_4}$	$\boldsymbol{C_5}$
$\boldsymbol{A_1}$	(0.1338)	(0.3894)	(0.3172)	(0.2718)	(0.0000)
$\boldsymbol{A_2}$	(0.3593)	(0.2717)	(0.0000)	(0.0391)	(0.0000)
$\boldsymbol{A_3}$	(0.3386)	(0.0204)	(0.3421)	(0.3546)	(0.3291)
$\boldsymbol{A_4}$	(0.0865)	(0.0446)	(0.0597)	(0.0000)	(0.0698)

Table 3.18: NDA_{ij}(Soft Dombi OWA)

	$\boldsymbol{C_1}$	$\boldsymbol{C_2}$	$\boldsymbol{C_3}$	$\boldsymbol{C_4}$	$\boldsymbol{C_5}$
$\boldsymbol{A_1}$	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.7548)
$\boldsymbol{A_2}$	(0.0000)	(0.0000)	(0.0347)	(0.0000)	(0.2760)
$\boldsymbol{A_3}$	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
$\boldsymbol{A_4}$	(0.0000)	(0.0000)	(0.0000)	(0.1114)	(0.0000)

Table 3.19: PDA_{ij}(Soft Dombi OWG)

	$\boldsymbol{C_1}$	$\boldsymbol{C_2}$	$\boldsymbol{C_3}$	$\boldsymbol{C_4}$	$\boldsymbol{C_5}$
$\boldsymbol{A_1}$	(0.1019)	(0.7973)	(0.3550)	(1.3047)	(0.0000)
$\boldsymbol{A_2}$	(0.2686)	(0.6464)	(0.0000)	(0.0000)	(0.0000)
$\boldsymbol{A_3}$	(0.0000)	(0.0373)	(0.0000)	(1.8514)	(0.0000)
$\boldsymbol{A_4}$	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)

Table 3.20: NDA_{ij}(Soft Dombi OWG)

	$\boldsymbol{C_1}$	$\boldsymbol{C_2}$	$\boldsymbol{C_3}$	$\boldsymbol{C_4}$	$\boldsymbol{C_5}$
$\boldsymbol{A_1}$	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(1.2424)
$\boldsymbol{A_2}$	(0.0000)	(0.0000)	(0.2699)	(0.2739)	(0.7824)
$\boldsymbol{A_3}$	(0.0422)	(0.0000)	(0.5269)	(0.0000)	(0.4394)
$\boldsymbol{A_4}$	(0.0559)	(0.2761)	(0.1307)	(0.2099)	(0.2882)

Table 3.21: $SP_i(i\ =\ 1,\ 2,\ 3,\ 4)$

	$\boldsymbol{SP_1}$	$\boldsymbol{SP_2}$	$\boldsymbol{SP_3}$	$\boldsymbol{SP_4}$
WA	1.1461	0.8168	1.083	1.0956
OWA	0.2096	0.1202	0.0569	0.2985
WG	1.3465	0.2598	0.9114	1.2248
OWG	0.0000	0.1431	0.2826	0.4212

Table 3.22: $SN_i(i = 1, 2, 3, 4)$

	SN_1	SN_2	SN_3	SN_4
WA	0.2572	0.0948	0.0000	0.0000
OWA	0.1660	0.0704	0.0145	0.0000
WG	0.3455	0.3482	0.3416	0.2204
OWG	0.1644	0.2888	0.2535	0.1644

Table 3.23: $NSP_i(i = 1, 2, 3, 4)$

	NSP_1	NSP_2	NSP_3	NSP_4
WA	1.0000	0.7127	0.9449	0.9559
OWA	0.7022	0.4028	0.1906	1.0000
WG	1.0000	0.1929	0.6768	0.2204
OWG	0.0000	0.3355	0.6709	1.0000

Table 3.24: $NSN_i(i = 1, 2, 3, 4)$

	NSN_1	NSN_2	NSN_3	NSN_4
WA	0.0000	0.6314	1.0000	1.0000
OWA	0.0000	0.5757	0.9128	1.0000
WG	0.07734	0.0000	0.0189	0.3668
OWG	0.4307	0.0000	0.1222	0.0535

Table 3.25: $AS_i(i = 1, 2, 3, 4)$

	AS_1	AS_2	AS_3	AS_4
WA	0.5	0.6721	0.9725	0.9779
OWA	0.3511	0.4892	0.5507	1.0000
WG	0.5387	0.0965	0.3479	0.6382
OWG	0.2153	0.1678	0.3965	0.5268

Table 3.26: Ranking of the Alternatives

Operators	Ranking on the Basis of AS_i Values
WA	$A_4 > A_3 > A_2 > A_1$
OWA	$A_4 > A_3 > A_2 > A_1$
WG	$A_4 > A_1 > A_3 > A_2$
OWG	$A_4 > A_3 > A_1 > A_2$

- The proposed aggregation operators make the aggregation information more flexible with the involvement of the parameter R.

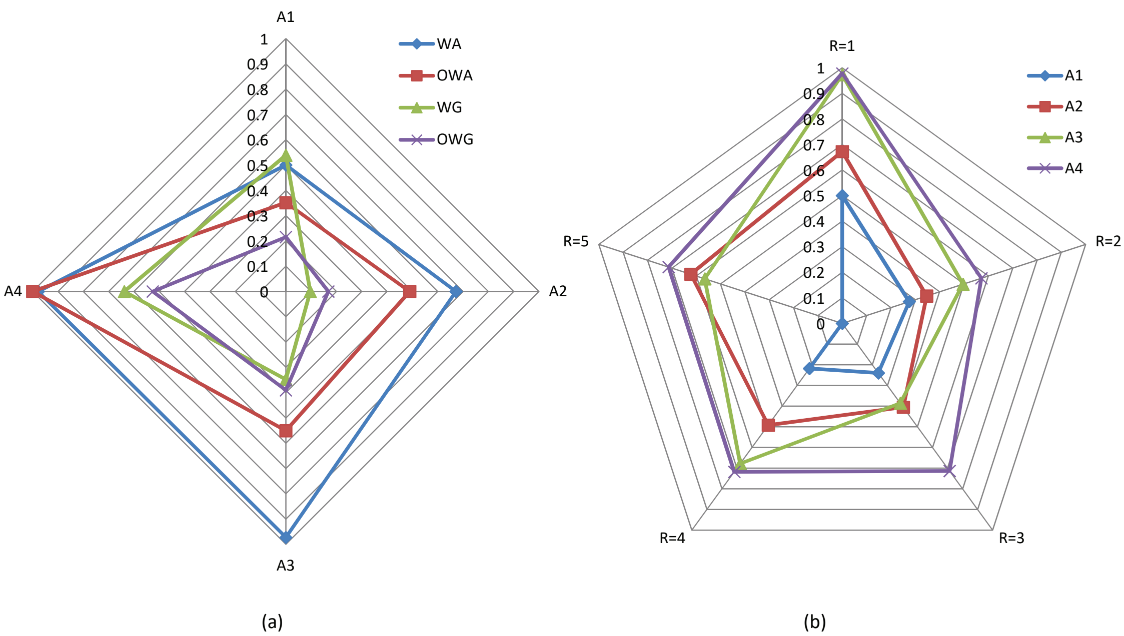

Figure 3.2: Sensitivity Analysis (a) Ranking of alternatives w.r.t. proposed operators (b) Ranking of alternatives w.r.t. values of R

Sensitivity Analysis:
On the basis of computations and the above sensitivity diagram, we observe that

- The effect of this parameter R on $PFSDWA$ operators is illustrated with the help of the sensitivity analysis as shown in Figure 3.2 (b). The role of the parameter R is very important in understanding the variability and reliability of the obtained result. We consider the value of $R = 1$ for our MCDM problem. However, the decision makers can choos the appropriate value o R according to their convenience.

- Figure 3.2 (b) clearly shows that if we keep on changing the values of R with respect to weighted aggregation operators, the obtained ranking reflects a kind of consistency in the selection of alternatives. This enables us to make sure that the uncertainty in the decision is reduced.

- Also, there is a comparison among the proposed aggregation operators which is shown by Figure 3.2 (a).

- Figure 3.2 (a) explains that if we carry on our computation with respect to weighted average and ordered weighted average operators and keep the value of R fixed, then the ranking of the alternatives slightly deviates at the least preference level but remains intact at the highest preference level.

3.6 Conclusions

In this chapter, we successfully proposed new aggregation operators called picture fuzzy soft Dombi aggregation operators. We devised the Dombi operational laws under picture fuzzy soft environment to develop some new aggregation operators, i.e., "*PFSDWA*, *PFSDOWA*, *PFSDHA*, *PFSDWG*, *PFSDOWG*, *PFSDHG*" aggregation operators by making use of Dombi norms. Also, we proposed some important results and properties along with their proofs which helps us in designing and validating the proposed methodology. On the basis of thes aggregation operators, we modified the EDAS methodology which involves the parametrization of attributes and interrelationship among the input arguments. Further, we have successfully applied the proposed methodology to the problem of robotic agrifarming problem. Also, the work in this chapter can further be modified to some dimensionality reduction problems, experimental studies [109], [110].

Chapter 4

Picture Fuzzy Hypersoft Sets

In this chapter, a new way of defining Picture Fuzzy Hypersoft Set (PFHSS) has been presented which contains an additional capacity of accommodating the components of neutral membership (abstain) and refusal compared to intuitionistic fuzzy hypersoft set. Some of the important properties and operational laws related to the introduced picture fuzzy hypersoft weighted average/ordered weighted average operator and weighted geometric/ordered weighted geometric operator have been proved in detail. Also, we have proposed the notion of similarity measure in the pictur fuzzy hypersoft sets along with some important theorems and their utilization in a decision-making problem. Based on these aggregation operators and obtained results, a new algorithm for solving a decision-making problem, involving the multi-sub attributes and their parametrization in the shade of abstain and refusal feature, has been proposed. A numerical example of the selection process of employees for a company has been solved to suitably ensure and validate the implementation of the proposed methodology. Some of the advantageous features of the proposed notions and algorithm have been listed along with the comparative analysis in contrast with the existing literature.

4.1 Picture Fuzzy Hypersoft Set & Operations

In this section, we introduce the novel notion of Picture Fuzzy Hypersoft Set (PFHSS) along with various important properties and fundamental operations. The following definition of PFHSS (the parametrization of multi-sub attributes and all the four components of picture fuzzy information) is being proposed:

Definition 25 ***(Picture Fuzzy Hypersoft Set).*** *Let V be the universal set and PFS(V) be the set of all picture fuzzy subsets of V. Consider $k_1, k_2, \ldots .k_n$* for $n \geq 1$, *be n well-defined attributes, whose corresponding attribute values are the sets $K_1, K_2, \ldots, K_n$* with $K_i \cap K_j = \varphi$ for $i \neq j$ and $i, j \in \{1, 2, \ldots, n\}$. *Let B_i be the non-empty subsets of K_i for each $i = 1, 2, \ldots, n$.*
A Picture Fuzzy Hypersoft Set (PFHSS) is defined as the pair $(R, B_1 \times B_2 \times \ldots . \times B_n)$; where $R : K_1 \times K_2 \times \ldots \times K_n \to \mathrm{PFS}(V)$ and

$$R(B_1 \times B_2 \times \ldots . \times B_n) = \left\{ < \vartheta, \left(\frac{v}{\rho_{R(\vartheta)}(v), \tau_{R(\vartheta)}(v), \omega_{R(\vartheta)}(v)} \right) > \mid v \in V \right\}.$$

It may be noted that $\vartheta \in B_1 \times B_2 \times \ldots . \times B_n \subseteq K_1 \times K_2 \times \ldots . K_n$ and ρ, τ and *ω represent the positive membership, neutral membership and negative membership degrees respectively, and satisfies the condition*

$$\rho_{R(\vartheta)}(v) + \tau_{R(\vartheta)}(v) + \omega_{R(\vartheta)}(v) \leq 1 \text{ where } \rho_{R(\vartheta)}(v), \tau_{R(\vartheta)}(v), \omega_{R(\vartheta)}(v) \in [0, 1].$$

The term $\complement_{R(\vartheta)}(v) = 1 - \rho_{R(\vartheta)}(v) - \tau_{R(\vartheta)}(v) - \omega_{R(\vartheta)}(v)$ is called the degree of refusal membership of v in PFS(V). *For the sake of simplicity, we denote $K_1 \times K_2 \times \ldots . K_n$ by Γ and $B_1 \times B_2 \times \ldots . \times B_n$ by Λ. We denote the set of all PFHSSs over V by $PFHSS(V)$.*

Particular Case: In particular, the proposed definition also directs that every picture fuzzy hypersoft set is also a picture fuzzy soft set. If we select the parameters from only one attribute set, say K_1, while forming the picture fuzzy hypersoft set, then the resulting set becomes the picture fuzzy soft set. In other words, the picture fuzzy hypersoft set is the generalized version of the picture fuzzy soft set. In view of the possible variability based on the extreme values of the four components of picture fuzzy information, we may categorize two sub-definitions of PFHSS as follows:

Definition 26 *A picture fuzzy hypersoft set (R, Γ) over the universe V is known as* ***void picture fuzzy hypersoft set*** *and denoted by $0_{(V_{PFH,\Gamma})}$ if for all $v \in V$ and $\vartheta \in \Gamma$, $\rho_{R(\vartheta)}(v) = 0$, $\tau_{R(\vartheta)}(v) = 0$* and *$\omega_{R(\vartheta)}(v) = 1$.*

Definition 27 *A picture fuzzy hypersoft set* $(R, \Gamma\ )$ *over the universe* V *is known as* ***absolute picture fuzzy hypersoft set*** *and denoted by* $1_{\left(V_{PFH,\Gamma}\right)}$ *if for all* $v \in V$ *and* $\vartheta \in \Gamma$, $\rho_{R(\vartheta)}(v) = 1$, $\tau_{R(\vartheta)}(v) = 0$ and $\omega_{R(\vartheta)}(v) = 0$.

Example 1 *Let* V *be the set of available four smart phones given as* $V = \{v_1,\ v_2,\ v_3,\ v_4\}$ *and the set of attributes given as* K_1 *= Display,* K_2 *= Storage1(ROM),* K_3 *= Storage2(RAM),* K_4 *= Colour. Further, in order to understand the framework of the proposed notion, assume that their respective sub-attributes are*

$$K_1 = \text{Display} = \{\text{OLED}\ (\alpha_1),\ \text{AMOLED}\ (\alpha_2),\ \text{super AMOLED}\ (\alpha_3)\}$$

$$K_2 = \text{Storage1} = \{\text{32GB}\ (\beta_1),\ \text{64GB}\ (\beta_2),\ \text{128GB}\ (\beta_3)\}$$

$$K_3 = \text{Storage2} = \{\text{4GB}\ (\gamma_1),\ \text{8G B}(\gamma_2),\ \text{16GB}\ (\gamma_3)\}$$

$$K_4 = \text{Colour} = \{\text{Black}\ (\delta_1),\ \text{Rose Gold}\ (\delta_2),\ \text{Space Grey}\ (\delta_3)\}$$

Also, assume that

$$B_1 = \{\alpha_3\},\ B_2 = \{\beta_1,\ \beta_2\},\ B_3 = \{\gamma_1,\ \gamma_3\},\ B_4 = \{\delta_1\}$$

$$C_1 = \{\alpha_1,\ \alpha_3\},\ B_2 = \{\beta_1\},\ B_3 = \{\gamma_1,\ \gamma_2\},\ B_4 = \{\delta_2\}$$

are the subsets of K_i for each $i = 1, 2, 3$. *Then the picture fuzzy hypersoft set* $(R,\ \Lambda_1\ )$ *and* $(R,\ \Lambda_2\ )$ *may have the following set-theoretic and tabular representation:*

$$(R,\ \Lambda_1\ ) = \left\{ \begin{array}{c} < (\alpha_3, \beta_1, \gamma_1, \delta_1),\ \left\{ \frac{v_1}{(0.1,0.3,0.5)}, \frac{v_2}{(0.2,0.4,0.2)}, \frac{v_3}{(0.1,0.2,0.4)} \right\} >, \\ < (\alpha_3, \beta_1, \gamma_3, \delta_1), \left\{ \frac{v_1}{(0.2,0.3,0.2)}, \frac{v_2}{(0.2,0.1,0.2)}, \frac{v_3}{(0.1,0.3,0.4)} \right\} >, \\ < (\alpha_3, \beta_2, \gamma_1, \delta_1),\ \left\{ \frac{v_1}{(0.1,0.1,0.5)}, \frac{v_2}{(0.2,0.2,0.3)}, \frac{v_3}{(0.1,0.4,0.4)}, \frac{v_4}{(0.1,0.2,0.3)} \right\} > \\ < (\alpha_3, \beta_2, \gamma_3, \delta_1),\ \left\{ \frac{v_1}{(0.1,0.2,0.5)}, \frac{v_2}{(0.2,0.2,0.2)}, \frac{v_3}{(0.2,0.4,0.4)}, \frac{v_4}{(0.1,0.2,0.6)} \right\} > \end{array} \right\}.$$

$$(R,\ \Lambda_2\ ) = \left\{ \begin{array}{c} < (\alpha_1, \beta_1, \gamma_1, \delta_2),\ \left\{ \frac{v_1}{(0.1,0.3,0.2)}, \frac{v_2}{(0.2,0.4,0.3)}, \frac{v_3}{(0.1,0.2,0.5)} \right\} >, \\ < (\alpha_1, \beta_1, \gamma_2, \delta_2), \left\{ \frac{v_1}{(0.2,0.3,0.1)}, \frac{v_2}{(0.2,0.1,0.1)}, \frac{v_3}{(0.1,0.3,0.5)} \right\} >, \\ < (\alpha_3, \beta_1, \gamma_1, \delta_2),\ \left\{ \frac{v_1}{(0.1,0.1,0.7)}, \frac{v_2}{(0.2,0.3,0.2)}, \frac{v_3}{(0.1,0.4,0.2)}, \frac{v_4}{(0.1,0.2,0.6)} \right\} > \\ < (\alpha_3, \beta_1, \gamma_2, \delta_2),\ \left\{ \frac{v_1}{(0.5,0.2,0.1)}, \frac{v_2}{(0.2,0.4,0.2)}, \frac{v_3}{(0.2,0.4,0.1)}, \frac{v_4}{(0.1,0.2,0.1)} \right\} > \end{array} \right\}.$$

Some Basic Operations on Picture Fuzzy Hypersoft Sets:

In view of the proposed definition of the picture fuzzy hypersoft set above, we formally define some of the fundamental set-theoretic operations for the sake of understanding and better readability.

Table 4.1: Tabular form of PFHSS $(R,\ \Lambda_1\ )$

$(R,\ \Lambda_1\ )$	v_1	v_2	v_3	v_4
$(\alpha_3, \beta_1,\ \gamma_1,\ \delta_1)$	(0.1, 0.3, 0.5)	(0.2, 0.4, 0.2)	(0.1, 0.2, 0.4)	(0, 0, 1)
$(\alpha_3,\ \beta_1,\ \gamma_3,\ \delta_1)$	(0.2, 0.3, 0.2)	(0.2, 0.1, 0.2)	(0.1, 0.3, 0.4)	(0, 0, 1)
$(\alpha_3, \beta_2,\ \gamma_1,\ \delta_1)$	(0.1, 0.1, 0.5)	(0.2, 0.2, 0.3)	(0.1, 0.4, 0.4)	(0.1, 0.2, 0.3)
$(\alpha_3, \beta_2,\ \gamma_3,\ \delta_1)$	(0.1, 0.2, 0.5)	(0.2, 0.2, 0.2)	(0.2, 0.4, 0.4)	(0.1, 0.2, 0.6)

Table 4.2: Tabular form of PFHSS $(R,\ \Lambda_2\ )$

$(R,\ \Lambda_2\ )$	v_1	v_2	v_3	v_4
$(\alpha_1, \beta_1,\ \gamma_1,\ \delta_2)$	(0.1, 0.3, 0.2)	(0.2, 0.4, 0.3)	(0.1, 0.2, 0.5)	(0, 0, 1)
$(\alpha_1, \beta_1,\ \gamma_2,\ \delta_2\ )$	(0.2, 0.3, 0.1)	(0.2, 0.1, 0.1)	(0.1, 0.3, 0.5)	(0, 0, 1)
$(\alpha_3, \beta_1,\ \gamma_1,\ \delta_2\ )$	(0.1, 0.1, 0.7)	(0.2, 0.3, 0.2)	(0.1, 0.4, 0.2)	(0.1, 0.2, 0.6)
$(\alpha_3, \beta_1,\ \gamma_2,\ \delta_2\ )$	(0.5, 0.2, 0.1)	(0.2, 0.4, 0.2)	(0.2, 0.4, 0.1)	(0.1, 0.2, 0.1)

Let $(R_1,\ \Lambda)$ and $(R_2,\ \Lambda')$ be two picture fuzzy hypersoft sets on V and $\Lambda,\ \Lambda' \subseteq \Gamma$ be the set of multi-parameters.

Complement . The complement of picture fuzzy hypersoft set over V is denoted by $(R_1,\ \Lambda)^{\mathbf{c}}$ and defined as $(R_1,\ \Lambda)^{\mathbf{c}} = (R_1{}^{\mathbf{c}}\ \Lambda)$, where $R_1{}^{\mathbf{c}}{:}\Gamma \rightarrow PFS(V)$ is a mapping given by $R_1{}^{\mathbf{c}}(\Lambda) = (R_1(\Lambda))^{\mathbf{c}}\ \forall\ \Lambda \subseteq \Gamma$.
Thus if,

$$(R_1, \Lambda\ ) = \left\{ < \vartheta, \left(\frac{v}{\rho_{R(\vartheta)}(v), \tau_{R(\vartheta)}(v), \omega_{R(\vartheta)}(v)} \right) > \mid v \in V,\ \ \vartheta \in \Lambda \right\}$$

then

$$(R_1, \Lambda\ )^{\mathbf{c}} = \left\{ < \vartheta, \left(\frac{v}{\omega_{R(\vartheta)}(v), \tau_{R(\vartheta)}(v), \rho_{R(\vartheta)}(v)} \right) > \mid v \in V,\ \ \vartheta \in \Lambda \right\}.$$

Subset . Let V be the universe of discourse and $(R_1,\ \Lambda)$ $(R_2,\ \Lambda')$ be any two picture fuzzy hypersoft sets over the set V. Then, $(R_1,\ \Lambda)$ is said to be a picture fuzzy hypersoft subset of $(R_2,\ \Lambda')$ and denoted by$(R_1,\ \Lambda) \subseteq (R_2,\ \Lambda')$ if $\Lambda \subseteq \Lambda'$ and for any $\vartheta \in \Lambda$, $R_1(\vartheta) \subseteq R_2(\vartheta)$, i.e., $\forall v \in V$ and $\vartheta \in \Lambda$,

$$\rho_{R_1(\vartheta)}(v) \leq \rho_{R_2(\vartheta)}(v), \quad \tau_{R_1(\vartheta)}(v) = \tau_{R_2(\vartheta)}(v) \text{ and } \omega_{R_1(\vartheta)}(v) \geq \omega_{R_2(\vartheta)}(v).$$

Equality . Let V be the universe of discourse and (R_1, Λ), (R_2, Λ') be any two picture fuzzy hypersoft sets over the set V. Then, (R_1, Λ) is said to be a *picture fuzzy hypersoft equal* (R_2, Λ') and denoted by$(R_1, \Lambda) = (R_2, \Lambda')$ if for all $v \in V$ and $\vartheta \in \Lambda$, $\rho_{R_1(\vartheta)}(v) = \rho_{R_2(\vartheta)}(v)$, $\tau_{R_1(\vartheta)}(v) = \tau_{R_2(\vartheta)}(v)$ and $\omega_{R_1(\vartheta)}(v) = \omega_{R_2(\vartheta)}(v)$.

Union . Let V be the universe of discourse, $\Lambda, \Lambda' \subseteq \tilde{A}$ and (R_1, Λ), (R_2, Λ') be any two picture fuzzy hypersoft sets over V. The union of (R_1, Λ) and (R_2, Λ') is denoted by $(R_1, \Lambda) \cup (R_2, \Lambda') = (R, \Lambda'')$, where $\Lambda'' = \Lambda \cup \Lambda'$ and $\vartheta \in \Lambda''$ Therefore, $\forall\, \vartheta \in \Lambda \cap \Lambda'$, we have

$$R(\vartheta) = \left\{v,\ \max\left(\rho_{R_1(\vartheta)}(v), \rho_{R_2(\vartheta)}(v)\right), \min\left(\tau_{R_1(\vartheta)}(v), \tau_{R_2(\vartheta)}(v)\right), \min(\omega_{R_1(\vartheta)}(v), \omega_{R_2(\vartheta)}(v))\right\}.$$

Intersection. Let V be the universe of discourse, $\Lambda, \Lambda' \subseteq \Gamma$ and (R_1, Λ), (R_2, Λ') be any two picture fuzzy hypersoft sets over V. The intersection of (R_1, Λ) and (R_2, Λ') is denoted by $(R_1, \Lambda) \cap (R_2, \Lambda') = (R, \Lambda^*)$, where $\Lambda^* = \Lambda \cap \Lambda'$ and $\vartheta \in \Lambda^*$ Therefore, $\forall \vartheta \in \Lambda \cap \Lambda'$, we have

$$R(\vartheta) = \left\{v, \min\left(\rho_{R_1(\vartheta)}(v), \rho_{R_2(\vartheta)}(v)\right), \min\left(\tau_{R_1(\vartheta)}(v), \tau_{R_2(\vartheta)}(v)\right), \max(\omega_{R_1(\vartheta)}(v), \omega_{R_2(\vartheta)}(v))\right\}.$$

Remarks: Let (R, Λ) be a picture fuzzy hypersoft set over the universal set V. Then,

- $((R, \Lambda)^c)^c = (R, \Lambda)$
- $0^c_{(V_{PFH}, \Gamma)} = 1_{(V_{PFH}, \Gamma)}$
- $1^c_{(V_{PFH}, \Gamma)} = 0_{(V_{PFH}, \Gamma)}$

Now, for the sake of methodological calculations and further simplifications, the notion of PFHSS can also be viewed as a Picture Fuzzy Hypersoft Number (PFHSN) given below:

$$R_{v_i}(\vartheta_j) = \left\{\rho_{R(\vartheta_j)}(v_i),\ \tau_{R(\vartheta_j)}(v_i),\ \omega_{R(\vartheta_j)}(v_i)\ |v_i \in V\right\}.$$

This structure would be known as **Picture Fuzzy Hypersoft Number (PFHSN).** Also, for convenience, PFHSN can also be described as $I_{\vartheta_{ij}} = \left(\rho_{R(\vartheta_{ij})}, \tau_{R(\vartheta_{ij})}, \omega_{R(\vartheta_{ij})}\right)$;

where the subscript ϑ_{ij} is used to establish the connection between the alternatives and the parameters for the calculation purposes.

In order to propose a new algorithm for ranking the alternatives based on the proposed PFHSS and their aggregation operators, we suitably reframe the notion of the *score function* and *accuracy function* for PFHSNs as follows:

Let $I_{\vartheta_{ij}} = \left(\rho_{R(\vartheta_{ij})},\ \tau_{R(\vartheta_{ij})},\ \omega_{R(\vartheta_{ij})}\right)$ be a PFHSN. The ***score function*** of $I_{\vartheta_{ij}}$ is given by $\mathbb{S}\left(I_{\vartheta_{ij}}\right) = \rho_{R(\vartheta_{ij})} - \tau_{R(\vartheta_{ij})} - \omega_{R(\vartheta_{ij})};\ i, j \in \{1,\ 2,\ \ldots,n\}$ and $\mathbb{S}\left(I_{\vartheta_{ij}}\right) \in [-1, 1]$.

Remarks:

- It may be noted that in some situations, the score function for two different PFHSNs may be the same, e.g.,
  if we take $I_{\vartheta_{11}} = (0.3,\ 0.5,\ 0.2)$ and $I_{\vartheta_{12}} = (0.6,\ 0.4,\ 0.6)$ as two PFHSNs then as per the definition of the score function, the score value would be -0.4.
- In such cases, it will not be easy to decide which one is the most appropriate $I_{\vartheta_{11}}$ or $I_{\vartheta_{12}}$. Therefore, in order to overcome such problems, the notion of accuracy function has to be further introduced.

The Accuracy function of $I_{\vartheta_{ij}}$ is given by

$$\mathbb{H}\left(I_{\vartheta_{ij}}\right) = \rho_{R(\vartheta_{ij})} + \tau_{R(\vartheta_{ij})} + \omega_{R(\vartheta_{ij})}.$$

It may also be noted that $\mathbb{H}\left(I_{\vartheta_{ij}}\right) \in [0, 1]$ and for the comparison of the two PFHSNs, $I_{\vartheta_{ij}}$ and $J_{\vartheta_{ij}}$, the following comparisons of the above-defined functions have been done.

- If $\mathbb{S}\left(I_{\vartheta_{ij}}\right) > \mathbb{S}\left(J_{\vartheta_{ij}}\right)$ then $I_{\vartheta_{ij}} > J_{\vartheta_{ij}}$.
- If $\mathbb{S}\left(I_{\vartheta_{ij}}\right) < \mathbb{S}\left(J_{\vartheta_{ij}}\right)$ then $I_{\vartheta_{ij}} < J_{\vartheta_{ij}}$.
- If $\mathbb{S}\left(I_{\vartheta_{ij}}\right) = \mathbb{S}\left(J_{\vartheta_{ij}}\right)$ then $I_{\vartheta_{ij}} \equiv J_{\vartheta_{ij}}$.
- If $\mathbb{H}\left(I_{\vartheta_{ij}}\right) > \mathbb{H}\left(J_{\vartheta_{ij}}\right)$ then $I_{\vartheta_{ij}} > J_{\vartheta_{ij}}$.
- If $\mathbb{H}\left(I_{\vartheta_{ij}}\right) < \mathbb{H}\left(J_{\vartheta_{ij}}\right)$ then $I_{\vartheta_{ij}} < J_{\vartheta_{ij}}$.
- If $\mathbb{H}\left(I_{\vartheta_{ij}}\right) = \mathbb{H}\left(J_{\vartheta_{ij}}\right)$ then $I_{\vartheta_{ij}} \equiv J_{\vartheta_{ij}}$.

Remark: From, the above definition, the score function is monotonically increasing with respect to its variables.

4.2 Average/Geometric Aggregation Operators

In the process of information fusion, the mathematical notion of aggregation operator which aggregates the interrelated multiple input values to solely one outturn value, is an essential tool and widely utilized for handling various decision-making problems. The problems are not only limited to the field of mathematics but also widely spread in physics, economics, engineering, social and other sciences. In this section, we devise two types of aggregation operators (Averaging aggregation operators and Geometric aggregation operators) for Picture Fuzzy Hypersoft Numbers and discuss various results based on this.

For the sake of defining the picture fuzzy hypersoft weighted averaging and weighted geometric operator, we first need to understand the notion of sum, product, scalar multiplication, exponent and complement of PFHSNs which have been defined as below: Let $I_{\vartheta_d} = (\rho_{\vartheta_d}, \tau_{\vartheta_d}, \omega_{\vartheta_d})$, $I_{\vartheta_{11}} = (\rho_{\vartheta_{11}}, \tau_{\vartheta_{11}}, \omega_{\vartheta_{11}})$ and $I_{\vartheta_{12}} = (\rho_{\vartheta_{11}}, \tau_{\vartheta_{12}}, \omega_{\vartheta_{12}})$ be three PFHSNs and κ be a positive real number. Then, the following operations are defined over three PFHSNs:

(a) $I_{\vartheta_{11}} \oplus \ I_{\vartheta_{12}} = \langle \rho_{\vartheta_{11}} + \rho_{\vartheta_{12}} - \rho_{\vartheta_{11}}\rho_{\vartheta_{12}}, \ \ \tau_{\vartheta_{11}}\tau_{\vartheta_{12}}, \ \omega_{\vartheta_{11}}\omega_{\vartheta_{12}} \rangle$.

(b) $I_{\vartheta_{11}} \otimes \ I_{\vartheta_{12}} = \langle (\rho_{\vartheta_{11}}\rho_{\vartheta_{12}}, \ \ \tau_{\vartheta_{11}} + \tau_{\vartheta_{12}} - \ \tau_{\vartheta_{11}}\tau_{\vartheta_{12}}, \omega_{\vartheta_{11}} + \omega_{\vartheta_{12}} - \ \omega_{\vartheta_{11}}\omega_{\vartheta_{12}} \rangle$.

(c) $\kappa I_{\vartheta_d} = \langle [1 - (1 - \rho_{\vartheta_d})^{\kappa}, \ (\tau_{\vartheta_d})^{\kappa}, (\omega_{\vartheta_d})^{\kappa}] \rangle$.

(d) $I^{\kappa}_{\vartheta_d} = \langle [(\rho_{\vartheta_d})^{\kappa}, \ 1 - (1 - \tau_{\vartheta_d})^{\kappa}, \ 1 - (1 - \omega_{\vartheta_d})^{\kappa}] \rangle$.

(e) $I^{c}_{\vartheta_d} = (\omega_{\vartheta_d}, \tau_{\vartheta_d}, \rho_{\vartheta_d})$.

4.2.1 Picture Fuzzy Hypersoft Weighted Averaging (PFHSWA) Aggregation Operator

Definition 28 *Let* $I_{\vartheta_d} = (\rho_{\vartheta_d},\ \tau_{\vartheta_d},\ \omega_{\vartheta_d})$ *be a PFHSN,* $\mathfrak{J}_i$ *and* $\mathfrak{F}_j$ *represent weight vectors for expert's and sub-attributes for selected parameters respectively with the constraint* $\mathfrak{J}_i > 0,\ \sum_{i=1}^{n} \mathfrak{J}_i = 1$ and $\mathfrak{F}_j > 0,\ \sum_{i=1}^{n} \mathfrak{F}_j = 1$. *Then, the* ***Picture Fuzzy Hypersoft Weighted Average (PFHSWA) aggregation operator*** *is a mapping* $\mathfrak{N}^n \rightarrow \mathfrak{N}$ *given by*

$$PFHSWA(I_{\vartheta_{11}},\ I_{\vartheta_{12}},\ \ldots,\ I_{\vartheta_{nm}}) = \oplus_{j=1}^{m} \mathfrak{F}_j \left(\oplus\ _{i=1}^{n} \mathfrak{J}_i I_{\vartheta_{ij}} \right). \tag{4.2.1}$$

where $\mathfrak{N}^n = (I_{\vartheta_{11}},\ I_{\vartheta_{12}},\ \ldots\ , I_{\vartheta_{nm}})$ *is a collection of all the PFHSNs.*

Theorem 8 *Let* $I_{\vartheta_d} = (\rho_{\vartheta_d},\ \tau_{\vartheta_d},\ \omega_{\vartheta_d})$ *be a PFHSN. Then from equation (4.2.1), the weighted average aggregation (fusion) of all the input values also gives a PFHSN, represented by,*

$$\begin{aligned} & PFHSWA(I_{\vartheta_{11}},\ I_{\vartheta_{12}},\ \ldots\ I_{\vartheta_{nm}}) \\ &= \left\langle 1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \rho_{\vartheta_{ij}}\right)^{\mathfrak{J}_i} \right)^{\mathfrak{F}_j},\ \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\tau_{\vartheta_{ij}}\right)^{\mathfrak{J}_i} \right)^{\mathfrak{F}_j}, \right. \\ & \qquad \left. \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\omega_{\vartheta_{ij}}\right)^{\mathfrak{J}_i} \right)^{\mathfrak{F}_j} \right\rangle. \end{aligned} \tag{4.2.2}$$

where $\mathfrak{J}_i$ *and* $\mathfrak{F}_j$ *represent weight vectors for expert's and sub-attributes for selected parameters respectively with the constraint* $\mathfrak{J}_i > 0,\ \sum_{i=1}^{n} \mathfrak{J}_i = 1$ and$\mathfrak{F}_j > 0,\ \sum_{j=1}^{n} \mathfrak{F}_j = 1$.

Proof 1 *The proof of the theorem follows from the principle of mathematical induction.*
For n = 1, we get $\mathfrak{J}_1 = 1$. *(because* $\sum_{i=1}^{n} \mathfrak{J}_i = 1$. *)*

Then, from equation (4.2.1), we have
$PFHSWA(I_{\vartheta_{11}}, I_{\vartheta_{12}}, \ldots, I_{\vartheta_{nm}}) = \oplus_{j=1}^{m} \mathfrak{F}_j I_{\vartheta_{1j}}$.

Now, using the above-stated functional laws (a)-(e), we get

$$PFHSWA(I_{\vartheta_{11}}, I_{\vartheta_{12}}, \ldots, I_{\vartheta_{nm}}) = \left\langle 1 - \prod_{j=1}^{m} \left(1 - \rho_{\vartheta_{1j}}\right)^{\mathfrak{F}_j},\ \prod_{j=1}^{m} \left(\tau_{\vartheta_{1j}}\right)^{\mathfrak{F}_j}, \prod_{j=1}^{m} \left(\omega_{\vartheta_{1j}}\right)^{\mathfrak{F}_j} \right\rangle$$

$$= \left\langle 1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{1}(1-\rho_{\vartheta_{ij}})^{\mathfrak{J}_i}\right)^{\mathfrak{F}_j}, \prod_{j=1}^{m}\left(\prod_{i=1}^{1}(\tau_{\vartheta_{ij}})^{\mathfrak{J}_i}\right)^{\mathfrak{F}_j}, \prod_{j=1}^{m}\left(\prod_{i=1}^{1}(\omega_{\vartheta_{ij}})^{\mathfrak{J}_i}\right)^{\mathfrak{F}_j} \right\rangle.$$

Also, For m = 1, we get $\mathfrak{F}_1 = 1$. *(because* $\sum_{j=1}^{m}\mathfrak{F}_j = 1$ *.)*

Then, from equation (4.2.1), we have $PFHSWA(I_{\vartheta_{11}}, I_{\vartheta_{12}}, \ldots, I_{\vartheta_{nm}}) = \oplus_{i=1}^{n}\mathfrak{J}_i I_{\vartheta_{i1}}$. *Again, using the above-stated functional laws (a)-(e), we get*

$$PFHSWA(I_{\vartheta_{11}}, I_{\vartheta_{12}}, \ldots, I_{\vartheta_{nm}}) = \left\langle 1 - \prod_{i=1}^{n}(1-\rho_{\vartheta_{i1}})^{\mathfrak{J}_i}, \prod_{i=1}^{n}(\tau_{\vartheta_{i1}})^{\mathfrak{J}_i}, \prod_{i=1}^{n}(\omega_{\vartheta_{i1}})^{\mathfrak{J}_i} \right\rangle$$

$$= \left\langle 1 - \prod_{j=1}^{1}\left(\prod_{i=1}^{n}(1-\rho_{\vartheta_{ij}})^{\mathfrak{J}_i}\right)^{\mathfrak{F}_j}, \prod_{j=1}^{1}\left(\prod_{i=1}^{n}(\tau_{\vartheta_{ij}})^{\mathfrak{J}_i}\right)^{\mathfrak{F}_j}, \prod_{j=1}^{1}\left(\prod_{i=1}^{n}(\omega_{\vartheta_{ij}})^{\mathfrak{J}_i}\right)^{\mathfrak{F}_j} \right\rangle.$$

This shows that equation (4.2.2) satisfies for n = 1 and m = 1. As per induction hypothesis, assume that equation (4.2.2) holds for $m = \alpha_1 + 1, n = \alpha_2$ *and* $m = \alpha_1, n = \alpha_2 + 1$*; i.e.,*

$$\oplus_{j=1}^{\alpha_1+1}\mathfrak{F}_j\left(\oplus_{i=1}^{\alpha_2}\mathfrak{J}_i I_{\vartheta_{ij}}\right) = \left\langle 1 - \prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2}(1-\rho_{\vartheta_{ij}})^{\mathfrak{J}_i}\right)^{\mathfrak{F}_j}, \prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2}(\tau_{\vartheta_{ij}})^{\mathfrak{J}_i}\right)^{\mathfrak{F}_j}, \prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2}(\omega_{\vartheta_{ij}})^{\mathfrak{J}_i}\right)^{\mathfrak{F}_j} \right\rangle;$$

$$\oplus_{j=1}^{\alpha_1}\mathfrak{F}_j\left(\oplus_{i=1}^{\alpha_2+1}\mathfrak{J}_i I_{\vartheta_{ij}}\right) = \left\langle 1 - \prod_{j=1}^{\alpha_1}\left(\prod_{i=1}^{\alpha_2+1}(1-\rho_{\vartheta_{ij}})^{\mathfrak{J}_i}\right)^{\mathfrak{F}_j}, \prod_{j=1}^{\alpha_1}\left(\prod_{i=1}^{\alpha_2+1}(\tau_{\vartheta_{ij}})^{\mathfrak{J}_i}\right)^{\mathfrak{F}_j}, \prod_{j=1}^{\alpha_1}\left(\prod_{i=1}^{\alpha_2+1}(\omega_{\vartheta_{ij}})^{\mathfrak{J}_i}\right)^{\mathfrak{F}_j} \right\rangle.$$

Now for $m = \alpha_1 + 1, n = \alpha_2 + 1$*, we get*

$$\oplus_{j=1}^{\alpha_1+1}\mathfrak{F}_j\left(\oplus_{i=1}^{\alpha_2+1}\mathfrak{J}_i I_{\vartheta_{ij}}\right) = \oplus_{j=1}^{\alpha_1+1}\mathfrak{F}_j\left(\oplus_{i=1}^{\alpha_2}\mathfrak{J}_i I_{\vartheta_{ij}} \oplus \mathfrak{J}_{\alpha_2+1} I_{\vartheta_{(\alpha_2+1)j}}\right)$$
$$= \oplus_{j=1}^{\alpha_1+1}\oplus_{i=1}^{\alpha_2}\mathfrak{F}_j\mathfrak{J}_i I_{\vartheta_{ij}} \oplus_{j=1}^{\alpha_1+1}\mathfrak{F}_j\mathfrak{J}_{\alpha_2+1} I_{\vartheta_{(\alpha_2+1)j}}$$

$$= \left\langle \left(1 - \prod_{j=1}^{\alpha_1+1} \left(\prod_{i=1}^{\alpha_2} \left(1-\rho_{\vartheta_{ij}}\right)^{\mathfrak{I}_i}\right)^{\mathfrak{F}_j}\right) \right.$$

$$\oplus \left(1 - \prod_{j=1}^{\alpha_1+1} \left(\left(1-\rho_{\vartheta_{(\alpha_2+1)j}}\right)^{\mathfrak{I}_{(\alpha_2+1)}}\right)^{\mathfrak{F}_j}\right),$$

$$\prod_{j=1}^{\alpha_1+1} \left(\prod_{i=1}^{\alpha_2} \left(\tau_{\vartheta_{ij}}\right)^{\mathfrak{I}_i}\right)^{\mathfrak{F}_j}$$

$$\oplus \prod_{j=1}^{\alpha_1+1} \left(\left(\tau_{\vartheta_{(\alpha_2+1)j}}\right)^{\mathfrak{I}_{(\alpha_2+1)}}\right)^{\mathfrak{F}_j},$$

$$\prod_{j=1}^{\alpha_1+1} \left(\prod_{i=1}^{\alpha_2} \left(\omega_{\vartheta_{ij}}\right)^{\mathfrak{I}_i}\right)^{\mathfrak{F}_j}$$

$$\left. \oplus \prod_{j=1}^{\alpha_1+1} \left(\left(\omega_{\vartheta_{(\alpha_2+1)j}}\right)^{\mathfrak{I}_{(\alpha_2+1)}}\right)^{\mathfrak{F}_j} \right\rangle$$

$$= \left\langle 1 - \prod_{j=1}^{\alpha_1+1} \left(\prod_{i=1}^{\alpha_2+1} \left(1-\rho_{\vartheta_{ij}}\right)^{\mathfrak{I}_i}\right)^{\mathfrak{F}_j}, \right.$$

$$\left. \prod_{j=1}^{\alpha_1+1} \left(\prod_{i=1}^{\alpha_2+1} \left(\tau_{\vartheta_{ij}}\right)^{\mathfrak{I}_i}\right)^{\mathfrak{F}_j}, \prod_{j=1}^{\alpha_1+1} \left(\prod_{i=1}^{\alpha_2+1} \left(\omega_{\vartheta_{ij}}\right)^{\mathfrak{I}_i}\right)^{\mathfrak{F}_j} \right\rangle.$$

Therefore, the result is true for $m = \alpha_1 + 1, n = \alpha_2 + 1$ and the theorem is proved.

Properties of PFHSWA Operator

- **Idempotency**

  If $I_{\vartheta_{ij}} = I_{\vartheta_\alpha} = \left(\rho_{\vartheta_{ij}},\ \tau_{\vartheta_{ij}},\ \omega_{\vartheta_{ij}}\right) \forall i, j$, then $PFHSWA(I_{\vartheta_{11}},\ I_{\vartheta_{12}},\ \ldots, I_{\vartheta_{nm}}) = I_{\vartheta_\alpha}$.

  Proof. Let $I_{\vartheta_{ij}} = I_{\vartheta_\alpha} = \left(\rho_{\vartheta_{ij}},\ \tau_{\vartheta_{ij}},\ \omega_{\vartheta_{ij}}\right)$ be a collection of PFHSNs, then with the use of equation (4.2.2), we get

$$PFHSWA(I_{\vartheta_{11}},\ I_{\vartheta_{12}},\ \ldots, I_{\vartheta_{nm}})$$

$$= \left\langle 1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1-\rho_{\vartheta_{ij}}\right)^{\mathfrak{I}_i}\right)^{\mathfrak{F}_j},\ \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\tau_{\vartheta_{ij}}\right)^{\mathfrak{I}_i}\right)^{\mathfrak{F}_j}, \right.$$

$$\left. \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\omega_{\vartheta_{ij}}\right)^{\mathfrak{I}_i}\right)^{\mathfrak{F}_j} \right\rangle$$

$$= \left\langle 1-\left(\left(1-\rho_{\vartheta_{ij}}\right)^{\sum_{i=1}^{n}\mathfrak{I}_i}\right)^{\sum_{j=1}^{m}\ \mathfrak{F}_j},\ \left(\left(\tau_{\vartheta_{ij}}\right)^{\sum_{i=1}^{n}\mathfrak{I}_i}\right)^{\sum_{j=1}^{m}\ \mathfrak{F}_j},\ \left(\left(\omega_{\vartheta_{ij}}\right)^{\sum_{i=1}^{n}\mathfrak{I}_i}\right)^{\sum_{j=1}^{m}\ \mathfrak{F}_j} \right\rangle$$

$$= \left\langle 1-((1-\rho_{\vartheta_{ij}}),\ (\tau_{\vartheta_{ij}}),\ \omega_{\vartheta_{ij}}\right\rangle = \left(\rho_{\vartheta_{ij}},\ \tau_{\vartheta_{ij}},\ \omega_{\vartheta_{ij}}\right) = I_{\vartheta_\alpha}.$$

Hence, th idempotency holds.

- **Boundedness**

Suppose $I_{\vartheta_{ij}}$ be a collection of picture fuzzy hypersoft numbers.

Let $I_{\vartheta_{ij}}^{-}=\left\langle \min\limits_{j}\min\limits_{i}\ \left\{\rho_{\vartheta_{ij}}\right\},\ \max\limits_{j}\max\limits_{i}\ \left\{\tau_{\vartheta_{ij}}\right\},\ \max\limits_{j}\max\limits_{i}\ \left\{\omega_{\vartheta_{ij}}\right\}\right\rangle$ and $I_{\vartheta_{ij}}^{+}=\left\langle \max\limits_{j}\max\limits_{i}\ \left\{\rho_{\vartheta_{ij}}\right\},\ \min\limits_{j}\min\limits_{i}\ \left\{\tau_{\vartheta_{ij}}\right\},\ \min\limits_{j} min_i\ \left\{\omega_{\vartheta_{ij}}\right\}\right\rangle$, then

$$I_{\vartheta_{ij}}^{-} \leq PFHSWA(I_{\vartheta_{11}}, I_{\vartheta_{12}}, \ldots, I_{\vartheta_{nm}} \leq I_{\vartheta_{ij}}^{+}.$$

Proof.

Let $I_{\vartheta_{ij}} = \left(\rho_{\vartheta_{ij}}, \tau_{\vartheta_{ij}}, \omega_{\vartheta_{ij}}\right)$ be a PFHSN, then $\min\limits_{j}\min\limits_{i}\ \left\{\rho_{\vartheta_{ij}}\right\} \leq \left\{\rho_{\vartheta_{ij}}\right\} \leq \max\limits_{j}\max\limits_{i}\ \left\{\rho_{\vartheta_{ij}}\right\}$

$$\Longrightarrow 1-\max_{j}\max_{i}\ \left\{\rho_{\vartheta_{ij}}\right\} \leq \left\{1-\rho_{\vartheta_{ij}}\right\} \leq 1-\min_{j}\min_{i}\ \left\{\rho_{\vartheta_{ij}}\right\}$$

$$\Longleftrightarrow \left(1-\ \max_{j}\max_{i}\ \left\{\rho_{\vartheta_{ij}}\right\}\right)^{\mathfrak{I}_i} \leq \left(1-\rho_{\vartheta_{ij}}\right)^{\mathfrak{I}_i} \leq \left(1-\ \min_{j}\min_{i}\ \left\{\rho_{\vartheta_{ij}}\right\}\right)^{\mathfrak{I}_i}$$

$$\Longleftrightarrow \left(1-\ \max_{j}\max_{i}\ \left\{\rho_{\vartheta_{ij}}\right\}\right)^{\sum_{i=1}^{n}\mathfrak{I}_i} \leq \prod_{i=1}^{n}\left(1-\rho_{\vartheta_{ij}}\right)^{\mathfrak{I}_i} \leq \left(1-\ \min_{j}\min_{i}\ \left\{\rho_{\vartheta_{ij}}\right\}\right)^{\sum_{i=1}^{n}\mathfrak{I}_i}$$

$$\Longleftrightarrow \left(1-\ \max_{j}\max_{i}\ \left\{\rho_{\vartheta_{ij}}\right\}\right) \leq \prod_{i=1}^{n}\left(1-\rho_{\vartheta_{ij}}\right)^{\mathfrak{I}_i} \leq \left(1-\ \min_{j}\min_{i}\ \left\{\rho_{\vartheta_{ij}}\right\}\right)$$

(as $\sum_{i=1}^{n}\mathfrak{I}_i = 1.$)

$$\Longleftrightarrow \left(1-\max_j \max_i \left\{\rho_{\vartheta_{ij}}\right\}\right)^{\sum_{j=1}^{m}\mathfrak{F}_j} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\rho_{\vartheta_{ij}})^{\mathfrak{I}_i}\right)^{\mathfrak{F}_j} \leq \left(1-\min_j \min_i \left\{\rho_{\vartheta_{ij}}\right\}\right)^{\sum_{j=1}^{m}\mathfrak{F}_j}$$

$$\Longleftrightarrow \left(1-\max_j \max_i \left\{\rho_{\vartheta_{ij}}\right\}\right) \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\rho_{\vartheta_{ij}})^{\mathfrak{I}_i}\right)^{\mathfrak{F}_j} \leq \left(1-\min_j \min_i \left\{\rho_{\vartheta_{ij}}\right\}\right)$$

(as $\sum_{j=1}^{m}\mathfrak{F}_j = 1.$)

$$\Longleftrightarrow \left(\min_j \min_i \left\{\rho_{\vartheta_{ij}}\right\}\right) \leq 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\rho_{\vartheta_{ij}})^{\mathfrak{I}_i}\right)^{\mathfrak{F}_j} \leq \left(\max_j \max_i \left\{\rho_{\vartheta_{ij}}\right\}\right). \tag{4.2.3}$$

Similarly,

$$\left(\min_j \min_i \left\{\tau_{\vartheta_{ij}}\right\}\right) \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\tau_{\vartheta_{ij}})^{\mathfrak{I}_i}\right)^{\mathfrak{F}_j} \leq \left(\max_j \max_i \left\{\tau_{\vartheta_{ij}}\right\}\right). \tag{4.2.4}$$

$$\left(\min_j \min_i \left\{\omega_{\vartheta_{ij}}\right\}\right) \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\omega_{\vartheta_{ij}})^{\mathfrak{I}_i}\right)^{\mathfrak{F}_j} \leq \left(\max_j \max_i \left\{\omega_{\vartheta_{ij}}\right\}\right). \tag{4.2.5}$$

Let$PFHSWA(I_{\vartheta_{11}}, I_{\vartheta_{12}}, \ldots, I_{\vartheta_{nm}})= (\rho_{\vartheta_\alpha}, \tau_{\vartheta_\alpha}, \omega_{\vartheta_\alpha}) = I_{\vartheta_\alpha}$, so that the inequalities (4.2.3), (4.2.4) and (4.2.5) could be transformed into following forms:

$$\min_j \min_i \left\{\rho_{\vartheta_{ij}}\right\} \leq \left\{\rho_{\vartheta_{ij}}\right\} \leq \left\{\rho_{\vartheta_\alpha}\right\} \leq \max_j \max_i \left\{\rho_{\vartheta_{ij}}\right\};$$

$$\min_j \min_i \left\{\tau_{\vartheta_{ij}}\right\} \leq \left\{\tau_{\vartheta_{ij}}\right\} \leq \left\{\tau_{\vartheta_\alpha}\right\} \leq \max_j \max_i \left\{\tau_{\vartheta_{ij}}\right\};$$

and

$$\min_j \min_i \ \{\omega_{\vartheta_{ij}}\} \le \{\omega_{\vartheta_{ij}}\} \le \{\omega_{\vartheta_\alpha}\}$$
$$\le \max_j \max_i \ \{\omega_{\vartheta_{ij}}\}$$

respectively. So, by making use of the earlier defined score function, we obtain the following values:

$$\mathbb{S}\left(I_{\vartheta_\alpha}\right) = \rho_{\vartheta_\alpha} - \tau_{\vartheta_\alpha} - \omega_{\vartheta_\alpha} \le \max_j \max_i \ \{\rho_{\vartheta_{ij}}\}$$
$$-\min_j \min_i \ \{\tau_{\vartheta_{ij}}\} - \min_j \min_i \ \{\omega_{\vartheta_{ij}}\} = \mathbb{S}\left(I^{+}_{\vartheta_{ij}}\right).$$

Also,

$$\mathbb{S}\left(I_{\vartheta_\alpha}\right) = \rho_{\vartheta_\alpha} - \tau_{\vartheta_\alpha} - \omega_{\vartheta_\alpha} \ge \min_j \min_i \ \{\rho_{\vartheta_{ij}}\}$$
$$-\max_j \max_i \ \{\tau_{\vartheta_{ij}}\} - \max_j \max_i \ \{\omega_{\vartheta_{ij}}\} = \mathbb{S}\left(I^{-}_{\vartheta_{ij}}\right).$$

Now, by making use of the order relation between these two PFHSNs, we get $I^{-}_{\vartheta_{ij}} \le$ PFHSWA$(I_{\vartheta_{11}},\ I_{\vartheta_{12}},\ \ldots\ , I_{\vartheta_{nm}} \le I^{+}_{\vartheta_{ij}}$ which is the proof of the boundedness.

- **Shift Invariance** If $I_{\vartheta_\alpha} = (\rho_{\vartheta_\alpha},\ \tau_{\vartheta_\alpha},\ \omega_{\vartheta_\alpha})$ be a PFHSN, then,

$$PFHSWA(I_{\vartheta_{11}} \oplus I_{\vartheta_\alpha}, I_{\vartheta_{12}} \oplus I_{\vartheta_\alpha}, \ldots, I_{\vartheta_{nm}} \oplus I_{\vartheta_\alpha}) = PFHSWA(I_{\vartheta_{11}}, I_{\vartheta_{12}}, \ldots, I_{\vartheta_{nm}}) \ \oplus I_{\vartheta_\alpha}.$$

Proof. Let I_{ϑ_α} and $I_{\vartheta_{ij}}$ be two PFHSNs. Then, by the operational law of direct sum defined above in (a), we get $I_{\vartheta_\alpha} \ \oplus \ I_{\vartheta_{ij}} =$ $\left\langle \rho_{\vartheta_\alpha} + \rho_{\vartheta_{ij}} - \rho_{\vartheta_\alpha}\rho_{\vartheta_{ij}},\ \tau_{\vartheta_\alpha}\tau_{\vartheta_{ij}},\ \omega_{\vartheta_\alpha}\omega_{\vartheta_{ij}} \right\rangle$. Therefore,

PFHSWA$(I_{\vartheta_{11}} \oplus I_{\vartheta_\alpha}, I_{\vartheta_{12}} \oplus I_{\vartheta_\alpha}, \ldots, I_{\vartheta_{nm}} \oplus I_{\vartheta_\alpha}) = \oplus_{j=1}^{m} \mathfrak{F}_j \left(\oplus_{i=1}^{n} \mathfrak{I}_i (I_{\vartheta_{ij}} \ \oplus \ I_{\vartheta_\alpha})\right)$

$$= \left\langle 1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1-\rho_{\vartheta_{ij}}\right)^{\mathfrak{I}_i} \left(1-\rho_{\vartheta_\alpha}\right)^{\mathfrak{I}_i} \right)^{\mathfrak{F}_j}, \right.$$
$$\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\tau_{\vartheta_{ij}}\right)^{\mathfrak{I}_i} \left(\tau_{\vartheta_\alpha}\right)^{\mathfrak{I}_i} \right)^{\mathfrak{F}_j},$$
$$\left. \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\omega_{\vartheta_{ij}}\right)^{\mathfrak{I}_i} \left(\omega_{\vartheta_\alpha}\right)^{\mathfrak{I}_i} \right)^{\mathfrak{F}_j} \right\rangle$$

$$= \left\langle 1-(1-\rho_{\vartheta_\alpha})\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\rho_{\vartheta_{ij}})^{\mathfrak{J}_i}\right)^{\mathfrak{F}_j}, \right.$$
$$(\tau_{\vartheta_\alpha})\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\tau_{\vartheta_{ij}})^{\mathfrak{J}_i}\right)^{\mathfrak{F}_j},$$
$$\left.(\omega_{\vartheta_\alpha})\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\omega_{\vartheta_{ij}})^{\mathfrak{J}_i}\right)^{\mathfrak{F}_j}\right\rangle;$$

(as $\sum_{i=1}^{n}\mathfrak{J}_i = 1.$)

$$=\left\langle 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\rho_{\vartheta_{ij}})^{\mathfrak{J}_i}\right)^{\mathfrak{F}_j}, \right.$$
$$\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\tau_{\vartheta_{ij}})^{\mathfrak{J}_i}\right)^{\mathfrak{F}_j},$$
$$\left.\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\omega_{\vartheta_{ij}})^{\mathfrak{J}_i}\right)^{\mathfrak{F}_j}\right\rangle \oplus \langle\rho_{\vartheta_\alpha},\ \tau_{\vartheta_\alpha},\ \omega_{\vartheta_\alpha}\rangle$$

$= PFHSWA(I_{\vartheta_{11}},\ I_{\vartheta_{12}},\ \ldots\ ,I_{\vartheta_{nm}}) \oplus I_{\vartheta_\alpha}$; which completes the proof of operator being shift invariant.

- **Homogeneity** For any positive real number κ,

$$PFHSWA(\kappa I_{\vartheta_{11}},\ \kappa I_{\vartheta_{12}},\ \ldots\ ,\kappa I_{\vartheta_{nm}}) = \kappa PFHSWA(I_{\vartheta_{11}},\ I_{\vartheta_{12}},\ \ldots\ ,I_{\vartheta_{nm}}).$$

Proof. Let $I_{\vartheta_{ij}}$ be a PFHSN and $\kappa > 0$ be a real number, then by the operational law of scalar multiplication defined above in (c), we get

$$\kappa I_{v_{ij}} = \left\langle\left[1-\left(1-\rho_{\vartheta_{ij}}\right)^{\kappa}, \quad \left(\tau_{\vartheta_{ij}}\right)^{\kappa}, \left(\omega_{\vartheta_{ij}}\right)^{\kappa}\right]\right\rangle.$$

Thus,

$PFHSWA(\kappa I_{\vartheta_{11}},\ \kappa I_{\vartheta_{12}},\ldots\ ,\kappa I_{\vartheta_{nm}}) =$
$$\left\langle 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\rho_{\vartheta_{ij}})^{\kappa\mathfrak{J}_i}\right)^{\mathfrak{F}_j},\ \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\tau_{\vartheta_{ij}})^{\kappa\mathfrak{J}_i}\right)^{\mathfrak{F}_j},\ \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\omega_{\vartheta_{ij}})^{\kappa\mathfrak{J}_i}\right)^{\mathfrak{F}_j}\right\rangle$$

$$=\left\langle 1-\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\rho_{\vartheta_{ij}})^{\mathfrak{J}_i}\right)^{\mathfrak{F}_j}\right)^{\kappa}, \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\tau_{\vartheta_{ij}})^{\mathfrak{J}_i}\right)^{\mathfrak{F}_j}\right)^{\kappa}, \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\omega_{\vartheta_{ij}})^{\mathfrak{J}_i}\right)^{\mathfrak{F}_j}\right)^{\kappa}\right\rangle$$

$= \kappa$ PFHSWA$(I_{\vartheta_{11}}, I_{\vartheta_{12}}, \ldots, I_{\vartheta_{nm}})$.

- **Monotonicity** Let $I_{\vartheta_{ij}}$ and $I'_{\vartheta_{ij}}$ be the collection of two PFHSNs. If $I_{\vartheta_{ij}} \leq I'_{\vartheta_{ij}}$ then,

$$PFHSWA(I_{\vartheta_{11}},\ I_{\vartheta_{12}},\ \ldots\ ,I_{\vartheta_{nm}}) \leq PFHSWA\left(I_{\vartheta'_{11}},\ I_{\vartheta'_{12}},\ \ldots\ I_{\vartheta'_{nm}}\right).$$

Proof. The proof follows by making use of the operational laws stated above.

Further, we introduce another type of average aggregation operator called the ordered weighted averaging operator for Picture Fuzzy Hypersoft Numbers as follows:

Definition 29 *Let $I_{\vartheta_d} = (\rho_{\vartheta_d},\ \tau_{\vartheta_d},\ \omega_{\vartheta_d})$ be a PFHSN, $\mathfrak{J}_i$ and $\mathfrak{F}_j$ represent weight vectors for expert's and sub-attributes for the selected parameters respectively with the constraint $\mathfrak{J}_i > 0,\ \sum_{i=1}^{n} \mathfrak{J}_i = 1$ and $\mathfrak{F}_j > 0,\ \sum_{i=1}^{n} \mathfrak{F}_j = 1$. Then, **Picture Fuzzy Hypersoft Ordered Weighted Average (PFHSOWA) Aggregation Operator** is a mapping $\mathfrak{N}^n \rightarrow \mathfrak{N}$ given by*

$$PFHSOWA(I_{\vartheta_{11}},\ I_{\vartheta_{12}},\ \ldots\ ,I_{\vartheta_{nm}}) = \oplus_{j=1}^{m}\mathfrak{F}_j\left(\oplus_{i=1}^{n}\mathfrak{J}_i I_{\vartheta_{\sigma(ij)}}\right). \qquad (4.2.6)$$

where $\mathfrak{N}^n = (I_{\vartheta_{11}},\ I_{\vartheta_{12}},\ \ldots\ ,I_{\vartheta_{nm}})$ is a collection of Picture Fuzzy Hypersoft Numbers and $\sigma(11), \sigma(12), \sigma(21), \ldots\ldots \sigma(nm)$ is a possible permutation of i and j with $i = 1,\ 2,\ \ldots .n$ and $j = 1, 2, \ldots .m$.

4.2.2 Geometric Aggregation Operators

In this subsection, we subsequently study and define the new geometric aggregation operators for the proposed picture fuzzy hypersoft numbers as follows:

Definition 30 *Let $I_{\vartheta_d} = (\rho_{\vartheta_d},\ \tau_{\vartheta_d},\ \omega_{\vartheta_d})$ be a PFHSN, $\mathfrak{J}_i$ and $\mathfrak{F}_j$ represent weight vectors for expert's and sub-attributes for selected parameters respectively with the constraint $\mathfrak{J}_i > 0,\ \sum_{i=1}^{n} \mathfrak{J}_i = 1$ and $\mathfrak{F}_j > 0,\ \sum_{i=1}^{n} \mathfrak{F}_j = 1$. Then the **Picture Fuzzy Hypersoft Weighted Geometric (PFHSWG) Aggregation Operator** is a mapping $\mathfrak{N}^n \rightarrow \mathfrak{N}$ given by*

$$PFHSWG(I_{\vartheta_{11}},\ I_{\vartheta_{12}},\ \ldots\ ,I_{\vartheta_{nm}}) = \otimes_{j=1}^{m}\left(\otimes_{i=1}^{n} I_{\vartheta_{ij}}{}^{\mathfrak{J}_i}\right)^{\mathfrak{F}_j}; \qquad (4.2.7)$$

where $\mathfrak{N}^n = (I_{\vartheta_{11}},\ I_{\vartheta_{12}},\ \ldots\ ,I_{\vartheta_{nm}})$ is a collection of PFHSNs.

Theorem 9 *Let $I_{\vartheta_d} = (\rho_{\vartheta_d},\ \tau_{\vartheta_d},\ \omega_{\vartheta_d})$ be a PFHSN. Then from equation (4.2.7), the weighted geometric aggregation(fusion) of all the input values also gives a PFHSN given by*

$$PFHSWG(I_{\vartheta_{11}}, I_{\vartheta_{12}}, \ldots, I_{\vartheta_{nm}}) = \left\langle \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\rho_{\vartheta_{ij}})^{\mathfrak{J}_i}\right)^{\mathfrak{F}_j}, 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\tau_{\vartheta_{ij}})^{\mathfrak{J}_i}\right)^{\mathfrak{F}_j}, 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\omega_{\vartheta_{ij}})^{\mathfrak{J}_i}\right)^{\mathfrak{F}_j}\right\rangle \quad (4.2.8)$$

where $\mathfrak{J}_i$ and $\mathfrak{F}_j$ represent weight vectors for expert's and sub-attributes for selected parameters respectively with the constraint $\mathfrak{J}_i > 0,\ \sum_{i=1}^{n}\mathfrak{J}_i = 1$ and $\mathfrak{F}_j > 0,\ \sum_{j=1}^{n}\mathfrak{F}_j = 1$.

Proof 2 *The proof of the theorem follows from the principle of mathematical induction.*

For $n = 1$, we get $\mathfrak{J}_1 = 1$. (because $\sum_{i=1}^{n}\mathfrak{J}_i = 1$.)

Then, from (4.2.7), we have
$PFHSWG(I_{\vartheta_{11}}, I_{\vartheta_{12}}, \ldots, I_{\vartheta_{nm}}) = \otimes_{j=1}^{m} {I_{\vartheta_{1j}}}^{\mathfrak{F}_j}$.

Now, using the above-stated functional laws (a)-(e), we get

$$PFHSWG(I_{\vartheta_{11}}, I_{\vartheta_{12}}, \ldots, I_{\vartheta_{nm}}) = \left\langle \prod_{j=1}^{m}(\rho_{\vartheta_{1j}})^{\mathfrak{F}_j},\ 1-\prod_{j=1}^{m}(1-\tau_{\vartheta_{1j}})^{\mathfrak{F}_j},\ 1-\prod_{j=1}^{m}(1-\omega_{\vartheta_{1j}})^{\mathfrak{F}_j} \right\rangle$$

$$= \left\langle \prod_{j=1}^{m}\left(\prod_{i=1}^{1}(\rho_{\vartheta_{ij}})^{\mathfrak{J}_i}\right)^{\mathfrak{F}_j},\ 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{1}(1-\tau_{\vartheta_{ij}})^{\mathfrak{J}_i}\right)^{\mathfrak{F}_j},\ 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{1}(1-\omega_{\vartheta_{ij}})^{\mathfrak{J}_i}\right)^{\mathfrak{F}_j} \right\rangle.$$

Also, for $m = 1$, we get $\mathfrak{F}_1 = 1$. (because $\sum_{j=1}^{m}\mathfrak{F}_j = 1$.)

Then, from (4.2.7), we have
$PFHSWG(I_{\vartheta_{11}}, I_{\vartheta_{12}}, \ldots, I_{\vartheta_{nm}}) = \otimes_{i=1}^{n} {I_{\vartheta_{i1}}}^{\mathfrak{I}_i}$.

Again, using the above-stated functional laws (a)-(e), we get

$$PFHSWG(I_{\vartheta_{11}}, I_{\vartheta_{12}}, \ldots, I_{\vartheta_{nm}}) = \left\langle \prod_{i=1}^{n} (\rho_{\vartheta_{i1}})^{\mathfrak{I}_i},\ 1 - \prod_{i=1}^{n} (1-\tau_{\vartheta_{i1}})^{\mathfrak{I}_i}, 1 - \prod_{i=1}^{n} (1-\omega_{\vartheta_{i1}})^{\mathfrak{I}_i} \right\rangle$$

$$= \left\langle \prod_{j=1}^{1} \left(\prod_{i=1}^{n} (\rho_{\vartheta_{ij}})^{\mathfrak{I}_i} \right)^{\mathfrak{F}_j},\ 1 - \prod_{j=1}^{1} \left(\prod_{i=1}^{n} (1-\tau_{\vartheta_{ij}})^{\mathfrak{I}_i} \right)^{\mathfrak{F}_j}, 1 - \prod_{j=1}^{1} \left(\prod_{i=1}^{n} (1-\omega_{\vartheta_{ij}})^{\mathfrak{I}_i} \right)^{\mathfrak{F}_j} \right\rangle.$$

This shows that equation (4.2.7) satisfies for $n = 1$ *and* $m = 1$.
Assume that equation (4.2.7) holds for $m = \alpha_1 + 1,\ n = \alpha_2$ and $m = \alpha_1, n = \alpha_2 + 1$; *i.e.,*

$$\otimes_{j=1}^{\alpha_1+1} \left(\otimes_{i=1}^{\alpha_2} {I_{\vartheta_{ij}}}^{\mathfrak{I}_i} \right)^{\mathfrak{F}_j} = \left\langle \prod_{j=1}^{\alpha_1+1} \left(\prod_{i=1}^{\alpha_2} (\rho_{\vartheta_{ij}})^{\mathfrak{I}_i} \right)^{\mathfrak{F}_j}, 1 - \prod_{j=1}^{\alpha_1+1} \left(\prod_{i=1}^{\alpha_2} (1-\tau_{\vartheta_{ij}})^{\mathfrak{I}_i} \right)^{\mathfrak{F}_j}, 1 - \prod_{j=1}^{\alpha_1+1} \left(\prod_{i=1}^{\alpha_2} (1-\omega_{\vartheta_{ij}})^{\mathfrak{I}_i} \right)^{\mathfrak{F}_j} \right\rangle;$$

$$\otimes_{j=1}^{\alpha_1} \left(\otimes_{i=1}^{\alpha_2+1} {I_{\vartheta_{ij}}}^{\mathfrak{I}_i} \right)^{\mathfrak{F}_j} = \left\langle \prod_{j=1}^{\alpha_1} \left(\prod_{i=1}^{\alpha_2+1} (\rho_{\vartheta_{ij}})^{\mathfrak{I}_i} \right)^{\mathfrak{F}_j}, 1 - \prod_{j=1}^{\alpha_1} \left(\prod_{i=1}^{\alpha_2+1} (1-\tau_{\vartheta_{ij}})^{\mathfrak{I}_i} \right)^{\mathfrak{F}_j}, 1 - \prod_{j=1}^{\alpha_1} \left(\prod_{i=1}^{\alpha_2+1} (1-\omega_{\vartheta_{ij}})^{\mathfrak{I}_i} \right)^{\mathfrak{F}_j} \right\rangle.$$

Now for $m = \alpha_1 + 1, n = \alpha_2 + 1,$ we get

$$\otimes_{j=1}^{\alpha_1+1}\left(\otimes_{i=1}^{\alpha_2+1} I_{\vartheta_{ij}}{}^{\mathfrak{I}_i}\right)^{\mathfrak{F}_j} = \otimes_{j=1}^{\alpha_1+1}\left(\otimes_{i=1}^{\alpha_2} I_{\vartheta_{ij}}{}^{\mathfrak{I}_i} \otimes I_{\vartheta_{(\alpha_2+1)j}}{}^{\mathfrak{I}_{\alpha_2+1}}\right)^{\mathfrak{F}_j}$$
$$= \otimes_{j=1}^{\alpha_1+1}\left(\otimes_{i=1}^{\alpha_2} I_{\vartheta_{ij}}{}^{\mathfrak{I}_i}\right)^{\mathfrak{F}_j} \otimes_{j=1}^{\alpha_1+1}\left(I_{\vartheta_{(\alpha_2+1)j}}{}^{\mathfrak{I}_{\alpha_2+1}}\right)^{\mathfrak{F}_j}$$

$$= \Bigg\langle \prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\acute{\alpha}_2}\left(\rho_{\vartheta_{ij}}\right)^{\mathfrak{I}_i}\right)^{\mathfrak{F}_j} \otimes$$
$$\prod_{j=1}^{\alpha_1+1}\left(\left(\rho_{\vartheta_{(\alpha_2+1)j}}\right)^{\mathfrak{I}_{(\alpha_2+1)}}\right)^{\mathfrak{F}_j},$$
$$1 - \prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2}\left(1 - \tau_{\vartheta_{ij}}\right)^{\mathfrak{I}_i}\right)^{\mathfrak{F}_j} \otimes$$
$$1 - \prod_{j=1}^{\alpha_1+1}\left(\left(1 - \tau_{\vartheta_{(\alpha_2+1)j}}\right)^{\mathfrak{I}_{(\alpha_2+1)}}\right)^{\mathfrak{F}_j},$$
$$1 - \prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2}\left(1 - \omega_{\vartheta_{ij}}\right)^{\mathfrak{I}_i}\right)^{\mathfrak{F}_j} \otimes$$
$$1 - \prod_{j=1}^{\alpha_1+1}\left(\left(1 - \omega_{\vartheta_{(\alpha_2+1)j}}\right)^{\mathfrak{I}_{(\alpha_2+1)}}\right)^{\mathfrak{F}_j} \Bigg\rangle$$

$$= \Bigg\langle \prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2+1}\left(\rho_{\vartheta_{ij}}\right)^{\mathfrak{I}_i}\right)^{\mathfrak{F}_j},$$
$$1 - \prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2+1}\left(1 - \tau_{\vartheta_{ij}}\right)^{\mathfrak{I}_i}\right)^{\mathfrak{F}_j},$$
$$1 - \prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2+1}\left(1 - \omega_{\vartheta_{ij}}\right)^{\mathfrak{I}_i}\right)^{\mathfrak{F}_j} \Bigg\rangle.$$

Therefore, the result is true for m= $\alpha_1 + 1,\ n = \alpha_2 + 1$ and the theorem is proved.

Remark: Further, the properties like Idempotency, Boundedness, Homogeneity, Shift Invariance and Monotonicity can be defined for geometric aggregation operators analogous to averaging aggregation operators.

4.2.3 Algorithm for Proposed Articulation and Devised Aggregation Operators under the surroundings of PFHSS.

For the sake of solving the proposed articulation outlined above, we present a new algorithm and list out the necessary steps with the help of the following Figure 4.1 as follows:

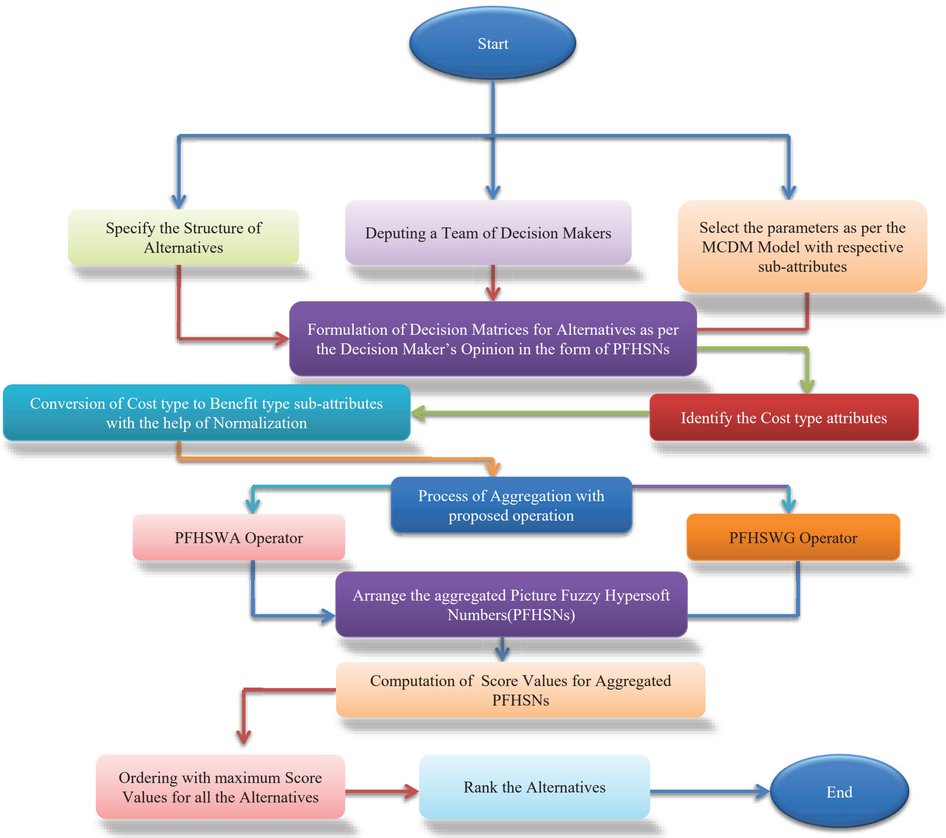

Figure 4.1: Flowchart of the Proposed Algorithm

The detailing of the outlined steps of the proposed methodology is being presented below:

Step 1: Assemble the data related to each alternative in the form of Picture Fuzzy Hypersoft Number in accordance with several conditions of multi-parameterizations and rearrange them to construct a Picture Fuzzy Hypersoft decision matrix for the available experts provided with respect to each alternative $\{\mathcal{Y}^{(q)} : q = 1, 2, \ldots, n\}$ as follows:

$$(\mathcal{Y}^{(q)}, \Omega^{'})_{w\times p} = \begin{array}{c} \\ \mathcal{Z}^1 \\ \mathcal{Z}^2 \\ \vdots \\ \mathcal{Z}^w \end{array} \begin{array}{c} \begin{array}{cccc} \vartheta_1^{'} & \vartheta_2^{'} & \cdots & \vartheta_p^{'} \end{array} \\ \begin{pmatrix} (\rho^{(q)}_{\vartheta_{11}^{'}}, \tau^{(q)}_{\vartheta_{11}^{'}}, \omega^{(q)}_{\vartheta_{11}^{'}}) & (\rho^{(q)}_{\vartheta_{12}^{'}}, \tau^{(q)}_{\vartheta_{12}^{'}}, \omega^{(q)}_{\vartheta_{12}^{'}}) & \cdots & (\rho^{(q)}_{\vartheta_{1p}^{'}}, \tau^{(q)}_{\vartheta_{1p}^{'}}, \omega^{(q)}_{\vartheta_{1p}^{'}}) \\ (\rho^{(q)}_{\vartheta_{21}^{'}}, \tau^{(q)}_{\vartheta_{21}^{'}}, \omega^{(q)}_{\vartheta_{21}^{'}}) & (\rho^{(q)}_{\vartheta_{22}^{'}}, \tau^{(q)}_{\vartheta_{22}^{'}}, \omega^{(q)}_{\vartheta_{22}^{'}}) & \cdots & (\rho^{(q)}_{\vartheta_{2p}^{'}}, \tau^{(q)}_{\vartheta_{2p}^{'}}, \omega^{(q)}_{\vartheta_{2p}^{'}}) \\ \vdots & \vdots & \ddots & \vdots \\ (\rho^{(q)}_{\vartheta_{n1}^{'}}, \tau^{(q)}_{\vartheta_{n1}^{'}}, \omega^{(q)}_{\vartheta_{n1}^{'}}) & (\rho^{(q)}_{\vartheta_{n2}^{'}}, \tau^{(q)}_{\vartheta_{n2}^{'}}, \omega^{(q)}_{\vartheta_{n2}^{'}}) & \cdots & (\rho^{(q)}_{\vartheta_{np}^{'}}, \tau^{(q)}_{\vartheta_{np}^{'}}, \omega^{(q)}_{\vartheta_{np}^{'}}) \end{pmatrix} \end{array}$$

Step 2: In the case of inconsistent sub-attributes, transformation of **cost** and **benefit** type sub-attributes is required. This can be done with help of the normalization rule and the resulting normalized decision matrix is as below:

$$\varsigma_{ij} = \begin{cases} I_{\vartheta_{ij}}{'}^{c} = \left(\omega_{\vartheta_{ij}}{'}^{(q)}, \ \tau_{\vartheta_{ij}}{'}^{(q)}, \ \rho_{\vartheta_{ij}}{'}^{(q)}\right); & \text{cost type parameter} \\ I_{\vartheta_{ij}}{'} = \left(\rho_{\vartheta_{ij}}{'}^{(q)}, \ \tau_{\vartheta_{ij}}{'}^{(q)}, \ \omega_{\vartheta_{ij}}{'}^{(q)}\right); & \text{benefit type parameter} \end{cases}$$

If the data is consistent then move to Step 3.

Step 3: Now with the use of the devised aggregation operators, we get a collective decision matrix $I_{\vartheta_{ij}}$ for each alternative $\mathcal{Y} = \{\mathcal{Y}^1, \ \mathcal{Y}^2, \ \dots \ , \mathcal{Y}^n\}$.

Step 4: For the collection of alternatives $\mathcal{Y} = \{\mathcal{Y}^1, \ \mathcal{Y}^2, \dots \ , \mathcal{Y}^n\}$, we compute the scoring values with the help of the formulae of scoring function.

Step 5: Choose the alternative with maximum score value & then rank the alternatives.

4.2.4 Numerical Illustration and Computation

In order to solve a MCDM problem based on the proposed methodology, a numerical problem of selecting the most suitable employee for a multi-national company from the set of employees by taking into account the choice of parameterizations is based on the following formulation

Let $\mathcal{Y} = \{\mathcal{Y}^1, \ \mathcal{Y}^2, \ \mathcal{Y}^3, \ \mathcal{Y}^4\}$ be a set of employees and Ω be the set of attributes given in the form of a hypersoft set as

$\Omega = \{\vartheta_1 = \text{Age}, \vartheta_2 = \text{Foreign Language knowledge}, \vartheta_3 = \text{Academic qualification}, \vartheta_4 = \text{Work experience}\}$and their further sub-parameters given by

- Age $= \vartheta_1 = \{\vartheta_{11} = 21 - 35, \vartheta_{12} = 35 - 52\}$,

- Foreign Language Knowledge $= \vartheta_2 = \{\vartheta_{21} = \text{English}, \ \vartheta_{22} = \text{French}\}$,
- Academic Qualification $= \vartheta_3 = \{\vartheta_{31} = \text{under-graduation}, \ \vartheta_{32} = \text{post-graduation}\}$,
- Work Experience $= \vartheta_4 = \{\vartheta_{41} = \text{atleast one year}\}$.

Let $\Omega' = \vartheta_1 \times \vartheta_2 \times \vartheta_3 \times \vartheta_4$ be a collection of sub-attributes, which is explicitly given by

$$= \left\{ \begin{pmatrix} (\vartheta_{11}, \vartheta_{21}, \vartheta_{31}, \vartheta_{41}), & (\vartheta_{11}, \vartheta_{21}, \vartheta_{32}, \vartheta_{41}), (\vartheta_{11}, \vartheta_{22}, \vartheta_{31}, \vartheta_{41}), (\vartheta_{11}, \vartheta_{22}, \vartheta_{32}, \vartheta_{41}) \\ (\vartheta_{12}, \vartheta_{21}, \vartheta_{31}, \vartheta_{41}), & (\vartheta_{12}, \vartheta_{21}, \vartheta_{32}, \vartheta_{41}), (\vartheta_{12}, \vartheta_{22}, \vartheta_{31}, \vartheta_{41}), (\vartheta_{12}, \vartheta_{22}, \vartheta_{32}, \vartheta_{41}) \end{pmatrix} \right\}$$

For the sake of simplicity collection of all sub-attributes can be restated as

$$\Omega' = \left\{ \vartheta'_1, \ \vartheta'_2, \ \vartheta'_3, \ \ \vartheta'_4, \ \vartheta'_5, \ \vartheta'_6, \ \vartheta'_7, \ \vartheta'_8 \right\}$$

and their respective weights are $(0.12, \ 0.18, \ 0.1, \ 0.15, \ 0.22, \ 0.08, \ 0.1)^T$.

Consider $\mathcal{Z} = \{\mathcal{Z}^1, \ \mathcal{Z}^2, \ \mathcal{Z}^3, \ \mathcal{Z}^4\}$ be a collection of experts with weight's $(0.2, \ 0.3, \ 0.4, \ 0.1)^T$ to examine the suitable alternative. The preferences are supposed to be given by experts in terms of PFHSNs by using multi sub-attributes. In order to obtain the most suitable choice, we go through the following process.

Productive Employees Selection Using Picture Fuzzy Hypersoft Weighted Average Operator

Step 1: The situations are examined by the experts in terms of PFHSNs. The multi-subattributes of the selected attributes, along with computation of score values are given in the following tables:

Step 2: Since al attributes are identical, so there is no need for normalization.
Step 3: By using equation(4.2.2), the opinion of expert's can be summarized as

$$\mathcal{J}_1 = \langle 0.309967, 0.231837, \ \ 0.200275 \rangle, \ \mathcal{J}_2 = \langle 0.269288, 0.275446, 0.196839 \rangle,$$

$$\mathcal{J}_3 = \langle 0.288827, 0.238588, 0.212493 \rangle, \ \mathcal{J}_4 = \langle 0.198194, 0.304841, 0.213194 \rangle.$$

Step 4: Now compute the scoring values by using the formulae of scoring functions.

$\mathbb{S}(\mathcal{J}_1) = -0.122145, \ \mathbb{S}(\mathcal{J}_2) = -0.202997, \ \mathbb{S}(\mathcal{J}_3) = -0.162254, \ \mathbb{S}(\mathcal{J}_4) = -0.319841.$

Table 4.3: Decision Matrix given by Experts for Alternative $\mathcal{Y}^1$

	$\mathcal{Z}^1$	$\mathcal{Z}^2$	$\mathcal{Z}^3$	$\mathcal{Z}^4$
ϑ_1'	(0.2, 0.5, 0.1)	(0.4, 0.3, 0.2)	(0.1, 0.2, 0.5)	(0.3, 0.5, 0.1)
ϑ_2'	(0.3, 0.4, 0.2)	(0.2, 0.4, 0.1)	(0.4, 0.1, 0.2)	(0.2, 0.4, 0.1)
ϑ_3'	(0.4, 0.1, 0.2)	(0.1, 0.2, 0.3)	(0.3, 0.2, 0.1)	(0.1, 0.2, 0.4)
ϑ_4'	(0.3, 0.5, 0.1)	(0.3, 0.2, 0.1)	(0.2, 0.4, 0.3)	(0.3, 0.4, 0.2)
ϑ_5'	(0.4, 0.1, 0.2)	(0.4, 0.2, 0.3)	(0.2, 0.1, 0.5)	(0.1, 0.3, 0.5)
ϑ_6'	(0.3, 0.2, 0.1)	(0.1, 0.3, 0.5)	(0.7, 0.1, 0.1)	(0.2, 0.4, 0.1)
ϑ_7'	(0.3, 0.2, 0.4)	(0.1, 0.2, 0.4)	(0.4, 0.2, 0.3)	(0.4, 0.1, 0.2)
ϑ_8'	(0.1, 0.2, 0.3)	(0.2, 0.3, 0.4)	(0.1, 0.5, 0.3)	(0.4, 0.3, 0.2)

Table 4.4: Decision Matrix given by Experts for Alternative $\mathcal{Y}^2$

	$\mathcal{Z}^1$	$\mathcal{Z}^2$	$\mathcal{Z}^3$	$\mathcal{Z}^4$
ϑ_1'	(0.4, 0.2, 0.3)	(0.1, 0.2, 0.5)	(0.4, 0.1, 0.2)	(0.4, 0.3, 0.2)
ϑ_2'	(0.1, 0.2, 0.6)	(0.2, 0.4, 0.1)	(0.1, 0.2, 0.4)	(0.2, 0.1, 0.5)
ϑ_3'	(0.2, 0.5, 0.1)	(0.3, 0.5, 0.1)	(0.3, 0.5, 0.1)	(0.1, 0.3, 0.5)
ϑ_4'	(0.1, 0.2, 0.3)	(0.2, 0.4, 0.3)	(0.4, 0.3, 0.2)	(0.1, 0.2, 0.4)
ϑ_5'	(0.3, 0.2, 0.1)	(0.1, 0.3, 0.5)	(0.1, 0.5, 0.3)	(0.3, 0.4 ,0.2)
ϑ_6'	(0.7, 0.1 0.1)	(0.2, 0.5, 0.1)	(0.3, 0.2, 0.1)	(0.2, 0.6, 0.1)
ϑ_7'	(0.2, 0.4, 0.1)	(0.1, 0.5, 0.3)	(0.3, 0.4, 0.2)	(0.1, 0.7, 0.1)
ϑ_8'	(0.2, 0.5, 0.2)	(0.4, 0.1, 0.2)	(0.2, 0.5, 0.1)	(0.3, 0.2, 0.1)

Table 4.5: Decision Matrix given by Experts for Alternative $\mathcal{Y}^3$

	$\mathcal{Z}^1$	$\mathcal{Z}^2$	$\mathcal{Z}^3$	$\mathcal{Z}^4$
ϑ_1'	(0.1, 0.3, 0.4)	(0.3, 0.5, 0.1)	(0.4, 0.1, 0.2)	(0.3, 0.5, 0.1)
ϑ_2'	(0.4, 0.3, 0.2)	(0.4, 0.3, 0.2)	(0.1, 0.2, 0.3)	(0.2, 0.4, 0.1)
ϑ_3'	(0.2, 0.6, 0.1)	(0.1, 0.5, 0.3)	(0.2, 0.1, 0.5)	(0.1, 0.2, 0.4)
ϑ_4'	(0.3, 0.2, 0.1)	(0.2, 0.4, 0.1)	(0.7, 0.1, 0.1)	(0.3, 0.4, 0.2)
ϑ_5'	(0.2, 0.5, 0.1)	(0.4, 0.2, 0.3)	(0.4, 0.1, 0.2)	(0.1, 0.3, 0.5)
ϑ_6'	(0.4, 0.1, 0.2)	(0.4, 0.3, 0.2)	(0.1, 0.3, 0.1)	(0.2, 0.4, 0.1)
ϑ_7'	(0.1, 0.2, 0.4)	(0.1, 0.2, 0.3)	(0.1, 0.2, 0.3)	(0.4, 0.1, 0.2)
ϑ_8'	(0.2, 0.4, 0.1)	(0.2, 0.3 0.5)	(0.1, 0.3, 0.6)	(0.4 0.3, 0.2)

Table 4.6: Decision Matrix given by Experts for Alternative $\mathcal{Y}^4$

	$\mathcal{Z}^1$	$\mathcal{Z}^2$	$\mathcal{Z}^3$	$\mathcal{Z}^4$
ϑ_1'	(0.3, 0.5, 0.1)	(0.2, 0.3, 0.4)	(0.2, 0.1, 0.5)	(0.1, 0.5, 0.2)
ϑ_2'	(0.2, 0.3, 0.4)	(0.1, 0.2, 0.3)	(0.2, 0.5, 0.2)	(0.2, 0.3, 0.1)
ϑ_3'	(0.1, 0.3, 0.2)	(0.3, 0.4, 0.2)	(0.1, 0.5, 0.2)	(0.2, 0.5, 0.1)
ϑ_4'	(0.3, 0.2, 0.1)	(0.1, 0.2, 0.6)	(0.3, 0.2, 0.1)	(0.1, 0.2, 0.3)
ϑ_5'	(0.2, 0.4, 0.3)	(0.1, 0.2, 0.2)	(0.1, 0.4, 0.2)	(0.3, 0.4, 0.2)
ϑ_6'	(0.2, 0.6, 0.1)	(0.2, 0.4, 0.3)	(0.2, 0.5, 0.1)	(0.1, 0.2, 0.3)
ϑ_7'	(0.5, 0.3, 0.2)	(0.2, 0.1, 0.4)	(0.1, 0.3, 0.5)	(0.3, 0.2, 0.1)
ϑ_8'	(0.1, 0.3, 0.4)	(0.3, 0.2, 0.1)	(0.2, 0.4, 0.3)	(0.2, 0.4, 0.1)

Step 5: Finally, on the basis of the obtained values of the score function, we observe that

$$\mathbb{S}(\mathcal{J}_1) \ > \ S(\mathcal{J}_3) \ > \ S(\mathcal{J}_2) \ > \ S(\mathcal{J}_4).$$

So, $\mathcal{Y}^1 \ > \ \mathcal{Y}^3 \ > \ \mathcal{Y}^2 \ > \ \mathcal{Y}^4$. Hence, the alternative $\mathcal{Y}^1$ is the most appropriate one.

Step 6: Also, by using equation (4.2.8), opinion of expert's can be summarized as

$$\mathcal{J}_1 = \langle 0.239208, 0.278116, \ \ 0.254269\rangle, \ \mathcal{J}_2 = \langle 0.216339, 0.335441, 0.259252\rangle,$$

$$\mathcal{J}_3 = \langle 0.217776, 0.284027, 0.27034\rangle, \ \mathcal{J}_4 = \langle 0.178452, 0.353784, 0.287766\rangle.$$

Step 7: Now compute the scoring values by using the formulae of scoring functions.

$$\mathbb{S}(\mathcal{J}_1) = -0.293177, \ \mathbb{S}(\mathcal{J}_2) = -0.378354, \ \mathbb{S}(\mathcal{J}_3) = -0.336591, \mathbb{S}(\mathcal{J}_4) = -0.463098$$

Step 8: Finally, on the basis of the obtained values of the score function, we observe that

$$\mathbb{S}(\mathcal{J}_1) \ > \ S(\mathcal{J}_3) \ > \ S(\mathcal{J}_2) \ > \ S(\mathcal{J}_4).$$

So, $\mathcal{Y}^1 \ > \ \mathcal{Y}^3 \ > \ \mathcal{Y}^2 \ > \ \mathcal{Y}^4$. Hence, the alternative $\mathcal{Y}^1$ is the most appropriate one.Therefore, by both the aggregation operators the alternative $\mathcal{Y}^1$ is the optimal one.

4.3 Comparative Analysis, Advantages and Discussions

In this section, we discuss the functionality, receptiveness, and conformity of the proposed notion and methodology in contrast with the existing techniques. In addition to this, some advantages and discussions over the obtained results have also been presented for better understanding and readability. In view of the numerical example under consideration and the results obtained through the existing techniques utilizing the intuitionistic fuzzy soft/hypersoft aggregation operators, we present the following Table 4.3, stating the ranking o the alternatives for the decision-making problem:

Table 4.7: Results of Comparative Analysis with Some Existing Aggregation Operators

Method	$\mathcal{Y}^1$	$\mathcal{Y}^2$	$\mathcal{Y}^3$	$\mathcal{Y}^4$	Ranking Order
IFSWA [111]	0.08158	0.07674	0.14762	0.09959	$\mathcal{Y}^3 > \mathcal{Y}^4 > \mathcal{Y}^1 > \mathcal{Y}^2$
IFSWG [111]	0.49830	0.41735	0.40935	0.46175	$\mathcal{Y}^1 > \mathcal{Y}^4 > \mathcal{Y}^2 > \mathcal{Y}^3$
IFHSWA [112]	-0.195086	-0.124363	0.084652	0.095501	$\mathcal{Y}^4 > \mathcal{Y}^3 > \mathcal{Y}^2 > \mathcal{Y}^1$
IFHSWG [112]	-0.259867	-0.242376	-0.141950	-0.035913	$\mathcal{Y}^4 > \mathcal{Y}^3 > \mathcal{Y}^2 > \mathcal{Y}^1$

Subsequently, on the basis of the obtained results by utilizing th proposed methodology involving the introduced PFHSWA/PFHSWG aggregation operators, we present the following respective computed values:

$$\mathbb{S}(\mathcal{J}_1) = -0.122145,\ \mathbb{S}(\mathcal{J}_2) = -0.202997,\ \mathbb{S}(\mathcal{J}_3) = -0.162254,\ \mathbb{S}(\mathcal{J}_4) = -0.319841.$$

and

$$\mathbb{S}(\mathcal{J}_1) = -0.293177,\ \mathbb{S}(\mathcal{J}_2) = -0.378354,\ \mathbb{S}(\mathcal{J}_3) = -0.336591,\ \mathbb{S}(\mathcal{J}_4) = -0.463098$$

On the basis of the computed score values, we finally conclude the following ranking of the alternatives (employee) which is certainly different due to the extra flexibility of sub-attributes:

$$\mathcal{Y}^1 > \mathcal{Y}^3 > \mathcal{Y}^2 > \mathcal{Y}^4.$$

Important Remarks and Advantages:

- Finally, we are able to state that the proposed notion o picture fuzzy hypersoft set (PFHSS) is a novel concept and a vali extension of fuzzy set/hypersoft set theories. The PFHSS has an added advantage to deal with the wider sense of applicability in uncertain situations with the incorporation of degree of refusal and abstain.

- The existing types of hypersoft sets - intuitionistic fuzzy hypersoft set [14], Pythagorean fuzzy hypersoft set [113], Neutrosophic hypersoft set [14] have their own limitations because of the exclusion of refusal and abstain component.
- It may be noted that the categorically designed information having the picture fuzzy relation would not be possible to address with the help of existing hypersoft set theory in order to ensure a kind of parametrization in the relation.
- The methodology implementing the proposed PFHSWA/PFHSWG aggregation operators can be well utilized for various group strategic MCDM models in a generalized framework effectively and consistently.
- As an overall critical aspect, we observe that eventually with the picture fuzzy information, it won't be possible to suitably address those membership values (given by the decision-makers/experts) whose sum exceeds one. Such restrictions in respect of decision-maker's opinion can be eradicated with the notion of T-spherical fuzzy information.

4.4 Similarity Measures of Picture Fuzzy Hypersoft Sets

Next, we define the similarity and weighted similarity measures between PFHSSs and some of its fundamental operations.

Definition 31 *Let V be the universal set and $PFHSS(V)$ be the set of all picture fuzzy hypersoft sets over V. Consider a mapping $S : PFHSS(V) \times PFHSS(V) \to [0,1]$, for any $\langle (R,\ \Lambda_1),\ (R',\ \Lambda_2) \rangle \in PFHSS(V)$, $S\langle (R,\ \Lambda_1),\ (R',\ \Lambda_2) \rangle$ is called a similarity measure between the picture fuzzy hypersoft sets $(R,\ \Lambda_1)$ and $(R',\ \Lambda_2)$ if it satisfies the following conditions:*

(i) $0 \leq S\langle (R,\ \Lambda_1),\ (R',\ \Lambda_2) \rangle \leq 1;$

(ii) $S\langle(R,\ \Lambda_1),\ (R',\ \Lambda_2)\rangle\ =\ 1 \Leftrightarrow R\ =\ R'$

(iii) $S\langle(R, \Lambda_1), (R', \Lambda_2)\rangle = S\langle(R', \Lambda_2), (R, \Lambda_1)\rangle$;

(iv) *Let* $(R'',\ \Lambda_3)$ *be a picture fuzzy hypersoft set, if* $(R,\ \Lambda_1)\ \subseteq\ (R',\ \Lambda_2)$ *and* $(R',\ \Lambda_2)\ \subseteq\ (R'',\ \Lambda_3)$, *then* $S\langle(R,\ \Lambda_1),\ (R'',\ \Lambda_3)\rangle\ \leq\ S\langle(R,\ \Lambda_1), (R',\ \Lambda_2)\rangle$ *and* $S\langle(R,\ \Lambda_1),\ (R'',\ \Lambda_3)\rangle\ \leq\ S\langle(R', \Lambda_2), (R'',\ \Lambda_3)\rangle$.

Definition 32 *Let* $(R,\ \Lambda)$ *and* $(R',\ \Lambda)$ *be any two PFHSSs over the universe of discourse* $V = \{v^1,\ v^2,\ \dots, v^n\}$ *with the attribute set values* $K_1^a \times K_2^b \times \dots \times K_m^z$. *Then, a similarity measure between* $(R,\ \Lambda)$ *and* $(R',\ \Lambda)$ *can be defined as:*

$$\mathbb{S}_{PFHSS}(R, R') = \tag{4.4.1}$$
$$\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m} \frac{1-\frac{1}{2}\left[\begin{array}{c}\Big(\min\{|\rho_{R(\vartheta_j^s)}(v^i) - \rho_{R'(\vartheta_j^s)}(v^i)|, |\tau_{R(\vartheta_j^s)}(v^i) - \tau_{R'(\vartheta_j^s)}(v^i)|\} \\ +|\omega_{R(\vartheta_j^s)}(v^i) - \omega_{R'(\vartheta_j^s)}(v^i)|\Big)\end{array}\right]}{1+\frac{1}{2}\left[\begin{array}{c}\Big(\max\{|\rho_{R(\vartheta_j^s)}(v^i) - \rho_{R'(\vartheta_j^s)}(v^i)|, |\tau_{R(\vartheta_j^s)}(v^i) - \tau_{R'(\vartheta_j^s)}(v^i)|\} \\ +|\omega_{R(\vartheta_j^s)}(v^i) - \omega_{R'(\vartheta_j^s)}(v^i)|\Big)\end{array}\right]}$$

where, $j\ =\ 1,\ 2,\ \dots m; i\ =\ 1,\ 2,\ \dots, n; s\ =\ a, b, \dots, z; a, b, \dots, z\ =\ 1, 2, \dots, n$ *and* $\vartheta_j^s\ \in\ K_1^a\ \times\ K_2^b\ \times\ \dots\ \times\ K_m^z$.

Definition 33 *Let* $(R,\ \Lambda)$ *and* $(R',\ \Lambda)$ *be any two PFHSSs over the universe of discourse* $V = \{v^1,\ v^2,\ \dots, v^n\}$ *with the attribute set values* $K_1^a \times K_2^b \times \dots \times K_m^z$. *Then, a tangent similarity measure between* $(R,\ \Lambda)$ *and* (R', Λ) *can be defined as:*

$$\mathbb{T}_{PFHSS}(R, R') = \tag{4.4.2}$$
$$\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\left\{1 - \tan\frac{\pi}{12}\left[\begin{array}{c}\Big(|\rho_{R(\vartheta_j^s)}(v^i) - \rho_{R'(\vartheta_j^s)}(v^i)| + |\tau_{R(\vartheta_j^s)}(v^i) - \tau_{R'(\vartheta_j^s)}(v^i)| \\ +|\omega_{R(\vartheta_j^s)}(v^i) - \omega_{R'(\vartheta_j^s)}(v^i)|\Big)\end{array}\right]\right\}$$

where, $j\ =\ 1,\ 2,\ \dots m; i\ =\ 1,\ 2,\ \dots, n; s\ =\ a, b, \dots, z; a, b, \dots, z\ =\ 1, 2, \dots, n$ *and* $\vartheta_j^s\ \in\ K_1^a\ \times\ K_2^b\ \times\ \dots\ \times\ K_m^z$.

Theorem 10 *Let* $(R,\ \Lambda)$, $(R',\ \Lambda)$ *and* $(R'',\ \Lambda)$ *be three* $PFHSSs$ *over the universal set* V. *Then* $\mathbb{S}_{PFHSS}$ *satisfies the four axioms of similarity measures as follows:*

(i) $0\ \leq\ S(R,\ R')\ \leq\ 1$;

(ii) $S(R,\ R') \ = \ S(R', R)$

(iii) $S(R, R') \ = \ 1 \Leftrightarrow R \ = \ R'$

(iv) If $(R,\ \Lambda) \ \subseteq \ (R',\ \Lambda)$ *and* $(R',\ \Lambda) \ \subseteq \ (R'',\ \Lambda)$*, then*
$\mathbb{S}_{PFHSS}(R,\ R'') \ \leq \ \mathbb{S}_{PFHSS}(R,\ R')$ *and* $\mathbb{S}_{PFHSS}(R,\ R'') \ \leq \ \mathbb{S}_{PFHSS}(R',\ R'')$.

Proof 3 *Proof of (i) and (ii) can be easily done by making use of the definition of the proposed measure.*
(iii) For the proof this part, let us suppose $R = R'$
Then, $\rho_{R(\vartheta_j^s)}(v^i) = \rho_{R'(\vartheta_j^s)}(v^i), \tau_{R(\vartheta_j^s)}(v^i) = \tau_{R'(\vartheta_j^s)}(v^i), \omega_{R(\vartheta_j^s)}(v^i) = \omega_{R'(\vartheta_j^s)}(v^i)$.
$\Rightarrow S(R, R') = 1$.
Conversely, let $S(R, R') = 1$.

$$\Rightarrow \frac{1 - \frac{1}{2}\left[\begin{array}{c}\Big(\min\{|\rho_{R(\vartheta_j^s)}(v^i) - \rho_{R'(\vartheta_j^s)}(v^i)|, |\tau_{R(\vartheta_j^s)}(v^i) - \tau_{R'(\vartheta_j^s)}(v^i)|\} \\ +|\omega_{R(\vartheta_j^s)}(v^i) - \omega_{R'(\vartheta_j^s)}(v^i)|\Big)\end{array}\right]}{1 + \frac{1}{2}\left[\begin{array}{c}\Big(\max\{|\rho_{R(\vartheta_j^s)}(v^i) - \rho_{R'(\vartheta_j^s)}(v^i)|, |\tau_{R(\vartheta_j^s)}(v^i) - \tau_{R'(\vartheta_j^s)}(v^i)|\} \\ +|\omega_{R(\vartheta_j^s)}(v^i) - \omega_{R'(\vartheta_j^s)}(v^i)|\Big)\end{array}\right]} = 1$$

$\Rightarrow 1 - \frac{1}{2}[\min\{|\rho_{R(\vartheta_j^s)}(v^i) - \rho_{R'(\vartheta_j^s)}(v^i)|, |\tau_{R(\vartheta_j^s)}(v^i) - \tau_{R'(\vartheta_j^s)}(v^i)|\} + |\omega_{R(\vartheta_j^s)}(v^i) - \omega_{R'(\vartheta_j^s)}(v^i)|] =$
$1 + \frac{1}{2}[\max\{|\rho_{R(\vartheta_j^s)}(v^i) - \rho_{R'(\vartheta_j^s)}(v^i)|, |\tau_{R(\vartheta_j^s)}(v^i) - \tau_{R'(\vartheta_j^s)}(v^i)|\} + |\omega_{R(\vartheta_j^s)}(v^i) - \omega_{R'(\vartheta_j^s)}(v^i)|]$

$\Rightarrow \frac{1}{2}[\min\{|\rho_{R(\vartheta_j^s)}(v^i) - \rho_{R'(\vartheta_j^s)}(v^i)|, |\tau_{R(\vartheta_j^s)}(v^i) - \tau_{R'(\vartheta_j^s)}(v^i)|\} + |\omega_{R(\vartheta_j^s)}(v^i) - \omega_{R'(\vartheta_j^s)}(v^i)|] +$
$\frac{1}{2}[\max\{|\rho_{R(\vartheta_j^s)}(v^i) - \rho_{R'(\vartheta_j^s)}(v^i)|, |\tau_{R(\vartheta_j^s)}(v^i) - \tau_{R'(\vartheta_j^s)}(v^i)|\} + |\omega_{R(\vartheta_j^s)}(v^i) - \omega_{R'(\vartheta_j^s)}(v^i)|] = 0$

$\Rightarrow |\rho_{R(\vartheta_j^s)}(v^i) - \rho_{R'(\vartheta_j^s)}(v^i)| = 0, |\tau_{R(\vartheta_j^s)}(v^i) - \tau_{R'(\vartheta_j^s)}(v^i)| = 0$ *and* $|\omega_{R(\vartheta_j^s)}(v^i) - \omega_{R'(\vartheta_j^s)}(v^i)| = 0$
$\Rightarrow R = R'$.

(iv) $\langle R, \Lambda\rangle \subseteq \langle R', \Lambda\rangle \subseteq \langle R'', \Lambda\rangle$.

$\Rightarrow |\rho_{R(\vartheta_j^s)}(v^i) - \rho_{R'(\vartheta_j^s)}(v^i)| \leq |\rho_{R(\vartheta_j^s)}(v^i) - \rho_{R''(\vartheta_j^s)}(v^i)|, |\tau_{R(\vartheta_j^s)}(v^i) - \tau_{R'(\vartheta_j^s)}(v^i)| \leq |\tau_{R(\vartheta_j^s)}(v^i) - \tau_{R''(\vartheta_j^s)}(v^i)|$ *and* $|\omega_{R(\vartheta_j^s)}(v^i) - \omega_{R'(\vartheta_j^s)}(v^i)| \leq |\omega_{R(\vartheta_j^s)}(v^i) - \omega_{R''(\vartheta_j^s)}(v^i)|$.

$\Rightarrow \min\{|\rho_{R(\vartheta_j^s)}(v^i)-\rho_{R'(\vartheta_j^s)}(v^i)|, |\tau_{R(\vartheta_j^s)}(v^i)-\tau_{R'(\vartheta_j^s)}(v^i)|\}+|\omega_{R(\vartheta_j^s)}(v^i)-\omega_{R'(\vartheta_j^s)}(v^i)| \leq \min\{|\rho_{R(\vartheta_j^s)}(v^i) - \rho_{R''(\vartheta_j^s)}(v^i)|, |\tau_{R(\vartheta_j^s)}(v^i) - \tau_{R''(\vartheta_j^s)}(v^i)|\} + |\omega_{R(\vartheta_j^s)}(v^i) - \omega_{R''(\vartheta_j^s)}(v^i)|$

and $\max\{|\rho_{R(\vartheta_j^s)}(v^i)-\rho_{R'(\vartheta_j^s)}(v^i)|, |\tau_{R(\vartheta_j^s)}(v^i)-\tau_{R'(\vartheta_j^s)}(v^i)|\}+|\omega_{R(\vartheta_j^s)}(v^i)-\omega_{R'(\vartheta_j^s)}(v^i)| \leq \max\{|\rho_{R(\vartheta_j^s)}(v^i) - \rho_{R''(\vartheta_j^s)}(v^i)|, |\tau_{R(\vartheta_j^s)}(v^i) - \tau_{R''(\vartheta_j^s)}(v^i)|\} + |\omega_{R(\vartheta_j^s)}(v^i) - \omega_{R''(\vartheta_j^s)}(v^i)|$.

$\Rightarrow 1 - \frac{1}{2}[\min\{|\rho_{R(\vartheta_j^s)}(v^i) - \rho_{R'(\vartheta_j^s)}(v^i)|, |\tau_{R(\vartheta_j^s)}(v^i) - \tau_{R'(\vartheta_j^s)}(v^i)|\} + |\omega_{R(\vartheta_j^s)}(v^i) - \omega_{R'(\vartheta_j^s)}(v^i)|] \geq 1-\frac{1}{2}[\min\{|\rho_{R(\vartheta_j^s)}(v^i)-\rho_{R''(\vartheta_j^s)}(v^i)|, |\tau_{R(\vartheta_j^s)}(v^i)-\tau_{R''(\vartheta_j^s)}(v^i)|\}+|\omega_{R(\vartheta_j^s)}(v^i)-\omega_{R''(\vartheta_j^s)}(v^i)|]$

and

$1+\frac{1}{2}[\max\{|\rho_{R(\vartheta_j^s)}(v^i)-\rho_{R'(\vartheta_j^s)}(v^i)|, |\tau_{R(\vartheta_j^s)}(v^i)-\tau_{R'(\vartheta_j^s)}(v^i)|\}+|\omega_{R(\vartheta_j^s)}(v^i)-\omega_{R'(\vartheta_j^s)}(v^i)|] \leq 1+\frac{1}{2}[\max\{|\rho_{R(\vartheta_j^s)}(v^i)-\rho_{R''(\vartheta_j^s)}(v^i)|, |\tau_{R(\vartheta_j^s)}(v^i)-\tau_{R''(\vartheta_j^s)}(v^i)|\}+|\omega_{R(\vartheta_j^s)}(v^i)-\omega_{R''(\vartheta_j^s)}(v^i)|]$

$\Rightarrow \mathbb{S}_{PFHSS}(R, R'') \leq \mathbb{S}_{PFHSS}(R, R')$. *Likewise, we can prove*

$$\mathbb{S}_{PFHSS}(R, R'') \leq \mathbb{S}_{PFHSS}(R', R'').$$

Theorem 11 *Let* $(R,\ \Lambda)$, $(R',\ \Lambda)$ *and* $(R'',\ \Lambda)$ *be three* $PFHSSs$ *over the universal set* V*. Then* $\mathbb{T}_{PFHSS}$ *also satisfies the four axioms of similarity measures.*

Proof 4 *The proof can be done on the similar lines as above.*

Definition 34 *The two* $PFHSSs$ $(R,\ \Lambda)$ *and* $(R',\ \Lambda)$ *are said to be* $\approx^\alpha$ *−similar, denoted by* $(R,\ \Lambda) \approx^\alpha (R',\ \Lambda) \Leftrightarrow \mathbb{S}_{PFHSS}(R,\ R') \geq \alpha$ *for* $\alpha \in (0,1)$.

Definition 35 *The two* $PFHSSs$ (R, Λ) *and* (R', Λ) *are said to be significantly similar if* $\mathbb{S}_{PFHSS}(R, R') \geq 0.8$

4.5 Application of Proposed PFHSSs Similarity Measures in Medical Diagnosis

In this section, we proposed a methodology for the diagnosis of a medical problem on the basis of proposed similarity measures of $PFHSSs$. The methodology has been outlined in Figure 4.2. Further, a numerical illustration has been presented which involves the similarity measures of two $PFHSSs$ to detect whether a patient suffering from a particular disease or not. Let us consider that there ar two patients I_1, I_2 in a hospital having a symptoms of COVID-19. Suppose there are three stages of characterization of the symptoms as $severe(v^1)$, $mild(v^2)$ and $no(v^3)$, i.e., the universal set $V = \{v^1,\ v^2,\ v^3\}$.

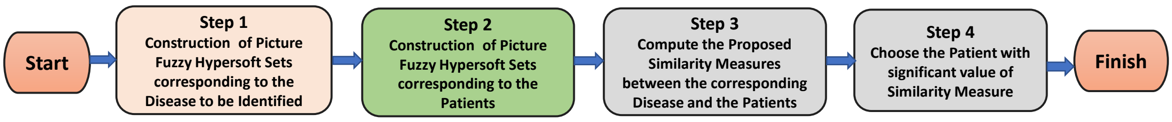

Figure 4.2: Proposed Methodology

Let $K = \{K^1 = sense\ of\ taste,\ K^2 = temperature,\ K^3 = chest\ pain,\ K^4 = flu\}$

be the set of symptoms which are classified into sub-attributes as:
$K^1 = sense\ of\ taste = \{no\ taste,\ Can\ taste\}$
$K^2 = temperature = \{97.5 - 98.5,\ 98.6 - 99.5,\ 99.6 - 101.5,\ 101.6 - 102.5\}$
$K^3 = chest\ pain = \{shortness\ of\ breath,\ no\ pain,\ normal\ pain\ angina\}$
$K^4 = flu = \{sore\ throat,\ cough,\ strep\ throat\}$
Now, let us define a relationv$R : (K_1^a \times K_2^b \times K_3^c \times K_4^d \longrightarrow P(V)$ defined as,
$R(K_1^a \times K_2^b \times K_3^c \times K_4^d) = \{\Im = shortness\ of\ breath,\ \wp = 101.3,\ \Re = sore\ throat,$ $\mho = no\ taste\}$ is the most prominent sample of the patient for the confirmation of the COVID-19. Two patients are randomly selected based on the above sample. Let $(R,\ \Lambda)$ be a $PFHSS$ over V for COVID-19 prepared with the help of a medical expert as given in Table 4.8

Next, the $PFHSSs$ for two patients under consideration is given in Table 4.9 and Table 4.10.

Now, by making use of the proposed similarity measure we get $\mathbb{S}_{PFHSS}(R,\ I_1) =$

Table 4.8: $PFHSS(R,\ \Lambda)$ for COVID-19

$(R,\ \Lambda)$	K_1^a	K_2^b	K_3^c	K_4^d
v^1	$(\Im(0.4,\ 0.1,\ 0.1))$	$(\wp(0.3,\ 0.0,\ 0.3))$	$(\Re(0.7,\ 0.0,\ 0.2))$	$(\mho(0.4,\ 0.2,\ 0.3))$
v^2	$(\Im(0.5,\ 0.1,\ 0.3))$	$(\wp(0.1,\ 0.3,\ 0.2))$	$(\Re(0.4,\ 0.3,\ 0.1))$	$(\mho(0.1,\ 0.2,\ 0.3))$
v^3	$(\Im(0.3,\ 0.5,\ 0.1))$	$(\wp(0.1,\ 0.3,\ 0.5))$	$(\Re(0.0,\ 0.4,\ 0.3))$	$(\mho(0.1,\ 0.2,\ 0.6))$

Table 4.9: $PFHSS(R,\ \Lambda)$ for the patient I_1

(I_1, Λ)	K_1^a	K_2^b	K_3^c	K_4^d
v^1	$(\Im(0.4,\ 0.2,\ 0.2))$	$(\wp(0.3,\ 0.1,\ 0.2))$	$(\Re(0.5,\ 0.1,\ 0.3))$	$(\mho(0.4,\ 0.1,\ 0.1))$
v^2	$(\Im(0.5,\ 0.2,\ 0.2))$	$(\wp(0.1,\ 0.2,\ 0.4))$	$(\Re(0.2,\ 0.3,\ 0.1))$	$(\mho(0.1,\ 0.5,\ 0.3))$
v^3	$(\Im(0.2,\ 0.5,\ 0.1))$	$(\wp(0.2,\ 0.3,\ 0.5))$	$(\Re(0.0,\ 0.3,\ 0.3))$	$(\mho(0.1,\ 0.2,\ 0.0))$

$0.8657\ >\ 0.75$ and $\mathbb{S}_{PFHSS}(R,\ I_2) = 0.6892\ <\ 0.75$.
Therefore, we conclude that the patient I_1 is suffering from COVID-19.

4.6 Conclusions

The processing of uncertain information in terms of multi sub-attributes parametrization with the help of proposed notion of Picture Fuzzy Hypersoft Set (PFHSS) is a novel and useful concept. PFHSSs and their aggregation operators can be a strong mathematical too to handle incomplete and inexact information with vagueness. Here, we could additionally address the components of neutral membership (abstain)and refusal in PFHSS and establish various important properties and operational laws helpful in a decision making problem. Also, the notion of similarity measure in the picture fuzzy hypersoft sets is also very useful for solving a decision-making problem. The concept of picture fuzzy hypersoft weighted average/ordered weighted average operator (PFHSWA/PFHSOWA) and weighted geometric/ordered weighted geometric operator (PFHSWG/PFHSOWG) have been proved and studied in detail.

Table 4.10: $PFHSS(R,\ \Lambda)$ for the patient I_2

(I_2, Λ)	K_1^a	K_2^b	K_3^c	K_4^d
v^1	$(\Im(0.3,\ 0.2,\ 0.3))$	$(\wp(0.2,\ 0.3,\ 0.1))$	$(\Re(0.5,\ 0.1,\ 0.0))$	$(\mho(0.4,\ 0.0,\ 0.1))$
v^2	$(\Im(0.4,\ 0.2,\ 0.2))$	$(\wp(0.3,\ 0.2,\ 0.1))$	$(\Re(0.1,\ 0.4,\ 0.2))$	$(\mho(0.2,\ 0.4,\ 0.5))$
v^3	$(\Im(0.1,\ 0.5,\ 0.1))$	$(\wp(0.2,\ 0.3,\ 0.0))$	$(\Re(0.0,\ 0.5,\ 0.3))$	$(\mho(0.4,\ 0.2,\ 0.2))$

Chapter 5

Picture Fuzzy Hypersoft Matrices

In this chapter, we first introduce the novel notion of picture fuzzy hypersoft matrix along with various important binary operations and properties. The proposition concentrates on presenting a robust decision-making framework for identifying the optimal and most suitable renewable energy source. In this regard, the revised definition of picture fuzzy hypersoft choice matrix/weighted choice matrix, value matrix, and total score matrix have been presented. Further, two algorithms of decision-making for the selection of the best renewable energy sources have been provided along with appropriate illustrations and ranking descriptions. A numerical example has also been worked out for the sake of illustrating the proposed algorithms. Finally, to establish the robustness of the MCDM algorithms, a necessary comparative analysis has been carried out successfully.

5.1 Picture Fuzzy Hypersoft Matrices & Operations

In this section, on the basis of the proposed notion of a picture fuzzy hypersoft set, we are also presenting the concept of a new type of hypersoft matrix termed a Picture Fuzzy Hypersoft Matrix (PFHSM) along with various binary operations and important properties.

Picture Fuzzy Hypersoft Matrix. Let $V = \{v^1, v^2, \ldots, v^n\}$ be the universe of discourse and PFS(V) be the collection of all picture fuzzy subsets of V. Suppose $K_1, K_2, \ldots, K_m$ for $m \geq 1$ be m well-defined attributes, whose respective attribute values are the sets $K_1^a, K_2^b, \ldots, K_m^z$ with the relation $K_1^a \times K_2^b \times \ldots \times K_m^z$ where $a, b, c, \ldots, z = 1, 2, \ldots, n$. The pair $\left(R, K_1^a \times K_2^b \times \cdots \times K_m^z\right)$ is called a picture fuzzy hypersoft set over V where $R : K_1^a \times K_2^b \times \ldots \times K_m^z \to \text{PFS}(V)$ defined by

$R\left(K_1^a \times K_2^b \times \cdots \times K_m^z\right)$

$= \left\{< v, \rho_\vartheta(v), \tau_\vartheta(v), \omega_\vartheta(v) > \mid v \in V, \vartheta \in K_1^a \times K_2^b \times \cdots \times K_m^z \right\}.$

Here, ρ, τ, ω represents the positive membership, neutral membership and negative membership degrees respectively. Let $Z_v = K_1^a \times K_2^b \times \ldots \times K_m^z$ be the relation with its characteristic function is $\chi_{Z_v} : K_1^a \times K_2^b \times \ldots \times K_m^z \to \text{PFS}(V)$ given by

$$\chi_{Z_v} = \left\{< v, \rho_\vartheta(v), \tau_\vartheta(v), \omega_\vartheta(v) > \mid v \in V, \vartheta \in K_1^a \times K_2^b \times \ldots \times K_m^z \right\}.$$

The tabular representation of Z_v is given in Table 6.1

Table 5.1: Tabular form of Z_v

	K_1^a	K_2^b	$\ldots$	K_m^z
v^1	$\chi_{Z_v}(v^1, K_1^a)$	$\chi_{Z_v}(v^1, K_2^b)$	$\ldots$	$\chi_{Z_v}(v^1, K_m^z)$
v^2	$\chi_{Z_v}(v^2, K_1^a)$	$\chi_{Z_v}(v^2, K_2^b)$	$\ldots$	$\chi_{Z_v}(v^2, K_m^z)$
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$
v^n	$\chi_{Z_v}(v^n, K_1^a)$	$\chi_{Z_v}(v^n, K_2^b)$	$\ldots$	$\chi_{Z_v}(v^n, K_m^z)$

If $B_{ij} = \chi_{Z_v}\left(v^i, K_j^s\right)$ where $i = 1, 2, .., n, j = 1, 2, ..., m$ and $s = a, b, c, ..., z$. Then a matrix is defined as

$$[B_{ij}]_{n \times m} = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1m} \\ B_{21} & B_{22} & \cdots & B_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ B_{n1} & B_{n2} & \cdots & B_{nm} \end{bmatrix}$$

which is called **Picture Fuzzy Hypersoft Matrix** of order $n \times m$, where

$$B_{ij} = \left(\rho_{K_j^s}(v_i), \tau_{K_j^s}(v_i), \omega_{K_j^s}(v_i), v_i \epsilon V, \left(K_j^s \epsilon K_1^a \times K_2^b \times \cdots \times K_m^z\right)\right) = \left(\rho_{ijs}^B, \tau_{ijs}^B, \omega_{ijs}^B\right).$$

Hence, it may be noted that any picture fuzzy hypersoft set can be represented in terms of the picture fuzzy hypersoft matrix. Throughout the chapter, we will denote the collection of all picture fuzzy hypersoft matrices by $PFHSM_{n \times m}$.

Example 1: Suppose a need arises for a School to hire a Mathematics teacher for the 10th class. A total of five candidates have applied to fill up the void space. The Human Resource cell of the school appoints an expert/decision-maker for this selection process. Let $V = \{v^1,\ v^2,\ v^3,\ v^4,\ v^5\}$ be the set of all five candidates with their set of attributes as $K_1 =$ Qualification, $K_2 =$ Experience, $K_3 =$ Age, $K_4 =$ Gender. Further, their respective sub-attributes are

$$K_1 = \text{Qualification} = \{\text{BS Hons., MS, M.Phil., Ph.D.}\}$$

$$K_2 = \text{Experience} = \{\text{3yr, 5yr, 7yr, 10yr}\}$$

$$K_3 = \text{Age} = \{\text{Less than twenty five, Great than twenty five}\}$$

$$K_4 = \text{Gender} = \{\text{Male, Female}\}.$$

Let the function be $R : K_1^a \times K_2^b \times \cdots \times K_m^z \to \text{PFS}(V)$. Based on some empirical-hypothetical assumptions and the decision maker's opinion, we present the following tables with respect to each attribute and with their further sub-attributes are given in Table 5.2, Table 5.3, Table 5.4 and Table 5.5

Table 5.2: Decision maker's opinion for Qualification

K_1^a (**Qualification**)	v^1	v^2	v^3	v^4	v^5
BS Hons.	$(0.2, 0.3, 0.4)$	$(0.1, 0.3, 0.5)$	$(0.3, 0.5, 0.1)$	$(0.3, 0.2, 0.1)$	$(0.4, 0.1, 0.2)$
MS	$(0.1, 0.2, 0.4)$	$(0.4, 0.2, 0.3)$	$(0.3, 0.5, 0.1)$	$(0.1, 0.1, 0.5)$	$(0.1, 0.3, 0.4)$
M.Phil.	$(0.1, 0.3, 0.5)$	$(0.1, 0.3, 0.5)$	$(0.1, 0.3, 0.5)$	$(0.1, 0.3, 0.5)$	$(0.1, 0.3, 0.5)$
Ph.D.	$(0.3, 0.1, 0.5)$	$(0.4, 0.3, 0.1)$	$(0.2, 0.5, 0.1)$	$(0.2, 0.1, 0.5)$	$(0.3, 0.1, 0.5)$

Table 5.3: Decision maker's opinion for Experience

K_2^b (**Experience**)	v^1	v^2	v^3	v^4	v^5
3*yr*	$(0.4, 0.3, 0.1)$	$(0.2, 0.3, 0.5)$	$(0.2, 0.2, 0.3)$	$(0.5, 0.1, 0.2)$	$(0.2, 0.3, 0.5)$
5*yr*	$(0.1, 0.3, 0.5)$	$(0.3, 0.3, 0.3)$	$(0.3, 0.3, 0.3)$	$(0.2, 0.4, 0.2)$	$(0.7, 0.1, 0.1)$
7*yr*	$(0.1, 0.7, 0.1)$	$(0.4, 0.3, 0.2)$	$(0.1, 0.3, 0.5)$	$(0.2, 0.2, 0.5)$	$(0.2, 0.5, 0.2)$
10*yr*	$(0.3, 0.5, 0.1)$	$(0.2, 0.4, 0.3)$	$(0.3, 0.3, 0.2)$	$(0.1, 0.3, 0.6)$	$(0.6, 0.3, 0.1)$

Table 5.4: Decision maker's opinion for Age

K_3^c (**Age**)	v^1	v^2	v^3	v^4	v^5
Less than twentyfive	$(0.2, 0.1, 0.5)$	$(0.3, 0.3, 0.3)$	$(0.5, 0.2, 0.1)$	$(0.6, 0.2, 0.1)$	$(0.2, 0.3, 0.4)$
Greater than twentyfive	$(0.5, 0.3, 0.1)$	$(0.4, 0.3, 0.1)$	$(0.2, 0.4, 0.3)$	$(0.5, 0.2, 0.2)$	$(0.4, 0.3, 0.1)$

Now, let us consider

$R\left(K_1^a \times K_2^b \times K_3^c \times K_4^d\right) = R\,(\text{MS, 7yr, Greater than twenty five, Male})$

Table 5.5: Decision maker's opinion for Gender

K_4^d (**Gender**)	v^1	v^2	v^3	v^4	v^5
Male	$(0.2, 0.1, 0.2)$	$(0.3, 0.2, 0.3)$	$(0.1, 0.2, 0.6)$	$(0.4, 0.2, 0.3)$	$(0.5, 0.2, 0.1)$
Female	$(0.2, 0.1, 0.5)$	$(0.4, 0.3, 0.1)$	$(0.3, 0.5, 0.1)$	$(0.2, 0.2, 0.2)$	$(0.4, 0.3, 0.1)$

$= \ (v^1,\ v^2,\ v^3,\ v^5)$.

For the above relational expression, the picture fuzzy hypersoft set can be expressed as

$$R\left(K_1^a \times K_2^b \times K_3^c \times K_4^d\right) =$$
$$\{< v^1, (MS(0.1, 0.2, 0.4), 7yr(0.1, 0.7, 0.1), \text{Greater than twenty five}(0.5, 0.3, 0.1), Male(0.2, 0.1, 0.2)) >$$
$$< v^2, (MS(0.4, 0.2, 0.3), 7yr(0.4, 0.3, 0.2), \text{Greater than twenty five}(0.4, 0.3, 0.1), Male(0.3, 0.2, 0.3)) >$$
$$< v^3, (MS(0.3, 0.5, 0.1), 7yr(0.1, 0.3, 0.5), \text{Greater than twenty five}(0.2, 0.4, 0.3), Male(0.1, 0.2, 0.6)) >$$
$$< v^5, (MS(0.1, 0.3, 0.4), 7yr(0.2, 0.5, 0.2), \text{Greater than twenty five}(0.4, 0.3, 0.1), Male(0.5, 0.2, 0.1)) >\}$$

The above example of picture fuzzy hypersoft set relational expression can be written in the following form:

$$[B]_{4\,\times\,4} = \begin{bmatrix} (MS(0.1,\ 0.2,\ 0.4)) & (7yr(0.1,\ 0.7,\ 0.1)) & (\text{Greater than twenty five}(0.5, 0.3, 0.1)) & (Male(0.2,\ 0.1,\ 0.2)) \\ (MS(0.4,\ 0.2,\ 0.3)) & (7yr(0.4,\ 0.3,\ 0.2)) & (\text{Greater than twenty five}(0.4, 0.3, 0.1)) & (Male(0.3,\ 0.2,\ 0.3)) \\ (MS(0.3,\ 0.5,\ 0.1)) & (7yr(0.1,\ 0.3,\ 0.5)) & (\text{Greater than twenty five}(0.2, 0.4, 0.3)) & (Male(0.1,\ 0.2,\ 0.6)) \\ (MS(0.1,\ 0.3,\ 0.4)) & (7yr(0.2,\ 0.5,\ 0.2)) & (\text{Greater than twenty five}(0.4, 0.3, 0.1)) & (Male(0.5,\ 0.2,\ 0.1)) \end{bmatrix}.$$

Various Types of Picture Fuzzy Hypersoft Matrices:

Let $B = [B_{ij}]$ be a picture fuzzy hypersoft matrix of order $n \times m$; where $B_{ij} = \left(\rho_{ijs}^B,\ \tau_{ijs}^B,\ \omega_{ijs}^B\right)$; then various kinds of important matrice can be presented as below:

- "**Picture fuzzy hypersoft zero matrix** if $\rho_{ijs}^B = 0,\ \tau_{ijs}^B = 0\ \&\ \omega_{ijs}^B = 0; \forall i, j, s$ and the matrix is denoted by $0 = [0, 0, 0]$."
- "**Picture fuzzy hypersoft square matrix** if $n = m$."
- "**Picture fuzzy hypersoft row matrix** if $m = 1$."
- "**Picture fuzzy hypersoft column matrix** if $n = 1$."
- "**Picture fuzzy hypersoft diagonal matrix** if all its non-diagonal entries are zero $\forall\ i, j, s$."

- "**Picture fuzzy hypersoft ρ-universal matrix** if $\rho_{ijs}^B = 1$, $\tau_{ijs}^B = 0$ & $\omega_{ijs}^B = 0$; $\forall\ i$, j & s, denoted by $\wp_\rho$."

- "**Picture fuzzy hypersoft τ-universal matrix** if $\rho_{ijs}^B = 0$, $\tau_{ijs}^B = 1$ & $\omega_{ijs}^B = 0$; $\forall\ i$, j & s, denoted by $\wp_\tau$."

- "**Picture fuzzy hypersoft ω-universal matrix** if $\rho_{ijs}^B = 0$, $\tau_{ijs}^B = 0$ & $\omega_{ijs}^B = 1$; $\forall\ i$ & j & s, denoted by $\wp_\omega$."

- "**Picture fuzzy hypersoft Scalar multiplication:** for any scalar m, we define $mB = [(m\rho_{ijs}^B, m\tau_{ijs}^B, m\omega_{ijs}^B)]$, $\forall\ i$, j & s."

- "**Picture fuzzy hypersoft Symmetric Matrix:** if $\left(\rho_{ijs}^B, \tau_{ijs}^B, \omega_{ijs}^B\right) = \left(\rho_{jsi}^B, \tau_{jsi}^B, \omega_{jsi}^B\right)$ i.e.$B^t = B$."

Further, we propose some set-theoretic relations for two given picture fuzzy hypersoft matrices, say, $B = [(\rho_{ijs}^B,\ \tau_{ijs}^B,\ \omega_{ijs}^B)]$ and $C = [(\rho_{ijs}^C, \tau_{ijs}^C,\ \omega_{ijs}^C)] \in PFHSM_{n\ \times\ m}$.

- "**Subsethood:** $B \subseteq C$ if $\rho_{ijs}^B \leq \rho_{ijs}^C$, $\tau_{ijs}^B \geq \tau_{ijs}^C$ & $\nu_{ijs}^B \geq \nu_{ijs}^C$; $\forall\ i$, j & s."

- "**Containment:** $B \supseteq C$ if $\rho_{ijs}^B \geq \rho_{ijs}^C$, $\tau_{ijs}^B \leq \tau_{ijs}^C$ & $\omega_{ijs}^B \leq \nu_{ijs}^C$; $\forall\ i$, j & s."

- "**Equality:** $B = C$ if $\rho_{ijs}^B = \rho_{ijs}^C$, $\tau_{ijs}^B = \tau_{ijs}^C$ & $\omega_{ijs}^B = \omega_{ijs}^C$; $\forall\ i$, j & s."

- "**Max Min Product:**
  Let $B = [B_{ij}] = [(\rho_{ijs}^B, \tau_{ijs}^B, \omega_{ijs}^B)] \in PFHSM_{n\times m}$ & $C = [C_{jt}] = [(\rho_{jst}^C, \tau_{jst}^C, \omega_{jst}^C)] \in PFHSM_{m\times p}$ be two picture fuzzy hypersoft matrices then
  $$B{*}C = [d_{it}]_{m\times p} = \left(\max(\min_{js}(\rho_{ijs}^B, \rho_{jst}^C)), \min(\min_{js}(\tau_{ijs}^B, \tau_{jst}^C)), \min(\max_{js}(\omega_{ijs}^B, \omega_{jst}^C))\right);$$
  $\forall\ i,\ j,\ s$ & t"

- "**Average Max Min Product:**
  Let $B = [B_{ij}] = [(\rho_{ijs}^B, \tau_{ijs}^B, \omega_{ijs}^B)] \in PFHSM_{n\times m}$ & $C = [C_{jt}] = [(\rho_{jst}^C, \tau_{jst}^C, \omega_{jst}^C)] \in PFHSM_{m\times p}$ be two picture fuzzy hypersoft matrices then
  $$B *_A C = [d_{it}]_{n\times p} = \left[\left(\max_j s(\tfrac{\rho_{ijs}^B + \rho_{jst}^C}{2}), \min_j s(\tfrac{\tau_{ijs}^B + \tau_{jst}^C}{2}), \min_j s(\tfrac{\omega_{ijs}^B, \omega_{jst}^C}{2})\right)\right];$$
  $\forall\ i,\ j,\ s$ & t."

Some Fundamental Binary Operations for Picture fuzzy hypersoft matrices:
Consider two Picture fuzzy hypersoft matrices $B_1 = [(\rho_{ijs}^{B_1},\ \tau_{ijs}^{B_1},\ \omega_{ijs}^{B_1})]$ and $B_2 = [(\rho_{ijs}^{B_2},\ \tau_{ijs}^{B_2},\ \omega_{ijs}^{B_2})] \in PFHSM_{n\times m}$. Some of the basic binary operations on these matrices can be presented as follows:

- $B_1^c = \left[\left(\omega_{ijs}^{B_1}, \tau_{ijs}^{B_1}, \rho_{ijs}^{B_1}\right)\right]$; $\forall$ i, j and s.
- $B_1 \cup B_2 = \left[\left(\max(\rho_{ijs}^{B_1}, \rho_{ijs}^{B_2}), \min(\tau_{ijs}^{B_1}, \tau_{ijs}^{B_2}), \min(\omega_{ijs}^{B_1}, \omega_{ijs}^{B_2})\right)\right]$; $\forall$ i, j and s.
- $B_1 \cap B_2 = \left[\left(\min(\rho_{ijs}^{B_1}, \rho_{ijs}^{B_2}), \min(\tau_{ijs}^{B_1}, \tau_{ijs}^{B_2}), \max(\omega_{ijs}^{B_1}, \omega_{ijs}^{B_2})\right)\right]$ $\forall$; i and j.
- $B_1 \otimes B_2 = \left[\left(\rho_{ijs}^{B_1} \cdot \rho_{ijs}^{B_2}, \tau_{ijs}^{B_1} \cdot \tau_{ijs}^{B_2}, \sqrt[n]{(\omega_{ijs}^{B_1})^2 + (\omega_{ijs}^{B_2})^2 - (\omega_{ijs}^{B_1})^2 \cdot (\omega_{ijs}^{B_2})^2}\right)\right]$; $\forall$ i, j amd s.
- $B_1 \oplus B_2 = \left[\left(\sqrt[n]{(\rho_{ijs}^{B_1})^2 + (\rho_{ijs}^{B_2})^2 - (\rho_{ijs}^{B_1})^2 \cdot (\rho_{ijs}^{B_2})^2}, \tau_{ijs}^{B_1} \cdot \tau_{ijs}^{B_2}, \omega_{ijs}^{B_1} \cdot \omega_{ijs}^{B_2}\right)\right]$; $\forall$ i, j and s.
- $B_1 @ B_2 = \left[\left(\frac{\rho_{ijs}^{B_1} + \rho_{ijs}^{B_2}}{2}, \frac{\tau_{ijs}^{B_1} + \tau_{ijs}^{B_2}}{2}, \frac{\omega_{ijs}^{B_1} + \omega_{ijs}^{B_2}}{2}\right)\right]$; $\forall$ i, j and s.
- $B_1 @_w B_2 = \left[\left(\frac{w_1\rho_{ijs}^{B_1} + w_2\rho_{ijs}^{B_2}}{w_1 + w_2}, \frac{w_1\tau_{ijs}^{B_1} + w_2\tau_{ijs}^{B_2}}{w_1 + w_2}, \frac{w_1\omega_{ijs}^{B_1} + w_2\omega_{ijs}^{B_2}}{w_1 + w_2}\right)\right]$; $\forall$ i, j and s ; where $w_1, w_2 > 0$ are the weights.
- $B_1 \$ B_2 = \left[\left(\sqrt[n]{\rho_{ijs}^{B_1} \cdot \rho_{ijs}^{B_2}}, \sqrt[n]{\tau_{ijs}^{B_1} \cdot \tau_{ijs}^{B_2}}, \sqrt[n]{\omega_{ijs}^{B_1} \cdot \omega_{ijs}^{B_2}}\right)\right]$; $\forall$ i, j and s.
- $B_1 \$_w B_2 = \left(((\rho_{ijs}^{B_1})^{w_1} \cdot (\rho_{ijs}^{B_2})^{w_2})^{\frac{1}{w_1+w_2}}, ((\tau_{ijs}^{B_1})^{w_1} \cdot (\tau_{ijs}^{B_2})^{w_2})^{\frac{1}{w_1+w_2}}, ((\omega_{ijs}^{B_1})^{w_1} \cdot (\omega_{ijs}^{B_2})^{w_2})^{\frac{1}{w_1+w_2}}\right)$; $\forall$ i, j and s, where $w_1, w_2 > 0$ are the weights.
- $B_1 \bowtie B_2 = \left[\left(2 \cdot \frac{\rho_{ijs}^{B_1} \cdot \rho_{ijs}^{B_2}}{\rho_{ijs}^{B_1} + \rho_{ijs}^{B_2}}, 2 \cdot \frac{\tau_{ijs}^{B_1} \cdot \tau_{ijs}^{B_2}}{\tau_{ijs}^{B_1} + \tau_{ijs}^{B_2}}, 2 \cdot \frac{\omega_{ijs}^{B_1} \cdot \omega_{ijs}^{B_2}}{\omega_{ijs}^{B_1} + \omega_{ijs}^{B_2}}\right)\right]$; $\forall$ i, j and s.
- $B_1 \bowtie_w B_2 = \left[\left(\frac{w_1 + w_2}{\frac{w_1}{\rho_{ijs}^{B_1}} + \frac{w_2}{\rho_{ijs}^{B_2}}}, \frac{w_1 + w_2}{\frac{w_1}{\tau_{ijs}^{B_1}} + \frac{w_2}{\tau_{ijs}^{B_2}}}, \frac{w_1 + w_2}{\frac{w_1}{\omega_{ijs}^{B_1}} + \frac{w_2}{\omega_{ijs}^{B_2}}}\right)\right]$; $\forall$ i, j and s ; where $w_1, w_2 > 0$ are the weights.

Proposition 1 *Let B_1 and $B_2 \in PFHSM_{n\times m}$ then the following laws hold:*

(i) $B_1 \cup B_2 = B_2 \cup B_1$

(ii) $B_1 \cap B_2 = B_2 \cap B_1$

(iii) $(B_1 \cup B_2)^c = B_1^c \cap B_2^c$

(iv) $(B_1 \cap B_2)^c = B_1^c \cup B_2^c$

(v) $(B_1^c \cap B_2^c)^c = B_1 \cup B_2$ *(vi)* $(B_1^c \cup B_2^c)^c = B_1 \cap B_2$.

Proof : Let $B_1 = [(\rho_{ijs}^{B_1}, \tau_{ijs}^{B_1}, \omega_{ijs}^{B_1})], B_2 = [(\rho_{ijs}^{B_2}, \tau_{ijs}^{B_2}, \omega_{ijs}^{B_2})] \in PFHSM_{n\times m}$.
Then $\forall$ i, j and s we get,

(i)

$$\begin{aligned} B_1 \cup B_2 &= \Big[\big(\max(\rho_{ijs}^{B_1}, \rho_{ijs}^{B_2}), \min(\tau_{ijs}^{B_1}, \tau_{ijs}^{B_2}), \min(\omega_{ijs}^{B_1}, \omega_{ijs}^{B_2})\big)\Big] \\ &= \Big[\big(\max(\rho_{ijs}^{B_2}, \rho_{ijs}^{B_1}), \min(\tau_{ijs}^{B_2}, \tau_{ijs}^{B_1}), \min(\omega_{ijs}^{B_2}, \omega_{ijs}^{B_1})\big)\Big] = B_2 \cup B_1. \end{aligned}$$

(ii)

$$\begin{aligned} B_1 \cup B_2 &= \Big[\big(\min(\rho_{ijs}^{B_1}, \rho_{ijs}^{B_2}), \min(\tau_{ijs}^{B_1}, \tau_{ijs}^{B_2}), \max(\omega_{ijs}^{B_1}, \omega_{ijs}^{B_2})\big)\Big] \\ &= \Big[\big(\min(\rho_{ijs}^{B_2}, \rho_{ijs}^{B_1}), \min(\tau_{ijs}^{B_2}, \tau_{ijs}^{B_1}), \max(\omega_{ijs}^{B_2}, \omega_{ijs}^{B_1})\big)\Big] = B_2 \cup B_1. \end{aligned}$$

(iii)

$$\begin{aligned} (B_1 \cup B_2)^c &= \Big(\big([(\rho_{ijs}^{B_1}, \tau_{ijs}^{B_1}, \omega_{ijs}^{B_1})] \cup [(\rho_{ijs}^{B_2}, \tau_{ijs}^{B_2}, \omega_{ijs}^{B_2})]\big)\Big)^c \\ &= [\max(\rho_{ijs}^{B_1}, \rho_{ijs}^{B_2}), \min(\tau_{ijs}^{B_1}, \tau_{ijs}^{B_2}), \min(\omega_{ijs}^{B_1}, \omega_{ijs}^{B_2})]^c \\ &= \big[\big(\min(\omega_{ijs}^{B_1}, \omega_{ijs}^{B_2}), \min(\tau_{ijs}^{B_1}, \tau_{ijs}^{B_2}), \max(\rho_{ijs}^{B_1}, \rho_{ijs}^{B_2})\big)\big] \\ &= \big[\big([(\omega_{ijs}^{B_1}, \tau_{ijs}^{B_1}, \rho_{ijs}^{B_1})] \cap [(\omega_{ijs}^{B_2}, \tau_{ijs}^{B_2}, \rho_{ijs}^{B_2})]\big)\big] = B_1^c \cap B_2^c. \end{aligned}$$

On similar lines, (iv), (v) and (vi) can be proved accordingly.

Proposition 2 *Let* $B_1 = [(\rho_{ijs}^{B_1}, \tau_{ijs}^{B_1}, \omega_{ijs}^{B_1})] \in PFHSM_{n\times m}$. *On the basis of the proposed definitions, the following laws hold:*

(i) $(B_1^c)^c = B_1$ *(vi)* $B_1 \cup \wp_\rho = \wp_\rho$

(ii) $(\wp_\rho)^c = \wp_\omega$ *(vii)* $B_1 \cap \wp_\nu = B_1$

(iii) $(\wp_\tau)^c = \wp_\tau$ *(viii)* $B_1 \cap B_1 = B_1$

(iv) $(\wp_\omega)^c = \wp_\rho$ *(ix)* $B_1 \cap \wp_\rho = B_1$

(v) $B_1 \cup B_1 = B_1$ *(x)* $B_1 \cap \wp_\omega = \wp_\omega$.

Proposition 3 *Let B_1 and $B_2 \in PFHSM_{n\times m}$. In view of the weighted form, the following laws hold:*

(i) $(B_1^c @_w B_2^c)^c = B_1 @_w B_2$

(ii) $(B_1^c \$_w B_2^c)^c = B_1 \$_w B_2$

(iii) $(B_1^c \bowtie_w B_2^c)^c = B_1 \bowtie_w B_2$

(iv) $B_1 @_w B_2 = B_2 @_w B_1$

(v) $B_1 \$_w B_2 = B_2 \$_w B_1$

(vi) $B_1 \bowtie_w B_2 = B_2 \bowtie_w B_1$.

Proof : Let $B_1 = [(\rho_{ijs}^{B_1}, \tau_{ijs}^{B_1}, \omega_{ijs}^{B_1})], B_2 = [(\rho_{ijs}^{B_2}, \tau_{ijs}^{B_2}, \omega_{ijs}^{B_2})] \in PFHSM_{n\times m}$. Then $\forall\ i, j,\ s\ \&\ w_1, w_2 > 0$, we get,

(i)

$$\begin{aligned}
(B_1^c @_w B_2^c)^c &= \left(\left[\left((\omega_{ijs}^{B_1}, \tau_{ijs}^{B_1}, \rho_{ijs}^{B_1}) @_w (\omega_{ij}^{B_2}, \tau_{ijs}^{B_2}, \rho_{ijs}^{B_2}) \right) \right] \right)^c \\
&= \left(\left[\left(\frac{w_1\omega_{ijs}^{B_1} + w_2\omega_{ijs}^{B_2}}{w_1 + w_2}, \frac{w_1\tau_{ijs}^{B_1} + w_2\tau_{ijs}^{B_2}}{w_1 + w_2}, \frac{w_1\rho_{ijs}^{B_1} + w_2\rho_{ijs}^{B_2}}{w_1 + w_2} \right) \right] \right)^c \\
&= \left[\left(\frac{w_1\rho_{ijs}^{B_1} + w_2\rho_{ijs}^{B_2}}{w_1 + w_2}, \frac{w_1\tau_{ijs}^{B_1} + w_2\tau_{ijs}^{B_2}}{w_1 + w_2}, \frac{w_1\omega_{ijs}^{B_1} + w_2\omega_{ijs}^{B_2}}{w_1 + w_2} \right) \right] = B_1 @_w B_2.
\end{aligned}$$

(ii)

$$\begin{aligned}
(B_1^c \$_w B_2^c)^c &= \left(\left[\left((\omega_{ijs}^{B_1}, \tau_{ijs}^{B_1}, \rho_{ijs}^{B_1}) \$_w (\omega_{ijs}^{B_2}, \tau_{ijs}^{B_2}, \rho_{ijs}^{B_2}) \right) \right] \right)^c \\
&= \left(\left[\left(((\omega_{ijs}^{B_1})^{w_1} \cdot (\omega_{ijs}^{B_2})^{w_2})^{\frac{1}{w_1+w_2}}, ((\tau_{ijs}^{B_1})^{w_1} \cdot (\tau_{ijs}^{B_2})^{w_2})^{\frac{1}{w_1+w_2}}, ((\rho_{ijs}^{B_1})^{w_1} \cdot (\rho_{ijs}^{B_2})^{w_2})^{\frac{1}{w_1+w_2}} \right) \right] \right)^c \\
&= \left[\left(((\rho_{ijs}^{B_1})^{w_1} \cdot (\rho_{ijs}^{B_2})^{w_2})^{\frac{1}{w_1+w_2}}, ((\tau_{ijs}^{B_1})^{w_1} \cdot (\tau_{ijs}^{B_2})^{w_2})^{\frac{1}{w_1+w_2}}, ((\omega_{ijs}^{B_1})^{w_1} \cdot (\omega_{ijs}^{B_2})^{w_2})^{\frac{1}{w_1+w_2}} \right) \right] \\
&= B_1 \$_w B_2.
\end{aligned}$$

Similar proof for (iii).

(iv)

$$\begin{aligned}B_1@_wB_2 &= \left[\left(\frac{w_1\rho_{ijs}^{B_1}+w_2\rho_{ijs}^{B_2}}{w_1+w_2}, \frac{w_1\tau_{ijs}^{B_1}+w_2\tau_{ijs}^{B_2}}{w_1+w_2}, \frac{w_1\omega_{ijs}^{B_1}+w_2\omega_{ijs}^{B_2}}{w_1+w_2}\right)\right] \\ &= \left[\left(\frac{w_2\rho_{ijs}^{B_2}+w_1\rho_{ijs}^{B_1}}{w_2+w_1}, \frac{w_2\tau_{ijs}^{B_2}+w_1\tau_{ijs}^{B_1}}{w_2+w_1}, \frac{w_2\omega_{ijs}^{B_2}+w_1\omega_{ijs}^{B_1}}{w_2+w_1}\right)\right] \\ &= B_2@_wB_1.\end{aligned}$$

(v)

$$\begin{aligned}B_1\$_wB_2 &= \left[\left(((\rho_{ijs}^{B_1})^{w_1}\cdot(\rho_{ijs}^{B_2})^{w_2})^{\frac{1}{w_1+w_2}}, ((\tau_{ijs}^{B_1})^{w_1}\cdot(\tau_{ijs}^{B_2})^{w_2})^{\frac{1}{w_1+w_2}}, ((\omega_{ijs}^{B_1})^{w_1}\cdot(\omega_{ijs}^{B_2})^{w_2})^{\frac{1}{w_1+w_2}}\right)\right] \\ &= \left[\left(((\rho_{ijs}^{B_2})^{w_2}\cdot(\rho_{ijs}^{B_1})^{w_1})^{\frac{1}{w_2+w_1}}, ((\tau_{ijs}^{B_2})^{w_2}\cdot(\tau_{ijs}^{B_1})^{w_1})^{\frac{1}{w_2+w_1}}, ((\omega_{ijs}^{B_2})^{w_2}\cdot(\omega_{ijs}^{B_1})^{w_1})^{\frac{1}{w_2+w_1}}\right)\right] \\ &= B_2\$_wB_1.\end{aligned}$$

On similar lines, (vi) can be verified accordingly.

Proposition 4 *For B_1, B_2 and $B_3 \in PFHSM_{n\times m}$, the following associative laws hold:*

(i) $(B_1 \cup B_2) \cup B_3 = B_1 \cup (B_2 \cup B_3)$

(ii) $(B_1 \cap B_2) \cap B_3 = B_1 \cap (B_2 \cap B_3)$

(iii) $(B_1@B_2)@B_3 = B_1@(B_2@B_3)$

(iv) $(B_1\$B_2)\$B_3 = B_1\$(B_2\$B_3)$

(v) $(B_1 \bowtie B_2) \bowtie B_3 = B_1 \bowtie (B_2 \bowtie B_3)$.

Proof: For all i & j we write,

(i)

$$\begin{aligned}(B_1\cup B_2)\cup B_3 &= \left[\left([(\max\{\rho_{ijs}^{B_1},\rho_{ijs}^{B_2}\}, \min\{\tau_{ijs}^{B_1},\tau_{ijs}^{B_2}\}), \min\{\omega_{ijs}^{B_1},\omega_{ijs}^{B_2}\}] \cup [(\rho_{ijs}^{B_3},\tau_{ijs}^{B_3},\omega_{ijs}^{B_3})]\right)\right] \\ &= \left[\left(\max\{(\rho_{ijs}^{B_1},\rho_{ijs}^{B_2}),\rho_{ijs}^{B_3}\}, \min\{(\tau_{ijs}^{B_1},\tau_{ijs}^{B_2}),\tau_{ijs}^{B_3}\}, \min\{(\omega_{ijs}^{B_1},\omega_{ijs}^{B_2}),\omega_{ijs}^{B_3}\}\right)\right] \\ &= \left[\left(\max\{(\rho_{ijs}^{B_1},(\rho_{ijs}^{B_2},\rho_{ijs}^{B_3}))\}, \min\{\tau_{ijs}^{B_1},(\tau_{ijs}^{B_2},\tau_{ijs}^{B_3})\}, \min\{\omega_{ijs}^{B_1},(\omega_{ijs}^{B_2},\omega_{ijs}^{B_3})\}\right)\right] \\ &= B_1\cup(B_2\cup B_3).\end{aligned}$$

(ii)

$$
\begin{aligned}
(B_1 \cap B_2) \cap B_3 &= \left[\left((\min\{\rho_{ijs}^{B_1}, \rho_{ijs}^{B_2}\}, (\min\{\tau_{ijs}^{B_1}, \tau_{ijs}^{B_2}\}, \max\{\omega_{ijs}^{B_1}, \omega_{ijs}^{B_2}\}) \cup (\rho_{ijs}^{B_3}, \tau_{ijs}^{B_3}, \omega_{ijs}^{B_3})\right)\right] \\
&= \left[\left(\min\{(\rho_{ijs}^{B_1}, \rho_{ijs}^{B_2}), \rho_{ijs}^{B_3}\}, (\min\{(\tau_{ijs}^{B_1}, \tau_{ijs}^{B_2}), \tau_{ijs}^{B_3}\}, , \max\{(\omega_{ijs}^{B_1}, \omega_{ijs}^{B_2}), \omega_{ijs}^{B_3}\})\right)\right] \\
&= \left[\left(\min\{(\rho_{ijs}^{B_1}, (\rho_{ijs}^{B_2}, \rho_{ijs}^{B_3}))\}, \big(\min\{(\tau_{ijs}^{B_1}, (\tau_{ijs}^{B_2}, \tau_{ijs}^{B_3}))\}, \max\{\omega_{ijs}^{B_1}, (\omega_{ijs}^{B_2}, \omega_{ijs}^{B_3})\}\big)\right)\right] \\
&= B_1 \cap (B_2 \cap B_3).
\end{aligned}
$$

On similar lines, (iii), (iv) and (v) can be proved accordingly.

Proposition 5 *For B_1, B_2 and $B_3 \in PFHSM_{n\times m}$, the following distributive laws hold:*

(i) $B_1 \cap (B_2 \cup B_3) = (B_1 \cap B_2) \cup (B_1 \cap B_3)$

(ii) $(B_1 \cap B_2) \cup B_3 = (B_1 \cup B_3) \cap (B_2 \cup B_3)$

(iii) $B_1 \cup (B_2 \cap B_3) = (B_1 \cup B_2) \cap (B_1 \cup B_3)$

(iv) $(B_1 \cup B_2) \cap B_3 = (B_1 \cap B_3) \cup (B_2 \cap B_3)$

(v) $(B_1 \cap B_2) @ B_3 = (B_1 @ B_3) \cap (B_2 @ B_3)$

(vi) $(B_1 \cap B_2) \bowtie B_3 = (B_1 \bowtie B_3) \cap (B_2 \bowtie B_3)$

(vii) $B_1 \cup (B_2 @ B_3) = (B_1 \cup B_2) @ (B_1 \cup B_3)$

(viii) $(B_1 \cup B_2) \bowtie B_3 = (B_1 \bowtie B_3) \cup (B_2 \bowtie B_3)$

(ix) $B_1 @ (B_2 \cup B_3) = (B_1 @ B_2) \cup (B_1 @ B_3)$

(x) $B_1 @ (B_2 \cap B_3) = (B_1 @ B_2) \cap (B_2 @ B_3)$

(xi) $B_1 \$ (B_2 \cup B_3) = (B_1 \$ B_2) \cup (B_1 \$ B_3)$

(xii) $(B_1 \cup B_2) \$ B_3 = (B_1 \$ B_3) \cup (B_2 \$ B_3)$

(xiii) $B_1 \cup (B_2 \bowtie B_3) = (B_1 \cup B_2) \bowtie (B_1 \cup B_3)$

(xiv) $B_1 \bowtie (B_2 \cup B_3) = (B_1 \bowtie B_2) \cup (B_1 \bowtie B_3)$

(xv) $B_1 \$ (B_2 \cap B_3) = (B_1 \$ B_2) \cap (B_2 \$ B_3)$

(xvi) $(B_1 \cap B_2) \$ B_3 = (B_1 \$ B_3) \cap (B_2 \$ B_3)$.

Proof :

(i)

$$B_1 \cap (B_2 \cup B_3) = \Bigg[\bigg(\big[(\rho_{ijs}^{B_1}, \tau_{ijs}^{B_1}, \omega_{ijs}^{B_1})\big] \cap \big[\big(\max\{\rho_{ijs}^{B_2}, \rho_{ijs}^{B_3}\}, \min\{\tau_{ijs}^{B_2}, \tau_{ijs}^{B_3}\}, \min\{\omega_{ijs}^{B_2}, \omega_{ijs}^{B_3}\}\big)\big]\bigg)\Bigg]$$
$$= \Bigg[\bigg(\min\{\rho_{ijs}^{B_1}, \max\{\rho_{ijs}^{B_2}, \rho_{ijs}^{B_3}\}\}, \min\{\tau_{ijs}^{B_1}, \min\{\tau_{ijs}^{B_2}, \rho_{ijs}^{B_3}\}\},$$
$$\max\{\omega_{ijs}^{B_1}, \min\{\omega_{ijs}^{B_2}, \omega_{ijs}^{B_3}\}\}\bigg)\Bigg].$$

Now,

$$(B_1 \cap B_2) \cup (B_1 \cap B_3) = \Big[\big(\min\{\rho_{ijs}^{B_1}, \rho_{ijs}^{B_2}\}, \min\{\tau_{ijs}^{B_1}, \tau_{ijs}^{B_2}\}, \max\{\omega_{ijs}^{B_1}, \omega_{ijs}^{B_2}\}\big)\Big] \cup \Big[\big(\min\{\rho_{ijs}^{B_1}, \rho_{ijs}^{B_3}\},$$
$$\min\{\tau_{ijs}^{B_1}, \tau_{ijs}^{B_3}\}, \max\{\omega_{ijs}^{B_1}, \omega_{ijs}^{B_3}\}\big)\Big]$$
$$= \Big[\big(\max(\min\{\rho_{ijs}^{B_1}, \rho_{ijs}^{B_2}\}, \min\{\rho_{ijs}^{B_1}, \rho_{ijs}^{B_3}\}), \min(\min\{\tau_{ijs}^{B_1}, \tau_{ijs}^{B_2}\}, \min\{\tau_{ijs}^{B_1}, \tau_{ijs}^{B_3}\}),$$
$$\min(\max\{\omega_{ijs}^{B_1}, \omega_{ijs}^{B_2}\}, \max\{\omega_{ijs}^{B_1}, \omega_{ijs}^{B_3}\})\big)\Big]$$
$$= \Big[\big(\max(\rho_{ijs}^{B_1}, \min\{\rho_{ijs}^{B_2}, \rho_{ijs}^{B_3}\}), \min(\tau_{ijs}^{B_1}, \min\{\tau_{ijs}^{B_2}, \tau_{ijs}^{B_3}\}),$$
$$\min(\omega_{ijs}^{B_1}, \max\{\omega_{ijs}^{B_2}, \omega_{ijs}^{B_3}\})\big)\Big]$$
$$= \Big[\big(\min(\rho_{ijs}^{B_1}, \max\{\rho_{ijs}^{B_2}, \rho_{ijs}^{B_3}\}), \min(\rho_{ijs}^{B_1}, \min\{\rho_{ijs}^{B_2}, \rho_{ijs}^{B_3}\}), \max(\omega_{ijs}^{B_1},$$
$$\min\{\omega_{ijs}^{B_2}, \omega_{ijs}^{B_3}\})\big)\Big] = B_1 \cap (B_2 \cup B_3).$$

Hence, $B_1 \cap (B_2 \cup B_3) = (B_1 \cap B_2) \cup (B_1 \cap B_3)$ holds.

(ii)

$$(B_1 \cap B_2) \cup B_3 = \big[\big(\min\{\rho_{ijs}^{B_1}, \rho_{ijs}^{B_2}\}, \min\{\tau_{ijs}^{B_1}, \tau_{ijs}^{B_2}\}, \max\{\omega_{ijs}^{B_1}, \omega_{ijs}^{B_2}\}\big)\big] \cup \big[\big(\rho_{ijs}^{B_3}, \tau_{ijs}^{B_3}, \omega_{ijs}^{B_3}\big)\big]$$
$$= \Big[\big(\max(\min\{\rho_{ijs}^{B_1}, \rho_{ijs}^{B_2}\}, \rho_{ijs}^{B_3}), \min(\min\{\tau_{ijs}^{B_1}, \tau_{ijs}^{B_2}\}, \rho_{ijs}^{B_3}),$$
$$\min(\max\{\omega_{ijs}^{B_1}, \omega_{ijs}^{B_2}\}, \omega_{ijs}^{B_3})\big)\Big].$$

Now,

$$(B_1 \cup B_3) \cap (B_2 \cup B_3) = \big[\big(\max\{\rho_{ijs}^{B_1}, \rho_{ijs}^{B_3}\}, \min\{\tau_{ijs}^{B_1}, \tau_{ijs}^{B_3}\}, \min\{\omega_{ijs}^{B_1}, \omega_{ijs}^{B_3}\}\big)\big] \cap \big[\big(\max\{\rho_{ijs}^{B_2}, \rho_{ijs}^{B_3}\},$$
$$\min\{\tau_{ijs}^{B_2}, \tau_{ijs}^{B_3}\}, \min\{\omega_{ijs}^{B_2}, \omega_{ijs}^{B_3}\}\big)\big]$$
$$= \Big[\big(\min(\max\{\rho_{ijs}^{B_1}, \rho_{ijs}^{B_3}\}, \max\{\rho_{ijs}^{B_2}, \rho_{ijs}^{B_3}\}), \min(\min\{\tau_{ijs}^{B_1}, \tau_{ijs}^{B_3}\}, \min\{\tau_{ijs}^{B_2}, \tau_{ijs}^{B_3}\}),$$
$$\max(\min\{\omega_{ijs}^{B_1}, \omega_{ijs}^{B_3}\}, \min\{\omega_{ijs}^{B_2}, \omega_{ijs}^{B_3}\})\big)\Big]$$
$$= \Big[\big(\min(\max\{\rho_{ijs}^{B_1}, \rho_{ijs}^{B_2}\}, \rho_{ijs}^{B_3}), \min(\min\{\tau_{ijs}^{B_1}, \tau_{ijs}^{B_2}\}, \tau_{ijs}^{B_3}),$$
$$\max(\min\{\omega_{ijs}^{B_1}, \omega_{ijs}^{B_2}\}, \omega_{ijs}^{B_3})\big)\Big]$$
$$= \Big[\big(\max(\min\{\rho_{ijs}^{B_1}, \rho_{ijs}^{B_2}\}, \rho_{ijs}^{B_3}), \min(\min\{\tau_{ijs}^{B_1}, \tau_{ijs}^{B_2}\}, \tau_{ijs}^{B_3}),$$
$$\min(\max\{\omega_{ijs}^{B_1}, \omega_{ijs}^{B_2}\}, \omega_{ijs}^{B_3})\big)\Big] = (B_1 \cap B_2) \cup B_3$$

Hence, $(B_1 \cap B_2) \cup B_3 = (B_1 \cup B_3) \cap (B_2 \cup B_3)$.

On a similar pattern, the rest of the laws can be proved accordingly.

5.2 Application of PFHSM in Renewable Energy Source Selection

In this section, we consider a basic framework of the renewable energy source selection problem where the formulation of the problem has been considered to be in the form of a picture fuzzy hypersoft matrix & proposed some revised definitions keeping the necessity of the problem into account.

Problem Statement (Renewable Energy Source Selection):
Suppose we have a set of m renewable energy resources $X = \{x_1, x_2, \ldots, x_m\}$ which are to be evaluated against n parameters (criteria) $Z = \{z_1, z_2, \ldots, z_n\}$ having further a set of k sub-attribute's parameters $Q = \{q_1, q_2, \ldots, q_k\}$. For the sake of the best possible selection of the available renewable energy sources, suppose that a committee gets constituted, say, with two experts (decision-makers) having adequate knowledge of the field of engineering, economics, management, government services and national energy policies, etc. The computation and the procedure of the decision-making should yield the best suitable source of renewable energy given all the interrelated parameters and sub-parameters. In case we take up a very formal selection process structure where the nature of information is accounted as a picture fuzzy hypersoft matrix then we need to propose some notions in a revised format that are essential for solving such MCDM problem. In view of the widely utilized structure of a decision-making problem and taking the proposed notion of picture fuzzy hypersoft matrices into consideration, we express the following revised definition of choice matrix and weighted choice matrix:

Definition 36 *If* $B_1 = [(\rho_{ijs}^{B_1}, \tau_{ijs}^{B_1}, \omega_{ijs}^{B_1})] \in PFHSM_{n \times m}$*,then the* ***choice matrix***

*of PFHSM (**PFHSCM**) B_1, in case the weights are same, is defined as*

$$C(B_1) = \left[\left(\frac{\sum_{js=1}^{n}(\rho_{ijs}^{B_1})^q}{n}, \frac{\sum_{js=1}^{n}(\tau_{ijs}^{B_1})^q}{n}, \frac{\sum_{js=1}^{n}(\omega_{ijs}^{B_1})^q}{n}\right)\right]_{n \times 1} ; \forall\, i.$$

Definition 37 *If $B_1 = [(\rho_{ijs}^{B_1}, \tau_{ijs}^{B_1}, \omega_{ijs}^{B_1})] \in PFHSM_{n \times m}$, then the **weighted choice matrix** of PFHSM (**PFHSWCM**) B_1, where $w_{js} > 0$ are weights, is defined by*

$$C_w(B_1) = \left[\left(\frac{\sum_{js=1}^{n} w_{js}(\rho_{ijs}^{B_1})^q}{\sum w_{js}}, \frac{\sum_{js=1}^{n} w_{js}(\tau_{ijs}^{B_1})^q}{\sum w_{js}}, \frac{\sum_{js=1}^{n} w_{js}(\omega_{ijs}^{B_1})^q}{\sum w_{js}}\right)\right]_{n \times 1} \forall\, i.$$

By making use of the revised choice matrix/weighted choice matrix, we present a new technique to handle the MCDM problem which is being presented with the help of Figure 5.1.

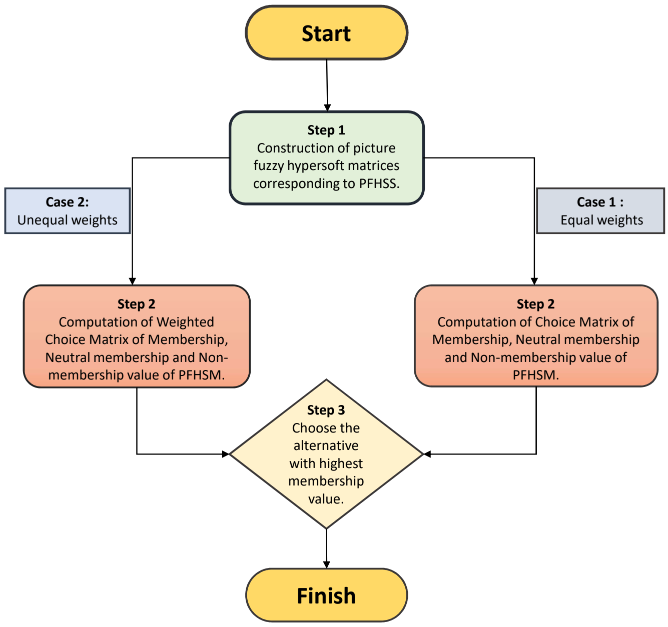

Figure 5.1: MCDM with Choice/Weighted Choice Picture Fuzzy Hypersoft Matrix

For the sake of better understanding and readability of the proposed methodology, the essential procedural steps are listed as follows:

Algorithm: I (MCDM Using Choice and Weighted Choice Picture Fuzzy Hypersoft Matrices)

Step 1: Construct the picture fuzzy hypersoft matrices corresponding to the picture fuzzy hypersoft sets.
Step 2: Compute the choice matrix of membership, neutral membership and non-membership value of the picture fuzzy hypersoft matrix.
Step 3: Compute the weighted choice matrix of membership, neutral membership and non-membership value of picture fuzzy hypersoft matrix.
Step 4: Choose the alternative with the highest membership value.

Remark: In case of any tie, we select the alternative with the highest membership value and the lowest non-membership value.

Further, in addition to the above methodology for MCDM, we propose an alternative technique where the notion of Value matrix and Score matrix in the form of picture fuzzy hypersoft information is utilized which is found to be more suitable and consistent.

Definition 38 *Let* $B = [B_{ij}]$ *be the PFHSM of order* $n \times m$, *where* $B_{ij} = (\rho^B_{ijs}, \tau^B_{ijs}, \omega^B_{ijs})$ *then the* ***value matrix*** *of* B *(**PFHSVM**) is denoted by* $\delta(B)$ *and is defined by* $\delta(B) = [B^B_{ij}]$ *of order* $n \times m$, *where* $B^B_{ij} = \rho^B_{ijs} - \tau^B_{ijs} - \omega^B_{ijs}$.

Definition 39 *Let* $B = [B_{ij}]$ *and* $C = [C_{ij}]$ *be two picture fuzzy hypersoft matrices of order* $n \times m$ *then the* ***score matrix*** *of* B *and* C *is given by* $\Gamma(B, C) = \delta(B) + \delta(C)$ *and* $\Gamma(B, C) = [\Gamma_{ij}]$ *where* $\Gamma_{ij} = \delta^B_{ij} + \delta^C_{ij}$. *The total score of every member is given by* $|\sum_{j=1}^{n} \Gamma_{ij}|$.

Based on the above definitions of value matrix and score matrix, we outline Algorithm II for solving the MCDM problem as given in Figure 5.2:

Similarly, for a better understanding and readability of the proposed methodology, the essential procedural steps are listed as follows:

Algorithm: II (MCDM Using Value & Score Picture Fuzzy Hypersoft Matrices)

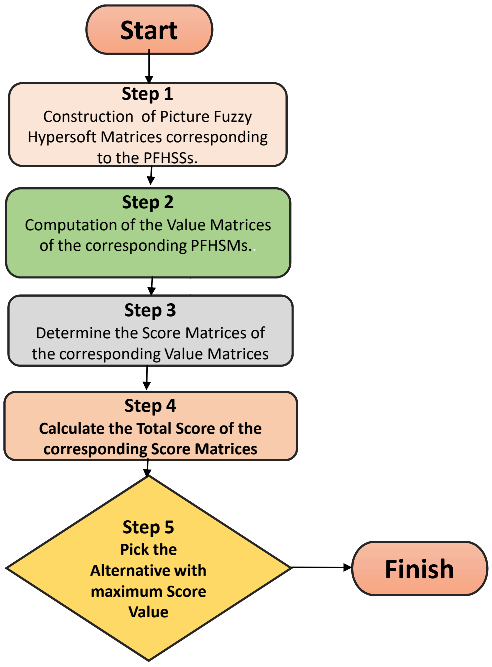

Figure 5.2: MCDM with Value/Score Picture Fuzzy Hypersoft Matrix

Step 1: Construct the picture fuzzy hypersoft matrices corresponding to the picture fuzzy hypersoft sets.

Step 2: Compute the value matrix obtained from PFHSM. Let $B = [B_{ij}]$ be the PFHSM of order $n \times m$, where $B_{ij} = (\rho^B_{ijs}, \tau^B_{ijs}, \omega^B_{ijs})$ then the **value matrix** of B is denoted by $\delta(B)$ and is defined by $\delta(B) = [B^B_{ij}]$ of order $n \times m$, where $B^B_{ij} = \rho^B_{ijs} - \tau^B_{ijs} - \omega^B_{ijs}$.

Step 3: Then compute the score matrix by making use of the value matrix obtained from step 2. Let $B = [B_{ij}]$ and $C = [C_{ij}]$ be two picture fuzzy hypersoft matrices of order $n \times m$ then the **score matrix** of B and C is given by $\Gamma(B, C) = \delta(B) + \delta(C)$ and $\Gamma(B, C) = [\Gamma_{ij}]$ where $\Gamma_{ij} = \delta^B_{ij} + \delta^C_{ij}$.

Step 4: Calculate the total score of every member obtained from the score matrix. The total score of every member is given by $|\sum_{j=1}^{n} \Gamma_{ij}|$.

Step 5: Select the alternative with the maximum score value obtained from the total score matrix (**PFHSTSM**).

5.3 Numerical Illustration of RES Selection Problem

Suppose $X = \{x_1,\ x_2,\ x_3,\ x_4,\ x_5\}$ be a set of renewable energy sources, where $x_1, x_2,\ x_3,\ x_4,\ x_5$ represents solar energy, wind energy, geothermal energy, hydropower and biomass energy, respectively. These renewable energy sources are to be examined against the criteria given by $Z = \{z_1,\ z_2,\ z_3, z_4,\ z_5,\ z_6\}$ and $z_1,\ z_2,\ z_3,\ z_4,\ z_5, z_6$ represents cost, environmental friendly, yields, maintenance, reliability and less number of peoples are effected from this project. A committee consists of two experts having knowledge of the field of engineering, economics, management, government services and policy-making for the best possible selection of the available resource. To formulate the problem into picture fuzzy hypersoft information let us consider the further sub-attributes of the above attributes given by

- Cost $= z_1 = \{z_{11} = average,\ z_{12} = moderate\}$,
- Environmental Friendly $= z_2$,
- Yields $= z_3$,
- Maintenance $= z_4 = \{z_{41} = \text{predictive}, z_{42} = \text{preventive},\}$,
- Reliability $= z_5 = \{z_{51} = \text{internal},\ z_{52} = \text{external},\}$,
- People effected from project $= z_6$.

Let $Z' = z_1 \times z_2 \times z_3 \times z_4 \times z_5 \times z_6$ be a set of sub-attributes which is explicitly given by

$$= \left\{ \begin{pmatrix} (z_{11}, z_2, z_3, z_{41}, z_{51}, z_6),\ (z_{11}, z_2, z_3, z_{41}, z_{52}, z_6), (z_{11}, z_2, z_3, z_{42}, z_{51}, z_6), (z_{11}, z_2, z_3, z_{42}, z_{52}, z_6) \\ (z_{12}, z_2, z_3, z_{41}, z_{51}, z_6),\ (z_{12}, z_2, z_3, z_{41}, z_{52}, z_6), (z_{12}, z_2, z_3, z_{42}, z_{51}, z_6), (z_{12}, z_2, z_3, z_{42}, z_{52}, z_6) \end{pmatrix} \right\}$$

For the calculation purpose set of all sub-attributes can be restated as

$$\mathbb{Z}' = \left\{ z_1',\ z_2',\ z_3',\ z_4',\ z_5',\ z_6',\ z_7',\ z_8' \right\}$$

Also, the study on various alternatives and criterions are also shown in Figure 5.3

Next, we illustrate the implementation of the proposed algorithms (Algorithm I and Algorithm II) by taking a numerical example existing in literature which has been

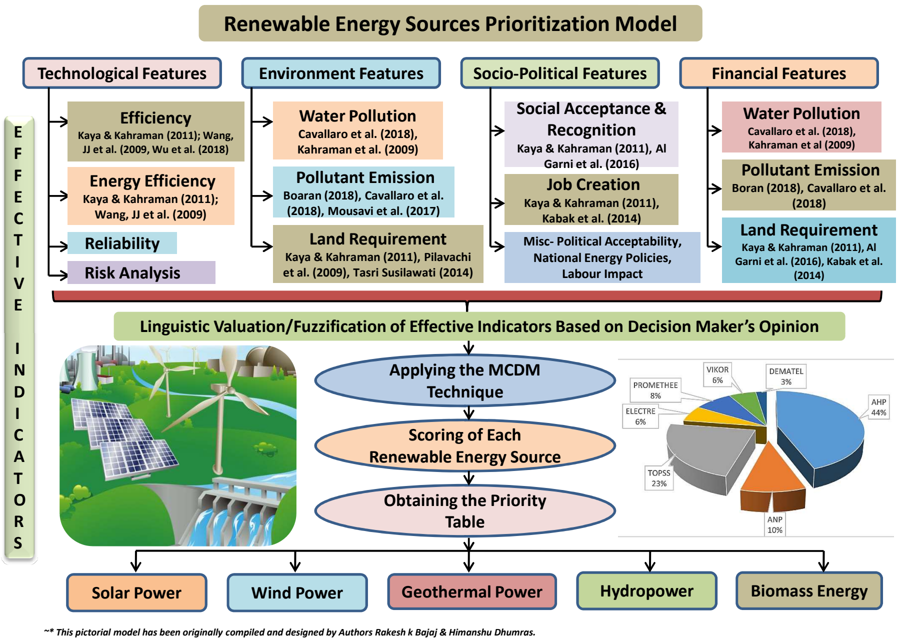

Figure 5.3: Role of Effective Indicators in RESs Prioritization Model

studied by Feng et al. [114] and Khan et al. [115], [116].

Algorithm I (MCDM Using Choice and Weighted Choice Picture Fuzzy Hypersoft Matrices)

Step 1: The situations are examined by the experts in terms of PFHSMs.

Note: In the picture fuzzy hypersoft matrix (Table 6.7), the first element $(0.1, 0.1, 0.8)$

Table 5.6: Decision Matrix given by First Panel of Experts

	x_1	x_2	x_3	x_4	x_5
z_1'	(0.1,0.1,0.8)	(0.2,0.1,0.6)	(0.3,0.0,0.7)	(0.1,0.1,0.8)	(0.4,0.0,0.5)
z_2'	(0.6,0.1,0.2)	(0.7,0.0,0.2)	(0.8,0.1,0.1)	(0.9,0.0,0.1)	(0.6,0.1,0.3)
z_3'	(0.4,0.1,0.4)	(0.4,0.1,0.5)	(0.6,0.1,0.3)	(0.6,0.1,0.2)	(0.7,0.1,0.2)
z_4'	(0.7,0.1,0.2)	(0.5,0.3,0.1)	(0.7,0.1,0.1)	(0.5,0.1,0.3)	(0.2,0.1,0.6)
z_5'	(0.6,0.2,0.1)	(0.6,0.1,0.2)	(0.2,0.2,0.5)	(0.7,0.1,0.2)	(0.7,0.1,0.1)
z_6'	(0.7,0.0,0.3)	(0.5,0.1,0.3)	(0.7,0.1,0.1)	(0.6,0.1,0.2)	(0.3,0.2,0.5)
z_7'	(0.3,0.2,0.4)	(0.1,0.2,0.4)	(0.4,0.2,0.3)	(0.4,0.1,0.2)	(0.5,0.1,0.3)
z_8'	(0.1,0.2,0.3)	(0.2,0.3,0.4)	(0.1,0.5,0.3)	(0.4,0.3,0.2)	(0.3,0.2,0.1)

defines the degree to which the criterion z_1' is satisfied by the alternative x_1 is 0.1 whereas the degree to which the criterion z_1' is not satisfied by the alternative x_1 is 0.8 and the degree of neutral membership is 0.1.

Step 2: Based on the above PFHSM, we construct the respective Choice matrices for the PFHSM given by the experts.

Case 1: (Equal Weights) Here, we assume the equal preference for all the criteria/subcriteria and we calculate the picture fuzzy hypersoft choice matrix as follows:

$$\begin{bmatrix} (\ 0.2463, 0.02, 0.1538\ ) \\ (\ 0.2, 0.0325, 0.1386\ ) \\ (\ 0.2850, 0.0463, 0.1300\ ) \\ (\ 0.3250, 0.12, 0.1175\ ) \\ (\ 0.2462, 0.0163, 0.1375\ ) \end{bmatrix}$$

Case 2: (UnequalCase weights) Based on the decision-makers opinion, if different weights $0.1,\ 0.1,\ 0.1,\ 0.1,\ 0.2,\ 0.1,\ 0.15, 0.15$ have been assigned for the set of all sub-attributes

$$\mathbb{Z}' = \left\{ z_1',\ z_2',\ z_3',\ z_4',\ z_5',\ z_6',\ z_7',\ z_8' \right\}$$

respectively, then the picture fuzzy hypersoft weighted choice matrix is being obtained as follows:

$$\begin{bmatrix} (\ 0.238, 0.024, 0.1365\ ) \\ (\ 0.1985, 0.0335, 0.131\ ) \\ (\ 0.2405, 0.0555, 0.138\ ) \\ (\ 0.325, 0.021, 0.102\ ) \\ (\ 0.263, 0.0165, 0.116\ ) \end{bmatrix}$$

Step 3: Analysis

Case 1: Equal Weights As per Step 2, if the equal preferences are given to all sub-attributes then from the choice matrix obtained having the highest membership value is **0.3250**, which is of renewable energy source x_4, i.e., Hydropower energy. Hence, the most suitable renewable energy source would be Hydropower energy.

Case 2: Unequal Weights However, if the preferences are not equal, i.e., if the sub-attribute z_5' is preferred more than other sub-attributes then from the choice matrix obtained above, the highest membership value is **0.325** which is of renewable energy source x_4, i.e., Hydropower energy. Hence, again the most suited renewable energy source would be Hydropower energy.

Algorithm II (Using Value & Score Picture Fuzzy Hypersoft Matrices)

Step 1: For using Algorithm II, we need at least two decision-matrices which are given by (Table 6.7) and (Table 6.8).

Table 5.7: Decision Matrix given by Second Panel of Experts

	x_1	x_2	x_3	x_4	x_5
$z_1^{'}$	(0.8,0.1,0.1)	(0.6,0.1,0.2)	(0.3,0.0,0.7)	(0.8,0.1,0.1)	(0.4,0.1,0.5)
$z_2^{'}$	(0.6,0.2,0.2)	(0.7,0.1,0.2)	(0.1,0.0,0.8)	(0.1,0.0,0.9)	(0.6,0.0,0.3)
$z_3^{'}$	(0.4,0.2,0.4)	(0.4,0.0,0.5)	(0.6,0.1,0.3)	(0.6,0.2,0.2)	(0.7,0.1,0.2)
$z_4^{'}$	(0.7,0.0,0.2)	(0.5,0.2,0.1)	(0.7,0.2,0.1)	(0.5,0.2,0.3)	(0.6,0.2,0.2)
$z_5^{'}$	(0.6,0.1,0.1)	(0.6,0.2,0.2)	(0.5,0.3,0.2)	(0.7,0.0,0.2)	(0.7,0.1,0.1)
$z_6^{'}$	(0.7,0.0,0.3)	(0.5,0.2,0.3)	(0.1,0.0,0.7)	(0.6,0.2,0.2)	(0.3,0.1,0.5)
$z_7^{'}$	(0.4,0.2,0.3)	(0.4,0.2,0.1)	(0.3,0.2,0.4)	(0.4,0.3,0.2)	(0.3,0.1,0.5)
$z_8^{'}$	(0.3,0.2,0.1)	(0.4,0.3,0.2)	(0.3,0.5,0.1)	(0.4,0.1.0.2)	(0.1,0.2,0.3)

Step 2: Next, we construct the value matrices from the provided picture fuzzy hypersoft matrices obtained in Step 1.

$$\delta(B) = \begin{bmatrix} -0.8 & 0.3 & -0.1 & 0.4 & 0.3 & 0.4 & -0.3 & -0.4 \\ -0.5 & 0.5 & -0.2 & 0.1 & 0.3 & 0.1 & -0.5 & -0.5 \\ -0.4 & 0.6 & 0.2 & 0.5 & -0.5 & 0.5 & -0.1 & -0.4 \\ -0.8 & 0.8 & 0.3 & 0.1 & 0.4 & 0.3 & 0.1 & -0.7 \\ -0.1 & 0.2 & 0.4 & -0.5 & 0.5 & -0.4 & 0.1 & 0.0 \end{bmatrix}$$

$$\delta(C) = \begin{bmatrix} 0.6 & 0.2 & -0.2 & 0.5 & 0.4 & 0.4 & -0.1 & 0.0 \\ 0.3 & 0.4 & -0.1 & 0.2 & 0.2 & 0.0 & 0.1 & -0.1 \\ -0.4 & -0.7 & 0.2 & 0.4 & 0.0 & -0.6 & -0.3 & -0.3 \\ 0.6 & -0.8 & 0.2 & 0.0 & 0.5 & 0.2 & -0.1 & 0.1 \\ -0.2 & 0.3 & 0.4 & 0.2 & 0.5 & -0.3 & -0.3 & -0.4 \end{bmatrix}$$

Step 3: Further, we calculate the score matrices by the above two value matrices:

$$\Gamma(B,C) = \begin{bmatrix} -0.2 & 0.5 & -0.3 & 0.9 & 0.7 & 0.8 & -0.4 & -0.4 \\ -0.2 & 0.9 & -0.3 & 0.3 & 0.5 & 0.1 & -0.4 & -0.6 \\ -0.8 & -0.1 & 0.4 & 0.9 & -0.5 & -0.1 & -0.4 & -1.0 \\ -0.2 & 0.0 & 0.5 & 0.1 & 0.9 & 0.5 & 0.0 & 0.0 \\ -0.3 & 0.5 & 0.8 & -0.3 & 1.0 & -0.7 & -0.2 & -0.4 \end{bmatrix}$$

Step 4: The total score of the above score matrix is given by

$$\begin{bmatrix} 1.6 \\ 0.3 \\ 1.2 \\ 1.8 \\ 1.1 \end{bmatrix}$$

Step 5: Now based on the above total score values, the maximum value comes out to be **1.8** which corresponds to the alternative x_4, i.e., Hydropower energy. Hence, the most suitable renewable energy source based on the total score value obtained by the proposed algorithm will be hydropower energy. The comparative score values and their ranking can be observed in Figure 5.4.

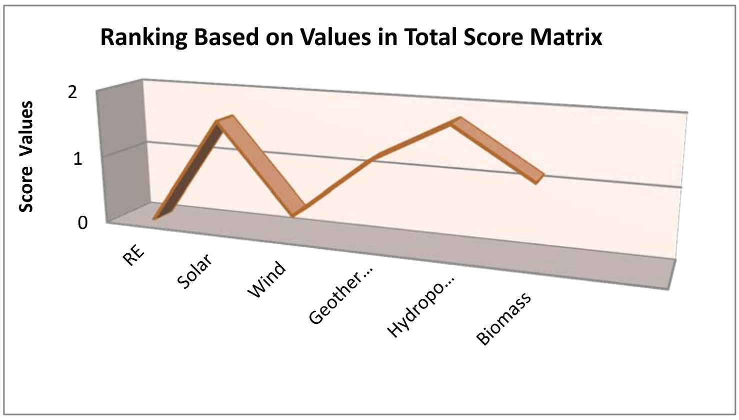

Figure 5.4: Ranking of Renewable Energy Sources Based on Score Matrix

5.4 Comparative Analysis & Advantages

In this section, we discuss the functionality, receptiveness, and conformity of the proposed notion and methodology in contrast with the existing techniques. In addition to this, some advantages and discussions over the obtained results have also been presented for better understanding and readability.

In view of the numerical example under consideration and the results obtained through the existing techniques by various researchers, we present the ranking of the alternatives for the decision-making problem as shown in Table 5.8.

Table 5.8: Comparative Analysis

Method	**Operators/Method Used**	**Developed Ranking**
Feng et al.[114]	Extended Intersection, IFWA	$x_4 > x_1 > x_3 > x_2 > x_5$
Khan et al.[115]	Soft Discernibility Matrix	$x_4 > x_3 > x_1 > x_2 > x_5$
Garg[117]	PFEWA Operator	$x_4 > x_3 > x_1 > x_2 > x_5$
Yager[118]	PFWA Operator	$x_4 > x_3 > x_1 > x_2 > x_5$
Yager[118]	PFWG Operator	$x_4 > x_1 > x_3 > x_2 > x_5$
Khan et al.[116]	VIKOR I	$x_4 > x_2 > x_3 > x_1 > x_5$
Khan et al.[116]	VIKOR II	x_4 , $x_2 > x_3 > x_1 > x_5$
Khan et al.[116]	VIKOR III	$x_4 > x_1 > x_3 > x_2 > x_5$
Khan et al.[116]	VIKOR IV	$x_4 > x_2 > x_3 > x_5 > x_1$
Proposed	PFHSCM	$x_4 > x_3 > x_1 > x_5 > x_2$
Proposed	PFHSWCM	$x_4 > x_5 > x_3 > x_1 > x_2$
Proposed	PFHSVM & PFHSTSM	$x_4 > x_1 > x_3 > x_5 > x_2$

Important Remarks and Advantages:

- Finally, we can state that the proposed notion of picture fuzzy hypersoft matrix PFHSM) is a novel concept and a valid extension of fuzzy set/hypersoft set theories. The PFHSM has the added advantage of dealing with the wider sense of applicability in uncertain situations with the incorporation of the degree of refusal and abstain.
- The existing types of hypersoft sets - intuitionistic fuzzy hypersoft set [14], Pythagorean fuzzy hypersoft set [113], Neutrosophic hypersoft set [14] have their limitations because of the exclusion of refusal and abstain component.
- It may be noted that the categorically designed information having the picture fuzzy relation would not be possible to address with the help of existing hypersoft set theory in order to ensure a kind of parametrization in the relation.
- For the sake of an overall critical aspect, we observe that eventually with the picture fuzzy information, it won't be possible to suitably address those membership values (given by the decision-makers/experts) whose sum exceeds one. Such restrictions with respect to the decision-maker's opinion can be eradicated with the notion of T-spherical fuzzy information.

5.5 Conclusions

The inherited diversity found in the information and various criteria for choosing the most suitable energy source alternative prove to be an important task for the decision-making process. The proposed decision-making algorithms involve the choice matrix, weighted choice matrix, followed by value and score matrix which span the variability of the problem more mathematically. The main purpose of the chapter lies in proposing new fuzzy decision-making methods for evaluating and ranking the available renewable nergy sources based on different criteria. Consequently, we successfully illustrated and implemented th formal procedure for solving the problem of renewable energy source selection by utilizing PFHSCM, PFHSWCM, PFHSVM and PFHSTSM. Since the real world is full of uncertainty with various parameters and sub-parameters, the proposed methodologies exhibit the capability to simultaneously span a wider coverage of information in terms of multi-sub-attribute features and comprehensiveness of the expert's opinion. The comparative analysis clearly shows the advantageous features in contrast with the recent existing techniques.

Chapter 6

q-Rung Picture Fuzzy AHP & WASPAS Decision-Making Model

This chapter discusses how to handle uncertainty in the green supply chain management system processes by using an integrated approach that converts information into quantitatively measurable fuzzy sets and then uses those sets to inform decision-making techniques such as the Weighted Aggregated Sum Product Assessment (WASPAS) and Analytic Hierarchy Process (AHP). The suggested model, which outlines potential strategies for green supply chain management in the energy sector, has been applied step-by-step. The ideal detailed analysis has been provided at each stage of the investigation in an integrated way to address the fundamental issues with ideal decision-making. For improved comprehension, consistency, and dependability, a quick sensitivity analysis and comparative analysis of the prospective strategy plans with regard to the deterministic parameters have been carried out. The results of the proposals show that, in terms of future strategic plans, prioritising the use of customer relationship management to meet customer needs is more important than looking at the process for creating new services and products.

6.1 Proposed AHP/WASPAS with q-RPFSs and GSCM Problem Formulation

The algorithmic details of the strategies used in the proposed study are described in this section.

The weights iforieach criterionian alternative are found using pairwise comparisons in this manner. There are, nevertheless, several variations to this approach, and fuzzy information-based AHP is the trick to prioritising more precisely.

6.1.1 Modified Analytic Hierarchy Process (AHP) with q-RPFS

In the domain of decision-making, Saaty [60] invented the AHP division of structure techniques, which divide a complex problem into several hierarchical levels. The weights iforieach criterionian alternative are found using pairwise comparisons in this manner. There are, nevertheless, several variations to this approach, and fuzzy information-based AHP is the trick to prioritising more precisely.

Consider $\mathbb{A} = \{A_1, A_2, \ldots, A_m\}$ be the set of alternatives which are available and $\mathbb{E} = \{E_1, E_2, \ldots, E_n\}$ be the set of criteria to be evaluated against the set of alternatives. In order to have different judgments on the available alternatives in reference to each criterion, let $\mathbb{D} = \{D_1, D_2, \ldots, D_l\}$ be the set of decision-makers who provide assessment for the different alternatives against various criteria and pass on their judgments in the form of linguistic parameters. The various steps involved in the first stage AHP method are given in Figure6.1.

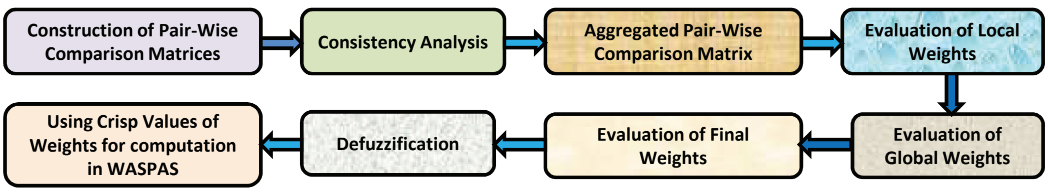

Figure 6.1: Procedural steps of modified AHP with q-RPFS

- **Step 1: Construction of Pair-wise Comparison Matrix**

  In the first step, construct the pair-wise comparison matrix $\mathtt{R} = (\mathtt{r}_{ij})_{n \times n}$ by converting the linguistic parameters into the corresponding q-rung picture fuzzy numbers (q-RPFNs) and Saaty's scale based on Table 6.3.

- **Step 2: Consistency Analysis**

  Next, we do consistency analysis for each of the pairwise comparison matrices given by the formula $CR = \frac{CI}{RI}$, where $CI = \frac{w_{max}}{n-1}$.
  Here, CR = Consistency Ratio, CI = Consistency Index, RI = Random Index and w_{max} = maximum eigenvalue of the pairwise comparison matrix.
  If $CR > 0.1$, then move to the next step else return to Step 1.

- **Step 3: Computing Aggregated Pair-wise Comparison Matrix**

  After doing consistency analysis, we aggregate all the pairwise comparison matrices by the decision-makers into a single matrix by taking average values of all the uncertainty components respectively.

- **Step 4: Evaluation of Local Weights**

  In this step, the local weights of each criterion are computed with the help of the following equation

$$\lambda_j{}^{local} = \left\{ \prod_{j=1}^{n} (\rho_{ij}^{q})^{1/n}, \sqrt[q]{1 - \prod_{j=1}^{n} (1 - \tau_{ij}^{q})^{1/n}}, \sqrt[q]{1 - \prod_{j=1}^{n} (1 - \omega_{ij}^{q})^{1/n}} \right\} \tag{6.1.1}$$

- **Step 5: Evaluation of Global Weights**

  In this step, the global weights of each criterion are computed with the help of the following equation

$$\lambda_j{}^{global} = \left\{ \prod_{j=1}^{l} (\rho_{ij}^{q})^{1/l}, \sqrt[q]{1 - \prod_{j=1}^{l} (1 - \tau_{ij}^{q})^{1/l}}, \sqrt[q]{1 - \prod_{j=1}^{l} (1 - \omega_{ij}^{q})^{1/l}} \right\} \tag{6.1.2}$$

- **Step 6: Evaluation of Final Weights**

  Next, the final q-rung picture fuzzy weights are computed for each criterion by making use of the following equation

$$\lambda_j{}^{final} = \lambda_j{}^{local} \otimes \lambda_j{}^{global} \tag{6.1.3}$$

- **Step 7: Defuzzification**
  In the final step, the q-rung picture fuzzy weights are defuzzified into crisp values by using the following identity

$$\lambda_j = \frac{\rho_{ij} + \tau_{ij} + \omega_{ij}}{3} \tag{6.1.4}$$

6.1.2 Modified Weighted Aggregated Sum Product Assessment (WASPAS) with q-RPFS

The WASPAS method was introduced by Zavadskas et al.[70] which fuses the weighted sum and weighted product models (WSM and WPM). This method is modified with q-rung picture fuzzy information to give more accurate and better results of prioritizing the available alternatives. The various steps involved in the second staged modified WASPAS method are given in Figure6.2.

- **Step 1: Construction of Expert Matrix**
  In the first step, the construction of an initial expert matrix from the linguistic variables obtained from the experts has been done. Then, the linguistic variables are converted into the q-rung fuzzy numbers (q-RPFNs).

- **Step 2: Computing Aggregated Expert Matrix**
  The second step is to aggregate all the q-rung picture fuzzy expert matrices into a single aggregated matrix as follows:

$$\mathbb{D} = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \cdots & E_n \end{array} \\ \begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{m1} & d_{m2} & \cdots & d_{mn} \end{pmatrix} \end{array} \tag{6.1.5}$$

  where $d_{ij} = \frac{1}{l}\left[\sum_{k=1}^{n} d_{ij}^{k}\right]$; l is the number of decision-makers.

- **Step 3: Utilizing Weighted Sum Model (WSM)**

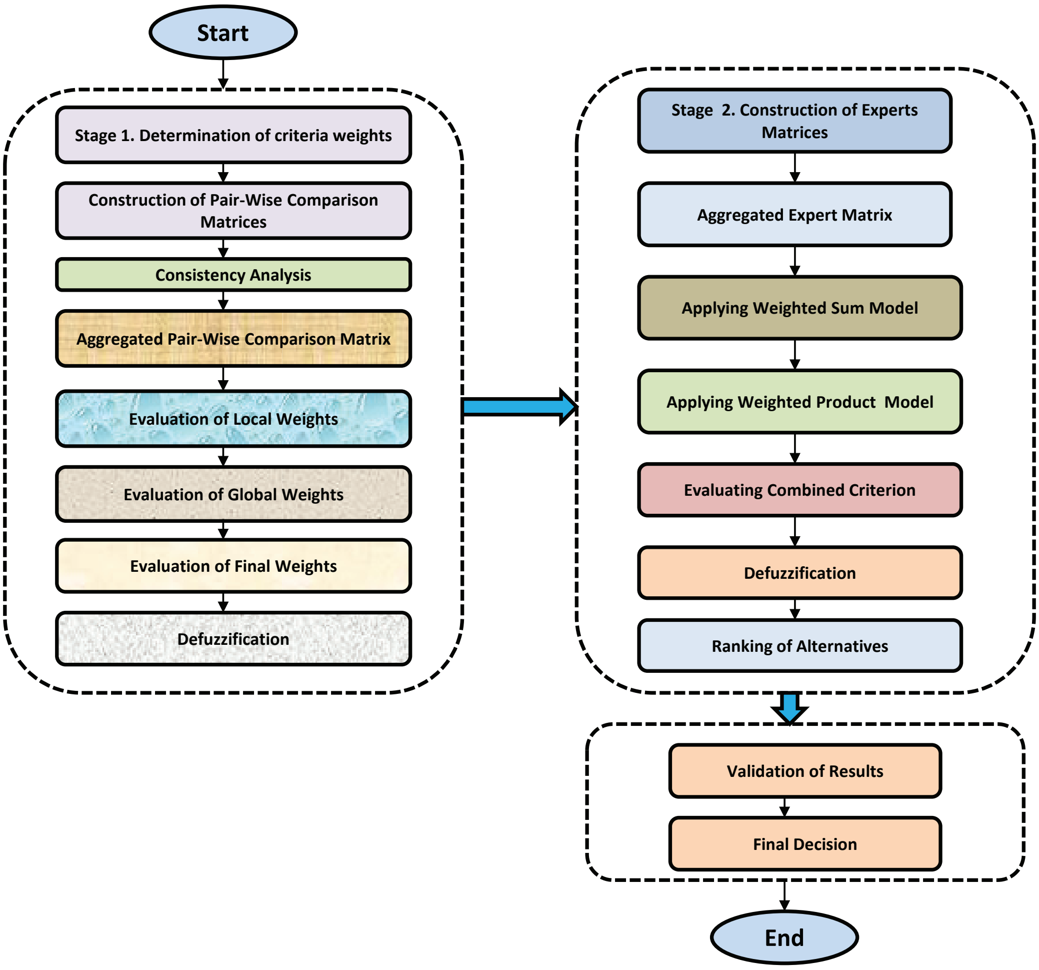

Figure 6.2: Procedural steps of 2nd staged modified WASPAS with q-RPFS

Compute the q-rung picture fuzzy weighted sum of every alternative as follows:

$$\mathcal{A}_i^{WSM} = \oplus \lambda_j d_{ij}) = \sqrt[q]{1 - \prod_{j=1}^{n}(\rho_{ij}^q)^{\lambda_j}}, \prod_{j=1}^{n}(\tau_{ij}^q)^{\lambda_j}, \prod_{j=1}^{n}(\omega_{ij}^q)^{\lambda_j}. \quad (6.1.6)$$

- **Step 4: Utilizing Weighted Product Model (WPM)**

  Compute the q-rung picture fuzzy weighted product of every alternative as follows:

$$\mathcal{A}_i^{WPM} = \otimes(d_{ij}^{\lambda_j}) = \prod_{j=1}^{n}(\rho_{ij}^q)^{\lambda_j}, \sqrt[q]{1 - \prod_{j=1}^{n}(1 - \tau_{ij}^q)^{\lambda_j}}, \sqrt[q]{1 - \prod_{j=1}^{n}(1 - \omega_{ij}^q)^{\lambda_j}}. \quad (6.1.7)$$

- **Step 5: Compute the Combined Criterion**

  Evaluate the q-rung picture fuzzy combined generalized criterion of every alternative as follows:

$$\mathcal{A}_i = y\mathcal{A}_i^{WSM} \oplus (1 - y)\mathcal{A}_i^{WPM}. \quad (6.1.8)$$

  where, y is a trade-off parameter between the two respective models and $y \in [0, 1]$.

- **Step 6: Defuzzification**

  In the final step, the q-rung picture fuzzy values are defuzzified into crisp values by using the following identity

$$\mathcal{S}_i = \rho_{\mathcal{A}_i} + \frac{\tau_{\mathcal{A}_i}}{2} + \frac{1 - (\rho_{\mathcal{A}_i} + \tau_{\mathcal{A}_i} + \omega_{\mathcal{A}_i})}{2}(1 + \rho_{\mathcal{A}_i} - \omega_{\mathcal{A}_i}). \quad (6.1.9)$$

- **Step 7: Ranking of Alternatives**

  The final ranking of the alternatives is to be performed based on the crisp values of the combined generalized criterion. The alternative with the maximum value will be preferred.

6.1.3 Proposed Model

In this section, we present a new hybrid fuzzy multi-criteria decision-making model by making use of various criteria for prioritizing the potential strategic plans of green supply chain management in the energy sector. The proposed model has been discussed with the help of the following stage-wise procedure shown in Figure 6.3.

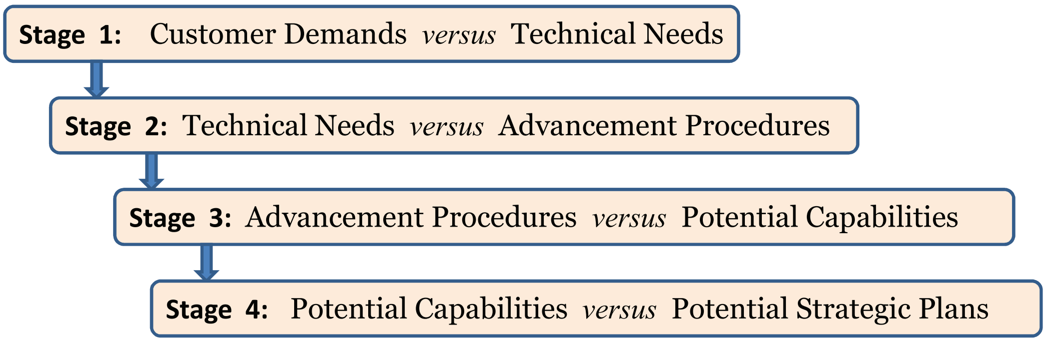

Figure 6.3: Stage-wise potential strategies for GSCM

The first stage involves the demands of the customers and their respective technical needs in the energy sector for green supply chain management. Firstly, the weighting of expectations of the customers is done by utilizing the q-RPF AHP. In the second stage, the technical needs along with the expectations of the customers are computed by utilizing q-RPF WASPAS. The obtained results are being utilized for the weights of the technical needs and the evaluation of the advancement procedures with the incorporation of q-RPF WASPAS. The third stage gives the results of innovating capabilities for the advancement procedures by making use of q-RPF WASPAS. In the final stage, the prioritization of potential strategic plans has been done on the basis of innovating capabilities utilizing q-RPF WASPAS. Therefore, the proposed method appropriately takes into account the successive computations that follow one another.

First, a variety of elements are taken into consideration when defining all the requirements and options for GSCM in the energy sector. These methodologies takes into account both the technical requirements of the various energy businesses and the desires of the customers [119]. Therefore, more feasible and agreeable possible strategy plans may be developed for these energy sectors [120]. Additionally, the computation of weights and the computation of prioritising results with different decision-making procedures are typically done using a hybrid MCDM technique [121]. This contribution opens the door to more objective contributions. Additionally, in the area of GSCM in the energy industry, q-RPF AHP and q-RPF WASPAS are considered.

6.2 Stage-wise Implementation of AHP/WASPAS in GSCM

In this section, various results are described in the following stages.

Stage 1. Customer demands versus technical needs:

The energy industries must now make the transition to customer-oriented management methods due to the rapidly changing markets and intense competition. The significance of a green supply chain's quality has also been considered in this regard. To substantially surpass their competitors in the market, energy sectors must be able to offer high-quality products to their customers at lower costs [122]. It must be necessary to dispense the items to be sold from the suppliers in the most ideal and appropriate circumstances, without delay in time, in order to make and dispatch to its clients on time, in order to achieve this goal. Thus, the importance of the supplier-customer connection has been recognised as having productive and effective GSCM. It will be easier for the energy sectors to have a strong level of competitiveness if this process is executed successfully. Furthermore, this situation will significantly enhance the energy sectors' performance. On the other hand, customer-focused supply chain management aids in lowering the procedure's customer-related hazards. For example, a company that understands that a customer needs a product on time may try to collaborate with its supplier to ensure that the required supplies arrive on time [123]. Using this strategy, one of the biggest supply chain risks—not being able to make timely goods purchases—might be mitigated. Furthermore, a company that pays close attention to customer expectations is aware of the demand for environmentally friendly products. By doing this, it will guarantee that the items must be environmentally sustainable and take this into account when selecting suppliers [124]. Consequently, supply chain management can mitigate environmental risk. Table 6.1 displays the aspects of consumer expectations as per the literature review. Table 6.1 lists customer requests towards green supply chain management. There have been significant differences in consumers' expectations for products, especially in the last several years. In the past, meeting consumer demand could be achieved simply by producing the goods, but more and more, new demands have emerged. Consumers prioritise these products' uniqueness in order to ensure that they completely meet

Table 6.1: Customer demands for the GSCM

Criteria	**Litearture**
Multifariousness of green distribution routes (CD1)	[153, 154]
Expenses for the sources of energy (CD2)	[155, 156]
Product waste recycling (CD3)	[157, 158]
Minimization of pollution and transmission (CD4)	[160, 162]
Effectiveness of energy sources with technological advancements (CD5)	[163, 164]

their needs [125]. Items ought to be extensive and reasonably priced to ensure client pleasure. Furthermore, the environmental friendliness of the items must be considered in the needs of the customers [126]. Environmentally friendly products are what the energy sectors strive to produce in order to simplify the GSCM. Concerns like reducing waste and carbon emissions will be given more attention in order to assist this [127]. Additionally, this will improve the company's reputation in the industry [128]. Firstly, the diversity of green distribution channels to expedite client delivery [129]. Another crucial requirement of the clients is that the energy sources be affordable. But for environmentally conscious consumers, proper waste recycling is also crucial [130]. Furthermore, if the energy industries release less carbon emissions and are less polluting, consumers will perceive them favourably [131]. Furthermore, increasing energy efficiency also requires advancements in the technology infrastructure [132]. With the advancement of technology, it will be possible to provide a good that meets the needs of more people. Additionally, Table 6.2 describes the technological requirements needed to meet these client's needs.

The five distinct technological demands that have been determined to satisfy customer requirements are covered in Table 6.2. First and foremost, money ought to be allocated for the GSCM sector's route expansion [133]. Based on this, the clients will be able to use a variety of distribution channels to provide speedier and better service. Furthermore, constructing the necessary infrastructure would improve the efficacy of cost management. In this case, the right people should be employed, and the required technical investments should be made. Furthermore, it's critical to thor-

Table 6.2: Technical Needs for the GSCM

Criteria	**Litearture**
Maximization of route volumes (TN1)	[165, 166]
Minimization on management costs (TN2)	[167, 168]
Planning of recycling process (TN3)	[158, 159]
Steady expenses on the minimization of pollution & transmission (TN4)	[169, 160]
Devising research & development schemes for energy effectiveness (TN5)	[161, 170]

Table 6.3: Qualitative variables with respect to alternatives

Linguistic Terms	**q-RPFNs**	**Scale**
"Absolutely High (AH)"	(0.95, 0.2, 0.2)	$\frac{1}{7}$
"Very High (VH)"	(0.9, 0.1, 0.1)	$\frac{1}{5}$
"High (H)"	(0.75, 0.2, 0.1)	$\frac{1}{3}$
"Medium High (MH)"	(0.6, 0.4, 0.3)	1
"Medium (M)"	(0.5, 0.3, 0.4)	3
"Medium Low (ML)"	(0.3, 0.3, 0.6)	5
"Low (L)"	(0.25, 0.2, 0.6)	7
"Very Low (VL)"	(0.1, 0.1, 0.85)	9

oughly prepare for the recycling process' reuse phase. This will enable the recycling procedure to be implemented properly and save expenses [134]. Customers' pleasure will also increase as a result, and the efforts made to reduce carbon emissions are equally important. In this case, spending money on carbon capture and technological storage can reduce air pollution. As a result, increasing research and development spending will make achieving energy efficiency much simpler [135]. In addition, three highly experienced professionals with knowledge of the energy sector are assigned to obtain the computations. Table 6.3 provides the linguistic scale for this in terms of q-RPFNs and Saaty numbers. According to this, the decisions made by all three experts on customer demands and the expert matrix of technical needs in association with the customer demands are given in Table 6.4 and Table 6.5. The following stage involves converting linguistic calculations into q-RPFNs for every aspect, and the q-RPF AHP is used to weight GSCM's customer needs in the energy industry. Now, using the consistency ratio formula, each pairwise comparison matrix's consis-

Table 6.4: Linguistic computation for customer demands

	CD_1	CD_2	CD_3	CD_4	CD_5
CD_1	(M, M, M)	(H, MH, ML)	(MH, H, VH)	(AH, AH, H)	(M, VH, H)
CD_2	(MH, M, VL)	(M, M, M)	(VH, MH, MH)	(VH, VH, VH)	(VH, M, MH)
CD_3	(VL, L, ML)	(ML, L, ML)	(M, M, M)	(VH, VL, ML)	(VL, VL, L)
CD_4	(L, M, M)	(VL, ML, ML)	(MH, VL, ML)	(M, M, M)	(ML, VL, VL)
CD_5	(VL, VL, VL)	(ML, L, L)	(VH, M, ML)	(VH, AH, AH)	(M, M, M)

Table 6.5: Linguistic computation for the technical needs

	TN_1	TN_2	TN_3	TN_4	TN_5
CD_1	(ML, MH, H)	(H, L, VH)	(VL, VL, ML)	(MH, MH, AH)	(MH, MH, H)
CD_2	(L, ML, L)	(ML, ML, VL)	(M, AH, MH)	(AH, MH, AH)	(AH, AH, VH)
CD_3	(H, VL, ML)	(M, MH, ML)	(AH, VH, VH)	(VL, VL, MH)	(M, VL, VL)
CD_4	(L, L, M)	(ML, M, M)	(L, M, L)	(VL, VL, ML)	(M, L, L)
CD_5	(M, M, M)	(VH, H, H)	(MH, M, M)	(H, MH, VH)	(H, H, VH)

tency ratio is calculated. The results show that each pairwise comparison matrix's consistency is less than ten percent, indicating that each pairwise comparison matrix is consistent. Next, Table 6.6 provides the combined pairwise comparison matrix of all the experts. Now, by using the aggregated pairwise comparison matrix and

Table 6.6: Aggregated pairwise comparison matrix

	CD_1	CD_2	CD_3	CD_4	CD_5
CD_1	$(0.5, 0.3, 0.4)$	$(0.55, 0.3, 0.3)$	$(0.75, 0.23, 0.17)$	$(0.88, 0.2, 0.17)$	$(0.72, 0.2, 0.23)$
CD_2	$(0.4, 0.27, 0.52)$	$(0.5, 0.3, 0.4)$	$(0.7, 0.3, 0.23)$	$(0.9, 0.1, 0.1)$	$(0.67, 0.27, 0.27)$
CD_3	$(0.09, 0.2, 0.68)$	$(0.28, 0.27, 0.6)$	$(0.5, 0.3, 0.4)$	$(0.43, 0.17, 0.52)$	$(0.15, 0.13, 0.77)$
CD_4	$(0.42, 0.27, 0.47)$	$(0.23, 0.23, 0.68)$	$(0.3, 0.27, 0.58)$	$(0.5, 0.3, 0.4)$	$(0.17, 0.17, 0.77)$
CD_5	$(0.1, 0.1, 0.85)$	$(0.27, 0.23, 0.6)$	$(0.57, 0.23, 0.37)$	$(0.43, 0.17, 0.17)$	$(0.5, 0.3, 0.4)$

the above-mentioned steps of q-RPF AHP, the weights of the customer demands are given by Table 6.7. From the results, q-RPF AHP, recycling of product waste (CD3) is the most important criterion for the demands of the customers, and multifariousness of green distribution routes (CD1) is the least important in the set of criteria. The profits of the energy sectors are affected by the costs to pay for the effectiveness of the energy and the sustainability of the environment [136]. Generally, customers prefer products that are environmentally friendly and do not harm the surroundings. This factor is useful for short-term benefit and the sectors can maximize their margin of profit by concentrating more on the customer demands [137]. However, energy efficiency and environmental sustainability are not that easy to adopt because of some technological inadequacies. To overcome such shortcomings, the energy sectors must

Table 6.7: Weights for customer demands

Criterions	**Weights(q=2)**
CD_1	0.18657
CD_2	0.19303
CD_3	0.20707
CD_4	0.20684
CD_5	0.20649

give relative importance to the studies of research and development [138].

Additionally, for the effectiveness of the energy and the sustainability of the environment, it is required to give more attention towards the awareness of environmental issues in all the phases starting from the earliest inputs to the last delivery of the last product [139]. For this context, the alternatives of renewable sources of energy must be taken into account, which results in a decrease in carbon emissions significantly. Further, the energy sectors must give importance to recycling and waste management. By this, energy efficiency can be increased and with the incorporation of all these aspects the risks in the green supply chain can be minimized and which results in enhanced performance.

Furthermore, the ideas of effectiveness of the energy and the sustainability of the environment have a favorable effect on various types of performances. Within this framework, costs of the energy sectors can be minimized over time by taking into account the aspect of energy efficiency [140]. This circumstance has a great contribution to the economic evaluations of the sectors. In addition to this, when nations concentrate on environmental sustainability, the customers are more dedicated to the respective energy sectors and these sectors are majorly preferred. This also has a greater impact on the effectiveness of the institutions, because of the representatives who are influenced by the effectiveness of the energy and the sustainability of the environment will gradually grow their job performance [141]. Further, the relationship between the customers and the suppliers is also very crucial for a clear understanding of the demands of the customers and also very careful about meeting the required necessities of the customers. Otherwise, it becomes very difficult to survive in competitive surroundings. Clearly, the energy sectors are required to modify their green supply chains in accordance with the demands of the customers [142] and should be enough flexible to reshape their chains over time with respect to the customer demands.

Now,, the needs of the customers along with the demands of the customers are computed by utilizing q-RPF WASPAS. In this step, customer demands and technical needs are calculated in a matrix. Within this framework, the fuzzy technique will be one of the most prominent for evaluating customer demands under vagueness and technical needs [143, 119]. The results for the needs of the customers are given in Table 6.8. In context with this, the weights of the needs of the customers are

Table 6.8: Expert matrix of technical needs

	TN_1	TN_2	TN_3	TN_4	TN_5
CE_1	$(0.55, 0.3, 0.33)$	$(0.63, 0.17, 0.27)$	$(0.17, 0.17, 0.77)$	$(0.72, 0.33, 0.27)$	$(0.65, 0.33, 0.23)$
CE_2	$(0.27, 0.24, 0.6)$	$(0.23, 0.23, 0.68)$	$(0.68, 0.3, 0.3)$	$(0.83, 0.27, 0.23)$	$(0.93, 0.17, 0.17)$
CE_3	$(0.38, 0.2, 0.52)$	$(0.55, 0.3, 0.33)$	$(0.92, 0.13, 0.13)$	$(0.27, 0.2, 0.67)$	$(0.23, 0.17, 0.7)$
CE_4	$(0.33, 0.23, 0.53)$	$(0.43, 0.3, 0.47)$	$(0.33, 0.23, 0.53)$	$(0.17, 0.17, 0.77)$	$(0.33, 0.23, 0.53)$
CE_5	$(0.5, 0.3, 0.4)$	$(0.8, 0.17, 0.1)$	$(0.53, 0.33, 0.37)$	$(0.75, 0.23, 0.17)$	$(0.8, 0.17, 0.1)$

evaluated by the above-mentioned steps of q-RPF WASPAS and given in Table 6.9.

Table 6.9: Weights for technical requirements

Criterions	**Weights(q=2)**
TN_1	0.20026
TN_2	0.21892
TN_3	0.19788
TN_4	0.15531
TN_5	0.22763

From Table 6.9 devising research and development activities (TN5) has the supreme order in the need of the customer's criteria while steady expenses on the minimization of pollution & transmission (TN4) is the last in the technical requirements. These weights are further used for evaluating the weighted values for the advancement procedures for the new products.

Stage 2. Technical needs versus new product advancement procedure:

Table 6.10: Advancement procedures for the GSCM

Criteria	**Existing Litearture**
Scheming (AP1)	[171, 172]
Observance (AP2)	[173, 174]
Computing (AP3)	[175, 176]
Devising (AP4)	[177, 178]
Examining (AP5)	[179, 180]

In this stage, new product advancement procedures are computed based on their performance. New product advancement procedures for GSCM are demonstrated in Table 6.10. From Table 6.10, there are five distinct phases of new product advancement for the GSCM. First, there is scheming of new products based on the different ideas gathered from various sectors so that the best-suited product should be devised. The next phase is about the observation of the kind of product to be devised and the kind of members who will work proactively in the sector. In the third phase, all steps leading up to the product's finalization are managed. In the fourth phase, the product is finalized. In the final phase, before being made available to customers, the product undergoes one more round of testing. The method q-RPF WASPAS is utilized for evaluating the advancement procedures on the technical needs. The linguistic computations are illustrated in Table 6.11. In the next step, the expert

Table 6.11: Linguistic computation for the advancement procedures

	AP_1	AP_2	AP_3	AP_4	AP_5
TN_1	(M, L, M)	(VL, H, M)	(ML, MH, VH)	(M, H, VH)	(MH, VH, H)
TN_2	(VH, L, M)	(H, ML, H)	(M, MH, H)	(VH, MH, A)	(A, H, VH)
TN_3	(MH, VH, H)	(M, VL, H)	(M, MH, VH)	(M, L, VL)	(L, VL, H)
TN_4	(L, L, M)	(ML, M, M)	(M, M, L)	(ML, VH, L)	(M, M, M)
TN_5	(H, M, ML)	(MH, H, MH)	(MH, VL, ML)	(M, ML, L)	(VH, L, M)

matrix is utilized along with q-RPF WASPAS and the computation percentages are illustrated in Table 6.12 and the evaluation results of q-RPF WASPAS for the second stage are given in Table 6.13. The obtained results of the second stage show that the advancement procedure (AP2) i.e. observance is the best choice and devising (AP4) is the least-suited advancement procedure.

Stage 3. New product advancement procedure versus potential capabilities:

Table 6.12: Expert matrix of advancement procedure

	AP_1	AP_2	AP_3	AP_4	AP_5
TN_1	$(0.42, 0.27, 0.47)$	$(0.45, 0.27, 0.45)$	$(0.63, 0.27, 0.33)$	$(0.72, 0.2, 0.2)$	$(0.75, 0.23, 0.17)$
TN_2	$(0.55, 0.2, 0.37)$	$(0.6, 0.23, 0.27)$	$(0.62, 0.3, 0.27)$	$(0.82, 0.23, 0.2)$	$(0.87, 0.17, 0.13)$
TN_3	$(0.75, 0.23, 0.17)$	$(0.45, 0.2, 0.45)$	$(0.75, 0.23, 0.17)$	$(0.37, 0.17, 0.52)$	$(0.37, 0.17, 0.52)$
TN_4	$(0.33, 0.23, 0.53)$	$(0.43, 0.3, 0.47)$	$(0.42, 0.27, 0.47)$	$(0.48, 0.2, 0.43)$	$(0.5, 0.3, 0.4)$
TN_5	$(0.52, 0.27, 0.37)$	$(0.65, 0.33, 0.23)$	$(0.33, 0.27, 0.58)$	$(0.43, 0.23, 0.43)$	$(0.55, 0.2, 0.37)$

Table 6.13: Weights for advancement procedures

Criterions	**Weights(q=2)**
AP_1	0.20777
AP_2	0.22716
AP_3	0.19933
AP_4	0.17607
AP_5	0.18966

In the third stage, similar computations (q-RPF WASPAS) can be done for weighting the potential capabilities. The set of criteria for potential capabilities is given in Table 6.14. According to Table 6.14, five distinct factors can affect the potential capabilities of the GSCM. Firstly, the sectors must have a good institutional setup. For this, the sectors must have proper departments and intellectual staff members who can cope with the level of competition in the market. In context with this, the sector's effective observance of the market and prompt execution of important steps over time will help its growth in this procedure. In addition to this, the maximum profit margin in the energy sector's investment should be taken into account for its growth in GSCM. Also, the potential capabilities of the customers are another major aspect of this process. The sectors that are more interested in

Table 6.14: Potential capabilities for the GSCM

Criteria	**Existing Litearture**
Institution (PC1)	[159, 181]
Competitiveness (PC2)	[182, 183]
Accomplishment (PC3)	[184, 185]
Procreation (PC4)	[186, 187]
Client (PC5)	[185, 166]

responding to the demands of the customers are considerably more successful in the GSCM. The linguistic evaluations are defined in Table 6.15. Next, the expert

Table 6.15: Linguistic computation for the potential capabilities

	PC_1	PC_2	PC_3	PC_4	PC_5
AP_1	(M, VH, M)	(H, VH, H)	(ML, VH, VH)	(MH, H, H)	(VH, VL, AH)
AP_2	(MH, VH, AH)	(VH, AH, AH)	(A, VH, L)	(A, VH, H)	(AH, VH, MH)
AP_3	(L, ML, L)	(M, H, VH)	(VH, L, L)	(M, L, M)	(M, VH, M)
AP_4	(H, L, ML)	(VH, M, ML)	(AH, VH, VH)	(VL, L, VL)	(L, L, L)
AP_5	(H, VH, ML)	(ML, M, ML)	(L, ML, VH)	(ML, H, VH)	(H, L, M)

matrix of potential capabilities is computed and demonstrated in Table 6.16. After that, the weights of potential capabilities are evaluated by utilizing q-RPF WASPAS and given in Table 6.17. From Table 6.17, competitiveness (PC2) is the best

Table 6.16: Expert matrix of potential capabilities

	PC_1	PC_2	PC_3	PC_4	PC_5
AP_1	(0.63, 0.23, 0.3)	(0.8, 0.17, 0.1)	(0.7, 0.17, 0.27)	(0.7, 0.27, 0.17)	(0.65, 0.13, 0.38)
AP_2	(0.82, 0.23, 0.2)	(0.93, 0.17, 0.17)	(0.7, 0.17, 0.3)	(0.87, 0.17, 0.13)	(0.82, 0.23, 0.2)
AP_3	(0.27, 0.23, 0.6)	(0.72, 0.2, 0.2)	(0.47, 0.17, 0.43)	(0.42, 0.27, 0.47)	(0.63, 0.23, 0.3)
AP_4	(0.43, 0.23, 0.43)	(0.57, 0.23, 0.37)	(0.92, 0.13, 0.13)	(0.15, 0.13, 0.77)	(0.25, 0.2, 0.6)
AP_5	(0.65, 0.2, 0.27)	(0.37, 0.3, 0.53)	(0.48, 0.2, 0.43)	(0.65, 0.2, 0.27)	(0.5, 0.23, 0.37)

Table 6.17: Weights for potential capabilities

Criterions	**Weights(q=2)**
PC_1	0.21761
PC_2	0.22327
PC_3	0.18494
PC_4	0.18623
PC_5	0.18796

factor for potential capabilities while accomplishment (PC3) is the least important among the set of criteria.

Stage 4. Potential capabilities versus potential strategic plans:

The final stage is to compute the potential strategic plans against the potential capabilities. Within this framework, the criteria for potential strategic plans are defined in Table 6.18. After doing a thorough review of the literature, five dis-

Table 6.18: Potential strategic plans for the GSCM

Criteria	**Existing Litearture**
Collaborating with the sectors on the technical needs (PSP1)	[188]
Standardizing the atmosphere of a competitive market (PSP2)	[189]
Concentrating on investment ventures with high anticipated returns (PSP3)	[190, 191]
Examining the procedure for developing new services and products (PSP4)	[192]
Utilizing the customer relationship management to meet the needs of the customers (PSP5)	[193]

tinct potential strategic plans have been identified for the energy sectors. Firstly, the necessity of the relationship between the demands of the customers and the management of the energy sectors is very useful for gaining an advantage in the competitive environment of the market. The reason behind this is that satisfied customers will prefer these sectors more than others. Another strategic plan that the energy sectors can implement is to make investments in technical development. Through this, the sectors will be able to manufacture items that meet the demands of the customers. Examining new products and services according to the demands of the customers is another potential strategic plan that can be implemented by the energy sectors, which will help them gain a good position in the market. Also, the comparisons of the energy sectors among themselves are also very crucial for gaining significant importance in the market and it will also help to become familiar with new applications in the energy sectors. The financial viability of the proposed initiatives can also be used as a potential strategic plan and for which the energy sectors conduct a modified analysis involving the cost-benefit parameters for the launching of new projects. The linguistic evaluations are given in Table 6.19. Also, the expert matrix of

Table 6.19: Linguistic computation for the potential strategic plans

	PSP_1	PSP_2	PSP_3	PSP_4	PSP_5
PC_1	(MH, VH, H)	(H, AH, VL)	(MH, L, VL)	(M, AH, VL)	(MH, L, H)
PC_2	(VL, L, M)	(VL, L, L)	(L, M, H)	(MH, VH, VL)	(AH, L, MH)
PC_3	(H, H, H)	(L, M, MH)	(MH, VH, H)	(VH, VL, L)	(M, H, VL)
PC_4	(L, VL, H)	(H, A, VL)	(VH, MH, L)	(AH, L, H)	(H, VH, H)
PC_5	(M, VH, H)	(VL, H, H)	(VH, MH, L)	(VL, AH, H)	(H, A, H)

the potential strategic plans is illustrated by Table 6.20. Within this framework, the obtained results for potential strategic plans are demonstrated in Table 6.21. From Table 6.21, it is clear that the prioritization for potential strategic plans is given as concentrating on investment ventures with high anticipated returns (PSP5), standardizing the atmosphere of a competitive market (PSP4), examining the procedure for developing new services and products (PSP3), utilizing the customer relationship management to meet the needs of the customers (PSP1) and collaborating with the sectors on the technical needs (PSP2) is least preferred.

Table 6.20: Expert matrix of potential strategic plans

	PSP_1	PSP_2	PSP_3	PSP_4	PSP_5
PC_1	$(0.75, 0.23, 0.17)$	$(0.6, 0.17, 0.38)$	$(0.32, 0.23, 0.58)$	$(0.55, 0.23, 0.45)$	$(0.53, 0.27, 0.33)$
PC_2	$(0.28, 0.2, 0.62)$	$(0.2, 0.17, 0.68)$	$(0.5, 0.23, 0.37)$	$(0.53, 0.2, 0.42)$	$(0.6, 0.27, 0.37)$
PC_3	$(0.75, 0.2, 0.1)$	$(0.45, 0.3, 0.43)$	$(0.75, 0.23, 0.17)$	$(0.42, 0.13, 0.52)$	$(0.45, 0.2, 0.45)$
PC_4	$(0.37, 0.17, 0.52)$	$(0.6, 0.17, 0.38)$	$(0.58, 0.23, 0.33)$	$(0.65, 0.2, 0.3)$	$(0.8, 0.17, 0.1)$
PC_5	$(0.72, 0.2, 0.2)$	$(0.53, 0.17, 0.35)$	$(0.58, 0.23, 0.33)$	$(0.6, 0.17, 0.38)$	$(0.82, 0.2, 0.13)$

Table 6.21: Ranking for potential strategic plans

Alternatives	**Rank**
PSP_1	0.19912
PSP_2	0.17206
PSP_3	0.20289
PSP_4	0.20853
PSP_5	0.2174

6.3 Comprartive Analysis and Discussion

In this section, we briefly present the overall concluding analysis along with a discussion of the advantages and limitations of the proposed methodology. In recent years, customers have looked forward to demonstration strategies in terms of the designing process of the products [144] which should also fulfill the budget issue of the customer and certain products of their needs. However, it is quite apparent in the findings that the customers are more specific about the environment and keen to go ahead with the companies that are more eco-friendly while developing the products [145]. It is certainly quite obvious that the concept of GSCM plays a vital role in any kind of strategic planning for the companies to meet the customer's expectations.

The detailed analysis has already been presented stage-wise in the previous section for better understanding and readability.

Sensitivity Analysis:

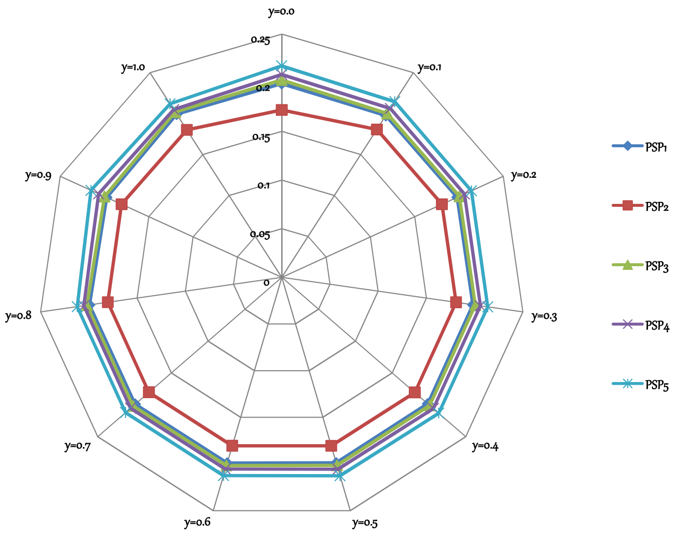

Figure 6.4: Effect of the parameter y on PSP's

On the basis of computations, we have presented some special findings in the form of the sensitivity diagram given in Figure 6.4. It is observed that the role of the parameter y is very important in understanding the variability and reliability of the obtained result. Suppose we consider the value of y to be 0.5 for our MCDM problem, but the decision makers can choose the appropriate value of y (ranging from 0 to 1) according to their convenience. The computational analysis shows that if we keep on changing the values of y in the proposed methodology, the obtained ranking reflects a kind of consistency in the selection of potential strategic plans. This enables us to make sure that the uncertainty in the decision is reduced. This gives the proposed methodology a kind of robustness in the calculation and validates the result and the process.

Also, the essential factors of a decision-making technique (i.e. criterion and decision maker's weights) and the evaluation of the alternatives under these criteria. These are very essential characteristics of an MCDM technique. Further, we compare our methodologies based on these terms with some existing methodologies tabulated in Table 6.22.

Table 6.22: Consistency with the MCDM Methods

	IVIF-DEMATEL & MOORA [146]	Spherical Fuzzy TOPSIS [198]	Fuzzy DE-MA-TEL [147]	Fuzzy COPRAS [148]	Proposed AHP & WASPAS
PSP_1	5	5	5	5	5
PSP_2	4	4	4	4	4
PSP_3	3	3	3	3	3
PSP_4	1	1	1	1	1
PSP_5	2	2	2	2	2

Table 6.23: Characteristic Comparison based on MCDM Methods

Literature	**Decision-Makers's Weightage**	**Criterion's Weights of the Criterions**	**Linguistic Computation**	**No. of Uncertainty Components**	**Restriction on the Uncertainty Components**	**Assessment Information of Alternatives**
[194]	Taken into account	Totally Unknown	✓	1	Yes	Fuzzy Set
[195]	Taken into account	Totally Unknown	✓	3	Yes	Pythagorean Fuzzy Set
[196]	Taken into account	Totally Unknown	✓	1	Yes	Fuzzy Set
[197]	Taken into account	Known	×	4	Yes	Picture Fuzzy Set
[198]	Taken into account	Totally Unknown	✓	4	Yes	Spherical Linear Diophantine Fuzzy Set
Proposed Methods	Calculated	Totally Unknown	✓	4	No	q-Rung Picture Fuzzy Set

Final rankings of the innovative/potential strategic plans for the GSCM by the variety of researchers (linguistic comparison) are aligned and tabulated in Table 6.23, which illustrates the consistency and reliability of the proposed technique. Although the outcomes are comparable statistically, this means that the suggested methodologies are different from others in the literature.

As a limitation, a different study can be carried out for the different industries focusing on the energy companies. In such cases, the utilization of quality function deployment (QFD) based indicators can be considered for measuring the different strategies for the energy-based sectors. Also, the technique of VIKOR/TOPSIS models for taking the different potential strategies can be implemented so that some quantitative/numerical-based analysis could be possible to comprehend whether the obtained results are compatible or non-compatible.

6.4 Conclusions

The significant contributions of the proposed work can be summarized as follows:

- The proposed model involving the q-rung picture fuzzy set in the AHP/WASPAS decision technique has been successfully presented and analyzed. The analysis comprises four types of interlinked different stages which have been presented in detail with observations. On the basis of the findings, it is clear that the prioritization for potential strategic plans is given as utilizing customer relationship management to meet the needs of the customers (PSP5) the most.
- Further, examining the procedure for developing new services and products (PSP4) and concentrating on investment ventures with high anticipated returns (PSP3) are the next level of priority. Collaborating with the sectors on the technical needs (PSP1) and standardizing the atmosphere of a competitive market (PSP2) is the least preferred in the order.
- It may be noted that the AHP is capable of considering the relative priorities of factors/alternatives and provides a wide range of usage for the systematics planning, effectiveness, benefit and risk analysis by choosing any kind of decision among alternatives. The work done in this chapter can further be applied to various other applications like social mediating technologies, green information technology for sustainability [199], [200].
- Also, the work can be expanded for handling various other types of real-world problems under different types of uncertain environments and various other sectors ([149], [150], [151], [152]).

Chapter 7

Conclusion

In the current thesis, we have explored and presented some new decision-making techniques in the picture fuzzy framework. The results of the research conducted in each of the chapters are mentioned along with a scope of future work:

- A bi-parametric (R, S)-norm picture fuzzy discriminant/cross-entropy measure has been effectively proposed along with its mathematical validation. Then, the proposed measure has been utilized for devising the modified VIKOR & TOPSIS decision-making methodologies.
- The proposed decision-making methodology has been successfully implemented for the assessment of hydrogen fuel cell technology problems along with suitable comparative and sensitivity analysis. These modified techniques give policy-makers useful information to help them for the selection of the best possible hydrogen fuel cell technology.
- Next, a new kind of picture fuzzy Dombi aggregation operators has been satisfactorily proposed along with some important properties and operational laws. With the help of picture fuzzy Dombi aggregation operators, we developed the modified EDAS decision-making techniques which include the parametrization feature of the attributes.
- The problem of sustainable agrifarming has been solved with the incorporation of picture fuzzy soft Dombi-based EDAS decision-making methodology. Also,

the validation of the proposed methodology is done by doing the comparative and consistency study.

- The novel notion of picture fuzzy hypersoft set (PFHSS) has been introduced with various essential properties and operational laws. Further, the aggregating operators of the form(PFHSWA/PFHSOWA/PFHSWG/PFHSOWG) have been detailed as well. The proposed decision-making methodology has been well illustrated by a numerical example under consideration.
- In continuation, by utilizing a picture fuzzy hypersoft set a new concept of picture fuzzy hypersoft matrix (PFHSM) has been effectively introduced along with different types, operations and properties.
- The variability of uncertain information has been spanned by the notions of choice matrix, weighted choice matrix, followed by value and score matrix. The proposed methodology has been applied in the assessment of renewable energy source selection problems. Then, a comparative study depicts the advantages over the contemporary techniques in the literature.
- Further, the q-rung picture fuzzy oriented AHP/WASPAS decision-making techniques have been presented which includes the stage-wise procedures for implementation.
- The problem of finding the best potential strategic plan for the energy sector has been remodeled with the AHP/WASPAS decision-making methodologies under q-rung picture fuzzy framework. Lastly, the viability and robustness of the presented techniques have been examined. Also, the overall organization of the proposed work is also shown in Figure 7.1.

7.1 Scope of Future Work

While the methodologies developed in this thesis have shown promising results, there are several avenues for future research that can extend and refine these techniques:

- Integration of Machine Learning Techniques: Future work could explore the integration of machine learning algorithms with picture fuzzy decision-making

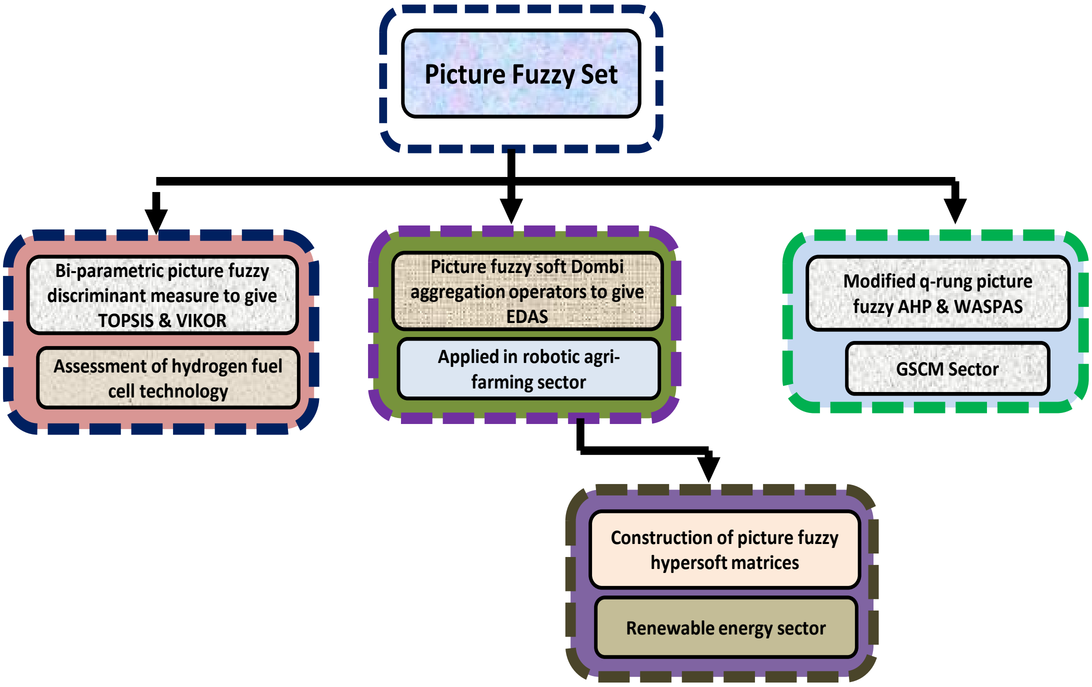

Figure 7.1: Organization of the Complete Proposed Work

methodologies to enhance the prediction and classification accuracy in dynamic environments, such as in renewable energy forecasting or agrifarming yield prediction.

- Multi-Criteria Decision-Making under Dynamic Conditions: Expanding the proposed frameworks to handle dynamic and time-varying data, which could be useful in applications like real-time energy management and adaptive agri-farming solutions.

- Hybrid Models for Complex Systems: Developing hybrid models that combine picture fuzzy decision-making with other soft-computing techniques (such as neural networks or genetic algorithms) to tackle highly complex decision-making problems with multiple layers of uncertainty.

- Extension to Group Decision-Making: Extending the picture fuzzy methodologies to multi-expert or group decision-making scenarios, where conflicting opinions and preferences can be effectively managed using the proposed aggregation operators.

- Applications in Other Domains: Applying the developed methods to other critical fields such as healthcare, transportation, and urban planning, where multi-criteria decision-making plays an essential role in strategic decision support systems.
- Software Implementation and Validation: Developing user-friendly software tools that implement the proposed decision-making methodologies, enabling practitioners and policymakers to easily apply these techniques in real-world scenarios.

These directions would further enhance the practical utility of the proposed methodologies and extend their applicability across diverse sectors.

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