STUDY OF SOME NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS FOR LIE SYMMETRIES AND EXACT SOLUTIONS

Thesis submitted in fulfillment of the requirements for the Degree of

DOCTOR OF PHILOSOPHY

BY

PREETI DEVI

DEPARTMENT OF MATHEMATICS

JAYPEE UNIVERSITY OF INFORMATION TECHNOLOGY

WAKNAGHAT, SOLAN-173 234, H.P. INDIA

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DECLARATION BY THE SCHOLAR

I hereby declare that the work reported in the Ph.D. thesis entitled **"Study of Some Nonlinear Partial Differential Equations for Lie Symmetries and Exact Solutions"** at **Jaypee University of Information Technology, Waknaghat, Solan (H.P.) India**, is an authentic record of my work carried out under the supervision of **Prof. Karanjeet Singh**. I have not submitted this work elsewhere for any other degree or diploma. I am fully responsible for the contents of my Ph.D. Thesis.

Preeti Devi
(Enrollment No.: 176851)
Department of Mathematics
Jaypee University of Information Technology,
Waknaghat, Solan, H.P., INDIA

Date:

SUPERVISOR'S CERTIFICATE

This is to certify that the work reported in the Ph.D. thesis entitled **"Study of Some Nonlinear Partial Differential Equations for Lie Symmetries and Exact Solutions"** submitted by **Preeti Devi** at **Jaypee University of Information Technology, Waknaghat**, is a bonafide record of her original work carried out under my supervision. This work has not been submitted elsewhere for any other degree or diploma.

Prof. Karanjeet Singh
Department of Mathematics
Jaypee University of Information Technology,
Waknaghat, Solan, H.P., INDIA

Date:

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(Preeti Devi)

Abstract

The objective of the thesis entitled, “Study of Some Nonlinear Partial Differential Equations for Lie Symmetries and Exact Solutions”, is to study the applications of Lie groups to the nonlinear partial differential equations (NLPDEs). The prime objective in this thesis is to examine the Lie symmetries of the NLPDEs in order to obtain the exact solutions, which are helpful in demonstrating the integrability and physical behavior of the nonlinear equations.

During the last few decades, investigations of exact solutions to nonlinear partial differential equations have played a vital role in the study of physical phenomena. Exact solutions provide the proper understanding of qualitative features of many nonlinear physical phenomena and processes in various areas of natural science.

In recent years, the generalization of the constant coefficients to variable coefficients has grown predominantly in research interest. Because the differential equations with variable coefficients characterize many nonlinear phenomena more realistically than equations with constant coefficients, but often, it is difficult to solve explicitly these nonlinear differential equations for exact solutions. However, there is much current interest in finding the exact explicit solutions of these nonlinear equations. The exact solutions provide much information about physical phenomena and various other aspects of these nonlinear systems. The exact solutions are also helpful to examine and discuss the sensitivity of physical phenomena with several important parameters described by variable coefficients. These solutions are also helpful in designing and testing numerical algorithms. Mathematical methods which generate a wide range of explicit solutions and applicable to all type of nonlinear differential equations are few. The group-theoretic techniques can be categorized in this class, and generally, it produces a variety of exact solutions, directly or via similarity solutions, classifying invariant equations and/or reducing the number of independent variables. The study carried out in this thesis is confined to the applications of Lie group theory to four NLPDEs, namely, the (2+1) dimensional dispersive long wave system, the (3+1) dimensional Kadomtsev-Petviashvilli (KP) equation with variable coefficients and an arbitrary nonlinear term, Schrödinger equation with variable coefficients, Gilson-Pickering equation with variable coefficients. Also, the (2+1) dimensional Boiti-Leon-Pempinelli equation has been investigated by utilizing the (G'/G^2)-expansion method and the first integral method.

Chapter 1 describes the introduction of nonlinear partial differential equations and exact solutions. Also, we have discussed the preliminaries of the Lie group of transformations, the first integral method, and the $G^{'}/G^2$-expansion method and relevant literature surveys.

Chapter 2, is devoted to the following (2+1) dimensional Dispersive Long Wave (DLW) equation of the form :

$$\begin{aligned} u_{yt} + v_{xx} + u_x u_y + u u_{xy} &= 0, \\ v_t + u_x v + v_x u + u_{xxy} &= 0. \end{aligned}$$

On applying the classical Lie group method, the infinite-dimensional symmetries have been derived along with the Kac-Moody-Virasoro algebra. The reduced PDEs are further investigated for their invariance properties to obtain the exact solutions.

Chapter 3, considers the study of following Schrödinger equation with variable coefficients :

$$iq_t + (a_1(t) + ia_2(t))q_{xx} + (b_1(t) + ib_2(t))(|q|^{2n}q)_x = 0,$$

where $a_1(t)$, $a_2(t)$, $b_1(t)$, $b_2(t)$ are arbitrary real functions and $q(x,t)$ is a complex-valued function that expresses wave envelope. The Lie symmetries of this equation have been worked out, and the reduced ordinary differential equations are further solved by employing the power series method.

Chapter 4, deals with the study of the following (3+1) dimensional Kadomtsev-Petviashvilli (KP) equation with variable coefficients :

$$(u_t + \lambda(t)(k(u))_x + \mu(t)u_{xxx})_x + \gamma(t)u_{yy} + \delta(t)u_{zz} = 0,$$

where $\mu(t)$, $\gamma(t)$, $\delta(t)$, $\lambda(t)$ are arbitrary functions of t only and $k(u)$ is an arbitrary nonlinear term.

For the KP equation, infinite-dimensional symmetries have been reported using the Lie group method. To illustrate the further process the variable coefficients have been taken as power functions of t. As a result the equation is reduced to nonlinear partial differential equations with three independent variables, which are further investigated using the group method and obtain the various new exact solutions.

Chapter 5, deals with the study of following Gilson-Pickering equation with variable coefficients

$$u_t - a(t)u_{xxt} + b(t)u_x - uu_{xxx} - c(t)uu_x - d(t)u_x u_{xx} = 0,$$

where $a(t), b(t), c(t), d(t)$ are arbitrary functions of their argument. This equation is reduced to nonlinear ordinary differential equation (ODE) by carrying out the classical Lie symmetries. These ODEs have been further investigated for obtaining the solutions with the help of the power series method.

Chapter 6, is devoted to the following (2+1) dimensional Boiti-Leon-Pempinelli equation

$$
\begin{aligned}
u_{ty} &= (u^2 - u_x)_{xy} + 2v_{xxx}, \\
v_t &= v_{xx} + 2uv_x.
\end{aligned}
$$

The first integral method and (G'/G^2)-expansion method have been employed to furnish the exact traveling wave solutions. The obtained solutions include the trigonometric, exponential, and rational functions.

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List of the Publications

Journal Articles

1. P. Devi and K. Singh, "Classical Lie symmetries and similarity reductions of the (2+1)-dimensional dispersive long wave system", Asian European Journal of Mathematics, vol. 14(4), pp. 2150052, 2020. **Indexed: Scopus, ESCI(Web of Science)**.

2. P. Devi and K. Singh, "Lie symmetry analysis of the nonlinear Schr$\ddot{o}$dinger equation with time dependent variable coefficients", International Journal of Applied and Computational Mathematics, vol.7(23), 2021. **Indexed: Scopus** .

3. P. Devi and K. Singh, "Symmetry analysis of the (3+1) dimensional Kadomtsev-Petviashvilli equation with variable coefficients and an arbitrary nonlinear term". **(Under Review)**

4. P. Devi and K. Singh, "Symmetry analysis of the Gilson-Pickering equation with time dependent Coefficients". **(Under Communication)**

5. P. Devi and K. Singh, "Classical Symmetries of the Klein-Gordon-Zakharov equations with time dependent variable Coefficients". **(Under Communication)**

Conference Paper

1. P. Devi and K. Singh, "Exact solutions of the (2+1)-dimensional Boiti-Leon-Pempinelli system by first integral method", AIP Conference Proceedings, 2061, 020014, 2019. **Indexed: Scopus**

2. P. Devi and K. Singh, "Exact traveling wave solutions of the (2+1)-dimensional Boiti-Leon-Pempinelli system using $(G^{'}/G^{2})$ expansion method, AIP Conference Proceedings, 2214, 020030, 2020. **Indexed: Scopus**.

Chapter 1

Introduction and Review of Literature

1.1 Background and Motivation

The study of differential equations has been playing a significant role in modeling the real-world problems and their applications not only in physics but approximately in every field of science and engineering for nearly five centuries. Most of the problems are nonlinear in nature and are often described by a single differential equation or a system of differential equations.

But in nature, there are complex dynamics, which cannot be elucidated by means of ordinary systems, and from the experimental observations and reality, it has been reported that there exists a lot of complex systems in real-world which have anomalous dynamics such as fluid mechanics, heat and mass transfer, the motion of an airplane, air flow in lungs, prediction of weather, modeling cancer growth, drug delivery to parts of the body, electromagnetic theory and so on.

The fact of studying nonlinear differential equations is regarded as a challenging and complex endeavor. On comparison with the variety of tools present in linear equations, the techniques for nonlinear equations are limited for some special categories. Due to the complexity in nature, there is no general method to solve the nonlinear

differential equations. This is the reason when dealing with a nonlinear differential equation, the first step is to linearize it or to avoid the nonlinear aspects completely. But in analyzing the behavior of the physical system, one often comes across situations when the linearized model is inadequate/inaccurate, and that is the time when the study of nonlinear models as such becomes imperative. In the nineteenth century, linear systems became the mathematical discipline and achieved remarkable success throughout the sciences, on the other hand, due to the complex nature of nonlinear differential equations, they remained much harder to understand. Consequently, some approximate solutions had been constructed by applying the asymptotic, numerical methods and perturbation methods. However, these solutions do not provide much information about the equation(s).
The analysis of the nonlinear partial differential equations (NLPDEs) has not only played an imperative role in modeling the physical phenomena but also helped in making more specific contributions in theories and concepts devised in the last centuries. Consequently, a strategy is normally adopted for obtaining the solutions of the nonlinear equations as follows:

(i) By applying the certain physical assumptions to linearize the given nonlinear differential equations.

(ii) Numerical integration of differential equations under certain boundary conditions.

(iii) To obtain the exact (explicit/implicit) solutions of the equations.

In fact, the first two steps have a great deal of contributions, whereas the third approach is usually avoided because of ponderous and complicated calculations. But the strong desire for exact analytical solutions to NLPDEs has made a formidable growth in the research interest. Thus, there is more interest on finding the exact solutions of nonlinear equations during last few decades, and these solutions provide the information about the various aspects of physical and nonlinear phenomena. Exact solutions can be used as models for physical experiments, as benchmarks for testing and designing numerical algorithms. These solutions can serve as a basis for perfecting and testing computer algebra software package for solving differential equations. The explicit solutions for NLPDEs are rare and the methods which generate the family of solutions are not only getting popular but also increasingly sought. So to find the exact solutions a number of effective methods have been developed such as the classical Lie symmetry method [28]-[31], nonclassical symmetry method [30], Clarkson Kruskal-direct method [111], nonlocal symmetry method [31], [78], [82], the ans$\ddot{a}$tze based method [26], [65], Painl$\grave{e}$ve approach [98], exp-function method [2], [50], (G'/G) - expansion method [70], Adomian decomposition method [43], Homotopy perturbation method [83], the first integral method [109], (G'/G^2) - expansion method [66], [110], modified extended tanh-function method [90], extended Jacobi elliptic function [105], elliptic equation rational expansion method [104] and many more. The methods which generate a variety of solutions and applicable to all types of nonlinear equations are very few. The group-theoretic techniques are categorized in this class and it produces a wide range of exact solutions in a systematic manner. Since the NLPDEs have been a powerful tool to model and study the dynamics of many physical processes of the applied sciences, the solutions of differential equations are compared with the actual behavior of the corresponding system to determine if or not the formulation in terms of differential equations is accurate.

The study carried out in this thesis is devoted to the applications of Lie group methods based on the theory of continuous groups, also known as Lie groups of transformations acting on the space of independent and dependent variables of the system. Norwegian mathematician Sophus Lie [96] introduced the notion of the Lie group method and established that the order of an ordinary differential equation (ODE) can be reduced by one if it remains invariant under a one-parameter point group of transformations and for a partial differential equation the invariance under a continuous group of point transformations leads directly to the superposition of solutions in terms of transformations [29]. Further, Bluman and Cole [28], Ovsiannikov [60], Ibragimov [71] and Olver [79] extended the theory of Lie point groups to wide range of problems.

The prime motive in the proposed work is to demonstrate the importance and adequacy of Lie group methods in solving nonlinear systems. In brief, a symmetry group of a single or a system of the partial differential equations which is a continuous group of point transformations acting on the space of independent and dependent variables and which leaves the equations invariant is determined algorithmically, and then the solutions of a partial differential equation(s) can be obtained by solving a reduced system of the differential equation(s) with lesser number of independent variables. The detailing theory and various applications of Lie groups may found in research notes/books of Bluman and Cole [28], Ovsiannikov [60] and Olver [79].

1.2 Methodology

During the period 1872-1899, Sophus Lie [94], [95], [96], established the concept of Lie group method of differential equations. Regardless of its important features, Lie's method to differential equations faded into obscurity and, the entire subject lay dormant for nearly half a century. It was in the fifties of the last century when the work of G. Birkhoff [27] and I. Sedov [58] on dimensional analysis gave relevant attention to the unexploited applications of Lie groups to the differential equations and then, in the late 1950s, it was successfully applied to a wide range of problems through the pioneering efforts of Ovsiannikov [60] and his co-workers. During the years 1960-1970, the whole field was active again and new applications of Lie group theory were being proposed by a number of researchers including Bluman and Anco [29], Bluman and Cole [30], [28], Bluman and Kumei [31], Hydon [25], Stephani [37], Cantwell [11], Olver and his co-workers [79, 81], Ibragimov [71, 72], Ibragimov and Kovalev [73], Hill et al. [44]-[47], Grundy [87], Gagnon and Winternitz [59], Clarkson and Mansfield [76, 77].

Symmetry method provides an essential tool to examine the wide range of topics in a systematic way, such as homogeneous and separable equations, the integration by quadrature of ODEs, methods of undetermined coefficients, the determination of invariant solutions of initial and boundary value problems, reduction of order, derivation of conservation laws, construction of links between different differential equations (DE) that turn out to be equivalent. Lie has shown that the invariance of an ODE

under the Lie group of point transformations, provides some special solutions called invariant solutions without the knowledge of the general solution of the ODE. For exhaustive reviews of Lie's, on this aspect we refer to the works of Lie and Engel [94], Cohen [8], Goursat [21], Ince [23], Hermann and Dickson [88].
The key idea of Lie's group theory of symmetry analysis of DE relies on the invariance of the latter under a transformation of independent and dependent variables. This transformation forms a local group of point transformations establishing a diffeomorphism on the space of independent and dependent variables, mapping solutions of the equations to other solutions. Lie proposed that the problem of obtaining the point symmetry of a differential equation leaving invariant a given differential equation, reduced to solving related linear homogeneous systems of determining equations for the infinitesimal generators. He also proposed that a point symmetry of a DE leads, in the case of an ordinary differential equation, to reduce the order of the DE and in the case of a partial differential equation (PDE), to finding special solutions known as the similarity (invariant) solutions of the differential equation. In this direction, some other important and significant contributions are from Rosati and Nucci [55], Gandarias and Bruzon [63]-[64], Anco and Dennis [89], Bihlo and Popvych [4], [5]. Among various generalizations of Lie's theory, there are the following approaches:

(i) Nonclassical method [30]

(ii) General method of differential constraints [22], [80]

(iii) Generalized symmetries [82]

(iv) Equivalence transformations [42]

(v) Nonlocal symmetries [31], [78], [82]

In recent years, Lie's classical theory has gained much interest of researchers in the field of NLPDEs. The prime objective in carrying out this work has been to demonstrate the importance and adequacy of the Lie group method over various other methods available in the literature. Some specific physical nonlinear systems have been considered to accomplish the task. The problems studied are dealt in two phases: In the first phase, Lie point symmetries of the nonlinear systems under investigation are derived by adopting the classical Lie group method, and then in the second phase, after successful deduction of the reduced systems of PDEs or ODEs, the efforts are confined to deduce the exact solutions. In some problems, we have also investigated the reduced ODEs using the power series method. Brief literature survey relevant to the work has also been put up in chapters. We reproduce in the following sections certain characteristic features of the techniques utilized and general notions essential for understanding and carryover of the Lie group method [28], [29].

1.3 Lie Group Method to Construct Solutions of NLPDEs

In this thesis, we are dealing with the method of group invariant solutions based on the theory of a continuous group of point transformations, also called the Lie group of transformations, acting on the space of independent and dependent variables. The notion of the Lie group method was originally introduced by Sophus Lie [94], [95], who proposed that the order of an ODE can be reduced by one if it remains invariant under a one-parameter Lie group of transformations, and for a PDE, the invariance under a continuous group of point transformations leads directly to the superposition of solutions in terms of transformations. In the following section, we first introduce the relevant concepts of Lie group transformations and then provide the algorithmic description of the methods applied in later chapters to derive the symmetry groups of the systems under investigation. Also, the G'/G^2-expansion method and first integral method, which are of interest in the present work to obtain the exact traveling wave solutions of the nonlinear PDEs have been described. For details on Lie groups, theorems and their proofs, we refer our reader to see (Bluman and Cole [28], [30], Bluman and Anco [29], Olver [79]-[81], and Stephani [37]). Also, the relevant details of the G'/G^2-expansion method and first integral method can be found in [54], [66],

[70], [109], [110].

1.3.1 Lie Group of Transformations

Definition 1.3.1 *[29] "Let $x = (x_1, x_2, x_3, \ldots, x_n)$ lies in region $D \subset R^n$. The set of transformations*

$$\tilde{x} = X(x; \epsilon), \tag{1.3.1}$$

defined for each x in D and parameter $\epsilon \in T \subset R$, with $\phi(\epsilon, \delta)$ defining a law of composition of parameters ϵ and δ in T, such that

(i) *For every ϵ in T, the transformations are bijective on D.*

(ii) *T with the law of composition ϕ forms a group.*

(iii) *For each x in D, $\tilde{x} = x$ when $\epsilon = \epsilon_o$ corresponds to the identity element e of T, i.e.,*

$$X(x; \epsilon_o) = x.$$

(iv) *If $\tilde{x} = X(x; \epsilon), \tilde{\tilde{x}} = X(\tilde{x}; \delta)$, then*

$$\tilde{\tilde{x}} = X(x; \phi(\epsilon, \delta)),$$

such family of transformation is known as the one-parameter group of transformations."

Definition 1.3.2 *[29] "A one-parameter group of transformations defines a one-parameter Lie group of transformations if, in addition to satisfying axioms $(i) - (iv)$ of definition (1.3.1), the followings hold:*

(i) *ϵ is a continuous parameter, i.e., T, is an interval in R. Without loss of generality, $\epsilon = 0$ corresponds to the identity element e.*

(ii) X is infinitely differentiable with respect to x in D and an analytic function of ϵ in T .

(iii) $\phi(\epsilon, \delta)$ is an analytic function of ϵ and δ in T."

1.3.1.1 Infinitesimal Transformation of Lie Group

Expanding, one-parameter group of point transformations $\tilde{x} = X(x;\epsilon)$, about $\epsilon = 0$, in some neighborhood of $\epsilon = 0$, we have

$$\begin{aligned}\tilde{x} = X(x;\epsilon) &= X(x;0) + \epsilon\Big(\frac{\partial X}{\partial \epsilon}\Big)|_{\epsilon=0} + \frac{1}{2}\epsilon^2\Big(\frac{\partial^2 X}{\partial \epsilon^2}\Big)|_{\epsilon=0} + \cdots \\ &= x + \epsilon\Big(\frac{\partial X}{\partial \epsilon}\Big)|_{\epsilon=0} + O(\epsilon^2). \end{aligned} \tag{1.3.2}$$

The transformation $\tilde{x} = x+\epsilon\tilde{\xi}(x)+O(\epsilon^2)$ is known as the infinitesimal transformation of the Lie group. The components of $\tilde{\xi}(x)$ are the infinitesimals of (1.3.1), where

$$\tilde{\xi}(x) = \frac{\partial X(x;\epsilon)}{\partial \epsilon}|_{\epsilon=0}. \tag{1.3.3}$$

1.3.1.2 Infinitesimal Generators

The infinitesimal generator of the one-parameter group of transformation (1.3.1) is the linear differential operator

$$V = \tilde{\xi}(x).\nabla = \sum_{i=1}^{n} \xi_i(x)\frac{\partial}{\partial x_i}, \tag{1.3.4}$$

where ∇ indicates the gradient operator

$$\nabla = \Big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots \frac{\partial}{\partial x_n}\Big). \tag{1.3.5}$$

1.3.1.3 Invariant Functions

An infinitely differentiable function $F(x)$ is defined to be an invariant of the Lie group of transformations (1.3.1) if and only if, for any group transformation (1.3.1),

$$F(\tilde{x}) = F(x). \tag{1.3.6}$$

Theorem 1.3.1 *$F(x)$ is invariant under a Lie group of point transformations (1.3.1) iff, $VF(x) = 0$.*

1.4 Point Transformations and Extended transformations

We will be concerned with the determination of one-parameter (ϵ) Lie groups of point transformations admitted by a given system S of differential equations. A one-parameter Lie group of transformations is a group of transformations of the form

$$\tilde{x} = X(x, u; \epsilon), \tag{1.4.1}$$

$$\tilde{u} = U(x, u; \epsilon), \tag{1.4.2}$$

acting on the space of $n + m$ variables

$$x = (x_1, x_2, \ldots, x_n), \tag{1.4.3}$$

$$u = (u^1, u^2, \ldots, u^m), \tag{1.4.4}$$

where x denotes n independent variables and u indicates the m dependent variables. A Lie group of transformations (1.4.1)-(1.4.2) introduced by S maps any solution $u = \Theta(x)$ of S into a one parameter family of solutions $u = \phi(x; \epsilon)$ of S. In other words, a group of point transformations (1.4.1)-(1.4.2) leaves S invariant in the sense that, the form of S remains same in terms of the transformed variables (1.4.1)-(1.4.2) for any solution $u = \Theta(x)$ of S.

Let ∂u represents the set on nm coordinates corresponding to all first order partial derivative of u with respect to x:

$$\partial u = \left(\frac{\partial u^1}{\partial x_1}, \frac{\partial u^1}{\partial x_2}, \ldots, \frac{\partial u^1}{\partial x_n}, \frac{\partial u^2}{\partial x_1}, \frac{\partial u^2}{\partial x_2}, \ldots, \frac{\partial u^2}{\partial x_n}, \ldots, \frac{\partial u^m}{\partial x_1}, \frac{\partial u^m}{\partial x_2}, \ldots, \frac{\partial u^m}{\partial x_n}\right). \tag{1.4.5}$$

In general, for $k \geq 1$, Let $\partial^k u$ represent the set of coordinates

$$u^{\mu}_{i_1, i_2, i_3, \ldots, i_k} = \frac{\partial^k u^{\mu}}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_k}},$$

with $\mu = 1, 2, \ldots, m$ and $i_j = 1, 2, \ldots, n$, for $j = 1, 2, \ldots, k$ corresponding to all kth-order partial derivatives of u with respect to x.

It turns out that the natural transformation of partial derivatives of the dependent variables leads successively to extensions (prolongations) of a one-parameter (ϵ) group of transformations (1.4.1)-(1.4.2) acting on (x, u)-space to one-parameter groups of transformations acting on $(x, u, \partial u)$-space, $(x, u, \partial u, \partial^2 u)$-space,…, $(x, u, \partial u, \partial^2 u, \ldots, \partial^k u)$-space, for any $k > 2$. Then the infinitesimal transformation of (1.4.1)-(1.4.2) is naturally extended or prolonged successively to infinitesimal transformations acting on $(x, u, \partial u, \ldots, \partial^l u)$-space, $l = 1, 2, 3, \ldots, k$.

1.4.1 Extended Infinitesimal Transformations

In studying the system of partial differential equations, the situation of m dependent variables $u = (u^1, u^2, \ldots, u^m)$ and n independent variables $x = (x_1, x_2, \ldots, x_n)$, $u = u(x)$, with $m \geq 2$ arises. This leads to consideration of prolonged transforms from (x, u)-space to $(x, u, \partial u, \ldots, \partial^k u)$-space, where $\partial^k u$ represents the components of all kth-order partial derivatives of u with respect to x. Consider the kth-extended transformation over $(x, u, \partial u, \ldots, \partial^k u)$-space

$$\tilde{x_i} = x_i + \epsilon \xi_i(x, u) + O(\epsilon^2), \tag{1.4.6}$$

$$\tilde{u^\mu} = u^\mu + \epsilon \eta^\mu(x, u) + O(\epsilon^2), \tag{1.4.7}$$

$$\tilde{u_i^\mu} = u_i^\mu + \epsilon \eta_i^{(1)\mu}(x, u, \partial u) + O(\epsilon^2), \tag{1.4.8}$$

$$\vdots$$

$$\tilde{u^\mu}_{i_1 i_2 \ldots i_k} = u^\mu_{i_1 i_2 \ldots i_k} + \epsilon \eta^{(k)\mu}_{i_1 i_2 \ldots i_k}(x, u, \partial u, \ldots, \partial^k u) + O(\epsilon^2), \tag{1.4.9}$$

with the extended infinitesimals given as

$$\eta_i^{(1)\mu} = D_i \eta^\mu - (D_i \xi_j) u_j^\mu, \tag{1.4.10}$$

$$\eta^{(k)\mu}_{i_1 i_2 \ldots i_k} = D_{i_k} \eta^{(k-1)\mu}_{i_1 i_2 \ldots i_{k-1}} - (D_{i_k} \xi_j) u^\mu_{i_1 i_2 \ldots i_{k-1} j}, \tag{1.4.11}$$

where $i_l = 1, 2, \ldots, n$ for $l = 1, 2, \ldots, k$ with $k \geq 2$ and D_i is total derivative operator defined as

$$D_i = \frac{\partial}{\partial x_i} + u_i^\mu \frac{\partial}{\partial u^\mu} + u_{ij}^\mu \frac{\partial}{\partial u_j^\mu} + \cdots + u_{i_1 i_2 \ldots i_n}^\mu \frac{\partial}{\partial u_{i_1 i_2 \ldots i_n}^\mu} + \cdots \tag{1.4.12}$$

Here, the kth-prolonged or (extended) infinitesimal generator is given as

$$\begin{aligned} V^{(k)} = \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta^{(\mu)}(x,u)\frac{\partial}{\partial u^\mu} + \eta^{(1)\mu}(x,u,\partial u)\frac{\partial}{\partial u_i^\mu} + \cdots \\ +\eta_{i_1 i_2 \ldots i_k}^{(k)\mu}(x,u,\partial u,\partial^2 u,\ldots,\partial^k u)\frac{\partial}{\partial u_{i_1 i_2 \ldots i_k}^\mu}, \qquad k \geq 1. \end{aligned} \tag{1.4.13}$$

1.4.1.1 The Invariance Condition for a System of PDEs

Lie symmetry of a given differential equation is a one-parameter group of point transformation under which the given differential equation remains invariant. Consider a system of N PDEs with n independent variables $x = (x_1, x_2, x_3, \ldots, x_n)$ and m dependent variables $u = (u^1, u^2, u^3, \ldots, u^m)$, given by

$$F^\mu(x, u, \partial u, \partial^2 u, \ldots, \partial^k u) = 0, \mu = 1, 2, \ldots, N. \tag{1.4.14}$$

Definition 1.4.1 *[29] "A one-parameter Lie group of point transformations (1.4.6)-(1.4.7) leaves invariant the system of PDEs (1.4.14), iff its kth extension, defined by (1.4.6)-(1.4.9) leaves invariant the N surfaces in $(x, u, \partial u, \partial^2 u, \ldots, \partial^k u)$-space, defined by (1.4.14)".*

Theorem 1.4.1 *[29] "(Infinitesimal Criterion for the Invariance of a System of PDEs). Let*

$$V = \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta^\mu(x,u)\frac{\partial}{\partial u^\mu}, \tag{1.4.15}$$

be the infinitesimal generator of the Lie group of point transformations (1.4.6)-(1.4.7). Let

$$\begin{aligned} V^k = \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta^\mu(x,u)\frac{\partial}{\partial u^\mu} + \eta_i^{(1)\mu}(x,u,\partial u)\frac{\partial}{\partial u_i^\mu}, + \cdots \\ +\eta_{i_1 i_2 \ldots i_k}^{(k)\mu}(x,u,\partial u,\partial^2 u,\ldots,\partial^k u)\frac{\partial}{\partial u_{i_1 i_2 \ldots i_k}^\mu}, \end{aligned} \tag{1.4.16}$$

be the kth extended infinitesimal generator of (1.4.15), with the extended infinitesimals defined by

$$\eta_i^{(1)\mu} = D_i\eta^\mu - (D_i\xi_j)u_j^\mu, \quad i = 1, 2, \ldots, n, \tag{1.4.17}$$

$$\eta_{i_1 i_2 \ldots i_k}^{(k)\mu} = D_{i_k}\eta_{i_1 i_2 \ldots i_{k-1}}^{(k-1)\mu} - (D_{i_k}\xi_j)u_{i_1 i_2 \ldots i_{k-1} j}^\mu, \tag{1.4.18}$$

where $i_l = 1, 2, \ldots, n$ for $l = 1, 2, \ldots, k$ with $k \geq 2$ and D_i is total derivative operator defined by (1.4.12). Then the one parameter Lie group of transformations (1.4.6)-(1.4.7) is admitted by the system of PDEs (1.4.14) iff

$$V^{(k)}F^\mu(x, u, \partial u, \partial^2 u, \ldots, \partial^k u) = 0, \quad \mu = 1, 2, \ldots, N, \tag{1.4.19}$$

when $\qquad F(x, u, \partial u, \partial^2 u, \ldots, \partial^k u) = 0.$*"*

1.4.1.2 Symmetry Determining Equations

Consider a system of N PDEs (1.4.14) with each of its equations given in a solved form

$$u_{i_1 i_2 \ldots i_k}^{\nu_\mu} = f^\mu(x, u, \partial u, \partial^2 u, \ldots, \partial^k u) \tag{1.4.20}$$

In terms of some specific l_μth-order partial derivative of u_μ^ν for some $\nu_\mu = 1, 2, \ldots, m$, where $f^\mu(x, u, \partial u, \partial^2 u, \ldots, \partial^k u)$ does not depend explicitly on any of the components $u_{i_1 i_2 \ldots i_k}^{\nu_\sigma}$, $\sigma = 1, 2, \ldots, N$, for each $\mu = 1, 2, \ldots, N$. From theorem (1.4.1) we note that the system (1.4.14) admits the Lie point symmetry (1.4.15) with the kth prolongation given by (1.4.16), iff

$$\eta_{i_1 i_2 \ldots i_{l_\mu}}^{(l_\mu)\nu_\mu} = \xi_j \frac{\partial f^\mu}{\partial x_j} + \eta^\nu \frac{\partial f^\mu}{\partial u^\nu} + \eta_j^{(1)\nu} \frac{\partial f^\mu}{\partial u_j^\nu}, + \cdots, + \eta_{j_1 j_2 \ldots j_k}^{(k)\nu} \frac{\partial f^\mu}{\partial u_{j_1 j_2 \ldots j_k}^\nu}, \tag{1.4.21}$$

with

$$u_{i_1 i_2 \ldots i_{k_\sigma}}^{\nu_\sigma} = f^\sigma(x, u, \partial u, \partial^2 u, \ldots, \partial^k u), \sigma = 1, 2, \ldots, N. \tag{1.4.22}$$

It is easy to see that $\eta_{j_1 j_2 \ldots j_p}^{(p)\nu}$ is a polynomial in the components of coordinates $\partial u, \partial^2 u, \ldots, \partial^p u$ with coefficients that are linear homogeneous in components of $\xi(x, u)$ and

$\eta(x, u)$ and their derivatives with respect to independent variable of order p. Therefore ξ and η appear linearly in equation (1.4.21). As is the situation for a given PDE, the system of symmetry determining equations (1.4.21)-(1.4.22) leads to a system of linear homogeneous PDEs for the determination of $\xi(x, u)$ and $\eta(x, u)$. First we eliminate the components $u^{\nu_\sigma}_{i_1 i_2 \dots i_{k_\sigma}}$ and their differential consequences from (1.4.21) by substitution from (1.4.22) and the differential consequences of (1.4.22), $\sigma = 1, 2, \dots, N$. Consequently, the components of x and u and the remaining components of $\partial u, \partial^2 u, \dots, \partial^p u$ that appear in the resulting system of symmetry determining equations (1.4.21) are themselves independent variables, that is, they take on arbitrary values. Since the resulting expression for (1.4.21) holds for any values of these independent variables, one can obtain a system of linear homogeneous PDEs for $\xi(x, u)$ and $\eta(x, u)$ that constitutes a set of determining equations for the infinitesimal generators V admitted by the given system (1.4.14). In particular, if every $f^\sigma(x, u, \partial u, \partial^2 u, \dots, \partial^k u), \sigma = 1, 2, \dots, N$, is a polynomial in the components of $\partial u, \partial^2 u, \dots, \partial^p u$, then the system of equations (1.4.21) yields polynomial equations in the independent components of $\partial u, \partial^2 u, \dots, \partial^p u$. Consequently, the coefficients of these polynomial equations must vanish separately. This yields the set of linear determining equations for the determination of $\tilde{\xi}$ and $\tilde{\eta}$. Typically, the numbers of determining equations are far greater than $n + m$, so that the system of determining equations is overdetermined.

1.4.1.3 Group Invariant Solutions

Consider a system of PDEs (1.4.14) which admits a one parameter Lie group of point transformations (1.4.6)-(1.4.7) with the infinitesimal generator (1.4.15). We assume that $\tilde{\xi}(x; u) \neq 0$.

Definition 1.4.2 *A solution $u = \theta(x)$, with components $u^\nu = \theta^\nu(x), \nu = 1, 2, \dots, m$, of the system of PDEs (1.4.14) is called a group invariant solution iff the surface*

$u = \theta(x)$ remains invariant under the point transformations (1.4.6)-(1.4.7), that is,

$$\xi(x, \theta(x))\frac{\partial \theta(x)}{\partial x_i} = \eta^\nu(x, \theta(x)), \nu = 1, 2, \ldots, m. \tag{1.4.23}$$

This equation (1.4.23) is called the invariant surface condition for the invariant solutions of the system (1.4.14) resulting from its invariance under the Lie point symmetry (1.4.6)-(1.4.7). As is the situation, the invariant solutions for scalar PDEs can be determined by the following procedure:

1.4.1.4 Invariant Form Method

Here, we first solve the invariant surface condition equations (1.4.23) by explicitly solving the corresponding characteristics equations for $u = \theta(x)$ given by

$$\frac{dx_1}{\xi_1(x,u)} = \frac{dx_2}{\xi_2(x,u)} = \ldots = \frac{dx_n}{\xi_n(x,u)} = \frac{du^1}{\eta^1(x,u)} = \frac{du^2}{\eta^2(x,u)} = \ldots = \frac{du^m}{\eta^m(x,u)}. \tag{1.4.24}$$

If $(y_1, y_2, \ldots, y_{n-1}), (h^1, h^2, \ldots, h^m)$, are $n+m-1$ functionally independent constants of integration that arise from solving the characteristic equations (1.4.24) with the non-zero Jacobian, i.e.

$$\frac{\partial(h^1, h^2 \ldots, h^m)}{\partial(u^1, u^2, \ldots, u^m)} \neq 0,$$

then the general solution $u = \theta(x)$ of the invariant surface conditions (1.4.23) is given implicitly by the following invariant form

$$u^\nu(x, u) = \Phi^\nu(y_1(x,u), y_2(x,u), \ldots, y_{n-1}(x,u)), \tag{1.4.25}$$

where Φ^ν is an arbitrary differentiable function of its arguments, for $\nu = 1, 2, \ldots, m$. Note that $(y_1(x,u), y_2(x,u), \ldots, y_{n-1}(x,u)), h^1(x,u), \ldots, h^m(x,u)$, are $n+m-1$ functionally independent invariants of the one-parameter Lie group of transformations with the infinitesimal generator V given by equation (1.4.15)), and hence are $n+m-1$ canonical coordinates for the one parameter group of transformations. Let $u_n(x,u)$ be the $(n+m)$th canonical coordinate satisfying $Vy_n = 1$. If the system of PDE (1.4.14)) is transformed by the corresponding invertible point transformation into a system of PDE with independent variables $(y_1, y_2, \ldots, y_n)$ and dependent variables $(h^1, h^2, \ldots, h^m)$, then the transformed system has the translation point symmetry given by

$$\tilde{y}_i = y_i, i = 1, 2, \ldots, n-1,$$
$$\tilde{y}_i = y_n + \epsilon,$$
$$\tilde{h}_\nu = h^\nu, \nu = 1, 2, \ldots, m.$$

Thus, the variable y_n does not appear explicitly in the transformed system of a differential equation, and hence the transformed system has exact solutions of the form (1.4.25) that in turn define, implicitly, specific functions $u = \theta(x)$ which are the invariant solutions of the system (1.4.14)), In particular, these invariant solutions are found by solving a reduced system of differential equations with $n-1$ independent variables $(y_1, y_2, \ldots, y_{n-1})$ and m dependent variables $(y_1, y_2, \ldots, y_n)$. The independent variables $(y_1, y_2, \ldots, y_{n-1})$ are commonly called the similarity variables. The reduced system of differential equations is obtained by substituting the invariant form (1.4.25) into the given system (1.4.14). It is assumed that this substitution does not lead to a system of differential equations with a singular equation. Note that if $\frac{\partial \xi_i}{\partial u^\mu} \equiv 0$ as is commonly the case, then $y_i = y_i(x)$, for $i = 1, 2, \ldots, n-1$. In the case when the system (1.4.14) has two independent variables, then, the reduced system is a system of ODE with independent variable y_1.

1.4.1.5 Lie Algebra

For the Lie group of point transformations with infinitesimal generators V_1, V_2, the commutator (Lie bracket) of V_1, V_2 is the first order operator defined by

$$[V_1, V_2] = V_1 V_2 - V_2 V_1 \tag{1.4.26}$$

Definition 1.4.3 *[28] "A Lie algebra is a vector space $\mathbb{L}$ over $\mathbb{R}$ or $\mathbb{C}$ with a bilinear bracket operation (the commutator) satisfying the following properties:*

1. *Bilinearity:*

$$[aV_1 + bV_2, V_3] = a[V_1, V_3] + b[V_2, V_3] \tag{1.4.27}$$

$$[V_1, aV_2 + bV_3] = a[V_1, V_2] + b[V_1, V_3] \tag{1.4.28}$$

2. *Skew-Symmetry:*

$$[V_1, V_2] = -[V_2, V_1] \tag{1.4.29}$$

3. *Jacobi Identity:*

$$[V_1, [V_2, V_3]] + [V_3, [V_1, V_2]] + [V_2, [V_3, V_1]] = 0. \tag{1.4.30}$$

The commutator of two vector fields is again a vector field. Moreover, if V_i and V_j be the two infinitesimal generators of a point transformation, then the commutator of both generators will again be a generator of a Lie group [28]. As a consequence, the set of all infinitesimal generators is closed under a commutation of vector fields, thus possessing more structure than just that of vector space. This additional closure property endows the space of infinitesimal generators with an additional algebraic structure, the so called Lie algebra. Hence, having found some of the infinitesimal generators V_i of an r-parameter symmetry group, it may be possible to find new generators by computing the commutators of the known ones. A common way to visualize the structure of a Lie algebra is the commutator table [28]. Let$V_1, V_2, \dots V_r$ be a basis of r-dimensional Lie algebra, then its commutator table has (i, j)-th entry $[V_i, V_j]$. Because the commutator is antisymmetric it suffices to compute just the part above the diagonal, as $[V_i, V_j] = -[V_j, V_i]$." The commutator table therefore reads as follows:

Table 1.1: Commutator table

	V_1	V_2	$\cdots$	V_r
V_1	0	$[V_1, V_2]$	$\cdots$	$[V_1, V_r]$
V_2	$-[V_1, V_2]$	0	$\cdots$	$[V_2, V_r]$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
V_r	$-[V_1, V_r]$	$-[V_2, V_r]$	$\cdots$	0

1.5 Classical Lie Group Method: An Algorithmic Overview

The classical method substantially consists of finding symmetry reduction of the PDEs with the help of determining equations obtained under the condition of invariance of the system of PDEs. More specifically, when a given system of PDEs (1.4.14) is subjected to invariance under one-parameter symmetry group of transformations (1.4.21)-(1.4.22) one arrives at an overdetermined linear homogeneous system of partial differential equations for the determination of group infinitesimals. These infinitesimals of the transformations help us attain the reduction of the system of PDEs. The procedural steps are as follows:

Consider a system of N PDEs with n independent variables $x = (x_1, x_2, \ldots, x_n)$ and m dependent variables $u = (u^1, u^2, \ldots u^m)$ given by

$$F^\mu(x, u, \partial u, \partial^2 u, \ldots, \partial^k u) = 0, \;\; \mu = 1, 2, \ldots, N. \qquad (1.5.1)$$

1. Let the one parameter group of transformations (1.4.21)-(1.4.22) leaves the system of equations (1.5.1) invariant.

2. Apply the extended infinitesimal operator $V^{(k)}$ given by (1.4.16) to each equation of the system (1.5.1) and require that

   $$V^{(k)} F^\mu(x, u, \partial u, \partial^2 u, \ldots, \partial^k u)|_{F^\nu} = 0, \quad \mu, \nu = 1, 2, \ldots, N. \qquad (1.5.2)$$

   The meaning of the this condition is that $V^{(k)}$ vanishes on the solution set of the original system (1.5.1). Precisely, this condition assures that $u(x)$ is solution of (1.5.1) whenever $\tilde{u}(\tilde{x})$ is one.

3. Following the procedure, as given in section (1.4.1.2), a system of linear PDEs for ξ and η that constitutes a set of determining equations for the infinitesimal generator V admitted by the system of PDEs (1.5.1) is obtained.

4. The solution of determining equations will lead to the explicit forms of ξ and η.

5. Construct the corresponding characteristic equations (1.4.24) and obtain u in terms of new $n-1$ independent variables.

6. Express the given system (1.5.1) in terms of new coordinates to get the reduced form of the given system.

1.6 First integral method

The first integral method was firstly coined by Feng [109] for obtaining the exact traveling solutions of the nonlinear partial differential equations. Various researchers have applied this method to solve NLPDEs. The method mainly consists of following steps [53] :

Step 1: Consider a general system of nonlinear partial differential equations with dependent variables u, v and independent variables x, y and t as follows:

$$\begin{aligned} &F_1(u, v, u_x, v_x, u_y, v_y, u_t, v_t, u_{xx}, v_{xx}, \ldots) = 0, \\ &F_2(u, v, u_x, v_x, u_y, v_y, u_t, v_t, u_{xx}, v_{xx}, \ldots) = 0. \end{aligned} \tag{1.6.1}$$

Step 2: Assume that system (1.6.1) has traveling wave solutions in the form $u(x, y, t) = U(\zeta), v(x, y, t) = V(\zeta), \zeta = lx + my + nt$. Substituting it into system (1.6.1), we get a system of nonlinear ODEs

$$\begin{aligned} &G_1(U, V, U^{'}, V^{'}, U^{''}, V^{''}, \ldots) = 0, \\ &G_2(U, V, U^{'}, V^{'}, U^{''}, V^{''}, \ldots) = 0, \end{aligned} \tag{1.6.2}$$

where the prime ($'$) indicates the differentiation with respect to the variable ζ. Further, by utilizing few mathematical operations, the equations (1.6.2) is converted into second-order ODE

$$R(U, U^{'}, U^{''}) = 0. \tag{1.6.3}$$

Step 3: Now, let us take

$$X(\zeta) = U(\zeta), Y = U_\zeta(\zeta), \tag{1.6.4}$$

then, the equation (1.6.3) is equivalent to

$$\begin{aligned} X_\zeta(\zeta) &= Y(\zeta), \\ Y_\zeta(\zeta) &= F(X(\zeta), Y(\zeta)). \end{aligned} \tag{1.6.5}$$

Step 4: Next, we introduced the Division Theorem, which depends on the group theory of commutative algebra. If we find the two first integral to equations (1.6.5) under the same conditions, then the general solutions to (1.6.5) can be expressed explicitly. However, in general, it is very difficult for us to realize this, even for one first integral, because for a given plane autonomous system, there is no systematic theory that can tell us how to find its first integrals, nor is there a logical way to tell us what these first integrals are. It is embraced to accomplish the one first integral of the equation (1.6.5), which reduces equation (1.6.3) to a first order integrable ordinary differential equation by using the Hilbert-Nullstellensatz Theorem. Thus, the exact solutions to equations (1.6.1) are obtained through solving the resulting first order integrable differential equation.

The Division Theorem is stated as follows :

Division Theorem [53]: "Suppose that $P(w, z)$ and $Q(w, z)$ are polynomials in two variable w and z and $P(w, z)$ is irreducible in $\mathbb{C}[w, z]$. If $Q(w, z)$ vanishes at all zero points of $P(w, z)$, then there exists a polynomial $G(w, z)$ in $\mathbb{C}[w, z]$ such that $Q(w, z) = P(w, z)G(w, z)$."

The division theorem comes from the Hilbert-Nullstellensatz theorem, the theorem stated as:

Hilbert-Nullstellensatz Theorem [53]: "Let K be a field and L be an algebraic closure of K. Then

1. Every ideal γ of $K[X_1, X_2, X_3, \ldots, X_n]$ not containing 1 admits at least one zero in L^n.

2. Let $x = (x_1, x_2, x_3, \ldots, x_n), y = (y_1, y_2, y_3, \ldots, y_n)$, be two elements of L^n. For the set of polynomials of $K[X_1, X_2, X_3, \ldots., X_n]$ zero at x to be identical with the set of polynomials of $K[X_1, X_2, X_3, \ldots, X_n]$ zero at y, it is necessary and

sufficient that there exists a K-automorphism S of L such that $y_i = S(x_i)$ for $1 \leq i \leq n$.

3. For an ideal α of $K[X_1, X_2, X_3, \ldots, X_n]$ to be maximal, it is necessary and sufficient that there exists an x in L^n such that α is the set of polynomials of $K[X_1, X_2, X_3, \ldots, X_n]$ zero at x.

4. For a polynomial Q of $K[X_1, X_2, X_3, \ldots, X_n]$ to be zero on the set of zeros in L^n of an ideal γ of $K[X_1, X_2, X_3, \ldots, X_n]$ it is necessary and sufficient that there exists an integer $m > 0$ such that $Q^m \in \gamma$."

1.7 The $(\frac{G'}{G^2})$-expansion method

Wang et al. [70] have proposed a simple method which is known as the (G'/G)-expansion method to look for exact traveling wave solutions of NLPDEs. The $(\frac{G'}{G^2})$-expansion method [66] is the extension of the $(\frac{G'}{G})$-expansion method. Here, we are providing the brief algorithm of the $(\frac{G'}{G^2})$-expansion method.

Step 1: Consider a general system of nonlinear PDEs as follows:

$$\begin{aligned} &F_1(u, v, u_x, v_x, u_t, v_t, u_y, v_y, u_{yy}, v_{yy}, u_{xx}, v_{xx}...) = 0, \\ &F_2(u, v, u_x, v_x, u_t, v_t, u_y, v_y, u_{yy}, v_{yy}, u_{xx}, v_{xx}...) = 0, \end{aligned} \tag{1.7.1}$$

where $u(x, y, t), v(x, y, t)$ are unknown functions of independent variables $x,\ y,\ t$.

Step 2: Applying the following traveling wave transformation

$$u(x, y, t) = U(\zeta), v(x, y, t) = V(\zeta) \text{alongwith} \zeta = lx + my + nt, \tag{1.7.2}$$

on equation (1.7.1), we obtained the system of nonlinear ordinary differential equations as follows:

$$\begin{aligned} &G_1(U, V, U', V', U'', V'', \ldots) = 0, \\ &G_2(U, V, U', V', U'', V'', \ldots) = 0. \end{aligned} \tag{1.7.3}$$

By means of some mathematical calculations, the system of equations (1.7.3) converted into a single nonlinear ordinary differential equation

$$H(U, U^{'}, U^{''}...) = 0, \tag{1.7.4}$$

where (′) prime indicates the derivative with respect to ζ.

Step 3: Suppose that the solution of equation (1.7.4) can be expressed in the powers of $(\frac{G^{'}}{G^2})$ as

$$U(\zeta) = a_0 + \sum_{k=1}^{M}\left[a_k\Big(\frac{G^{'}}{G^2}\Big)^k + b_k\Big(\frac{G^{'}}{G^2}\Big)^{-k}\right], \tag{1.7.5}$$

with $G = G(\zeta)$ satisfying the following nonlinear ODE

$$\Big(\frac{G^{'}}{G^2}\Big)^{'} = \mu + \lambda\Big(\frac{G^{'}}{G^2}\Big)^2, \tag{1.7.6}$$

where $\mu,\ \lambda$, are arbitrary constants. The unknown constants a_k or b_k can be zero, but both of these constants cannot be zero simultaneously.

The integral value of M can be determined by using the homogeneous balance principle, that is, by balancing the highest order derivative and nonlinear terms present in equation (1.7.4).

Step 5: Next, on using the equation (1.7.5) along with (1.7.6) in equation (1.7.4), we obtain a polynomial in $(\frac{G^{'}}{G^2})$. Collecting all the coefficients of same powers of $(\frac{G^{'}}{G^2})^n$, $(n = 0, \pm1, \pm2, \pm3, \ldots, \pm N)$ and equating them to zero, we obtain a nonlinear algebraic system of equations, which on solving gives the values of unknown parameters $a_0,\ b_k,\ a_k\ l,\ m,\ n$, for $k = 1, 2, 3, ...M$.

Step 6: The general solutions of equation (1.7.6) can be expressed into three cases :

(i) If $\mu\lambda > 0$, the general solution is given by

$$\frac{G^{'}}{G^2} = \sqrt{\frac{\mu}{\lambda}}\Big(\frac{Acos(\sqrt{\mu\lambda})\zeta + Bsin(\sqrt{\mu\lambda})\zeta}{Bcos(\sqrt{\mu\lambda})\zeta - Asin\sqrt{\mu\lambda})\zeta}\Big). \tag{1.7.7}$$

(ii) If $\mu\lambda < 0$, the general solution is as follows:

$$\frac{G^{'}}{G^2} = \frac{1}{2\lambda}\Big(2\sqrt{|\mu\lambda|} - \frac{4A\sqrt{|\mu\lambda|}e^{2\zeta\sqrt{|\mu\lambda|}}}{Ae^{2\zeta\sqrt{|\mu\lambda|}} - B}\Big), \tag{1.7.8}$$

which is equivalent to

$$\frac{G'}{G^2} = -\frac{\sqrt{|\mu\lambda|}}{\lambda}\Big(\frac{Asinh(2\sqrt{|\mu\lambda|})\zeta + Acosh(2\sqrt{|\mu\lambda|})\zeta + B}{Asinh(2\sqrt{|\mu\lambda|})\zeta + Acosh(2\sqrt{|\mu\lambda|})\zeta - B}\Big). \tag{1.7.9}$$

(iii) If $\lambda \neq 0, \mu = 0$, the general solution can be expressed as

$$\frac{G'}{G^2} = -\frac{A}{\lambda(A\zeta + B)}, \tag{1.7.10}$$

where $A,\ B$ are arbitrary constants. The exact solutions of system (1.7.1) can be obtained by substituting the values of the parameters $a_0,\ b_k,\ a_k,\ l,\ m,\ n$ and solutions from equations (1.7.7)-(1.7.10) into equation (1.7.5) alongwith the transformation.

Chapter 2

Invariant solutions of the (2+1) dimensional dispersive long wave system

2.1 Introduction

In 1975, L. F. Broer [57] presented the dispersive long wave equation

$$\begin{aligned} u_t + v_x + uu_x &= 0, \\ v_t + (uv + v + u_{xx})_x &= 0, \end{aligned} \tag{2.1.1}$$

which describes the evolution of the horizontal velocity component $u(x,t)$ of water waves of height $v(x,t)$, propagating along x, y-directions in an infinite narrow channel of finite constant depth. It plays an important role in nonlinear physics [13], considered as a good model for the study of bidirectional solitons in water waves. Various researchers have investigated this equation. Xiaomei Xue and Yushan Bai [103] have reported the Lie symmetries of equation (2.1.1). Multiple soliton solutions were presented by Zhang [52] using the Homogenous balance principle method.

Boiti et al. [61] has extended the equation (2.1.1) in higher-dimensional spaces, which is also called the (2+1) dimensional dispersive long wave equation of the form

$$u_{yt} + v_{xx} + u_x u_y + u u_{xy} = 0,$$
$$v_t + u_x v + v_x u + u_x + u_{xxy} = 0. \tag{2.1.2}$$

The dispersive long wave equation has been shown to be IST solvable by Kaup [57] in 1975 and Matveev and Yavor in 1979 using an energy-dependent Schr$\ddot{o}$dinger spectral problem. Later by Reyman in 1980 and Kupershmidt [13] in 1985 using instead an integro differential spectral problem. After that, in 1986, Konopelchenko obtained the recursion operator for this latter spectral problem. Paquin and Winternitz [34] have proposed the Lie point symmetries alongwith the Kac-Moody-Virasoro subalgebras of equation (2.1.2). Tang [101, 102] have presented the abundant propagating localization excitations by adopting the Painlev$\acute{e}$- B$\ddot{a}$cklund transformation and multilinear variable separation approach. Wanga et al. [39] have generated some interaction solutions. Some similarity reductions and exact solutions were furnished by Yue [56]. Lou [92] also showed that equation (2.1.2) is integrable but it has no Painlev$\acute{e}$ property. Hui et al. [97] proposed the symmetry groups and new exact solutions by utilizing the modified Clarkson Kruskal-direct method. Also, some traveling wave solutions were furnished in [62].

Some authors have considered the (2+1) dimensional dispersive long wave equation of the form [111]

$$u_{yt} + v_{xx} + u_x u_y + u u_{xy} = 0,$$
$$v_t + u_x v + v_x u + u_{xxy} = 0. \tag{2.1.3}$$

Ma et al. [111] studied this equation by utilizing the Clarkson Kruskal-direct method to obtain the symmetry reductions and some particular solutions. The generalized symmetry algebra with arbitrary functions of equation (2.1.3) has been taken up in [93].

In this chapter, we have carried out the Lie symmetry analysis of system (2.1.3) alongwith the Kac-Moody Virasoro type subalgebra and presented the various new exact solutions.

The organization of the chapter is as follows: In section 2.2, we present the Lie symmetry analysis alongwith the Kac-Moody Virasoro type subalgebra. Similarity reductions and exact solutions have been reported in section 2.3. Further, section 2.4 considers the invariance analysis of the reduced partial differential equations. Finally, conclusions are provided in the last section.

2.2 Lie symmetries

In the present section, we furnish the Lie symmetries of the equation (2.1.3) by utilizing the classical symmetry method. Let us consider the one-parameter Lie group of point transformations under which system (2.1.3) remains invariant of the form [44]

$$\begin{aligned} x^* &= x + \epsilon\xi(x,y,t,u,v) + O(\epsilon^2), \\ y^* &= y + \epsilon\eta^1(x,y,t,u,v) + O(\epsilon^2), \\ t^* &= t + \epsilon\eta^2(x,y,t,u,v) + O(\epsilon^2), \\ u^* &= u + \epsilon\tau(x,y,t,u,v) + O(\epsilon^2), \\ v^* &= v + \epsilon\zeta(x,y,t,u,v) + O(\epsilon^2), \end{aligned} \tag{2.2.1}$$

where ϵ is a group parameter and $\xi, \eta^1, \eta^2, \tau, \zeta$ are the infinitesimals. Apply the group of transformations (2.2.1) on the system (2.1.3) and then equating the coefficients of various partial derivative terms to zero. We obtain, from the first equation of system (2.1.3), the list of determining equations as follows :

$$\begin{aligned} \eta^1_u = \eta^1_v = \eta^1_x = \eta^1_t &= 0, \\ \eta^2_u = \eta^2_v = \eta^2_y = \eta^2_x = 0, \zeta_{vv} &= 0, \\ \xi_u = \xi_y = \xi_v = 0, \zeta_u = 0, \tau_v = 0, \tau_{uy} = 0, \tau_{uu} &= 0, \\ \tau_{yt} + u\tau_{xy} + \zeta_{xx} = 0, \tau_{ut} + u\tau_{ux} + \tau_x &= 0, \\ \tau_u - \eta^1_y - \eta^2_t - \zeta_v + 2\xi_x = 0, \tau_y + u\tau_{yu} &= 0, \\ 2\tau_u + \xi_x - \eta^1_y - \zeta_v = 0, 2\zeta_{xv} - \xi_{xx} &= 0, \\ \tau + u\tau_u - \xi_t - u\eta^1_y + u\xi_x - u\zeta_v &= 0. \end{aligned} \tag{2.2.2}$$

Similarly, the second equation of the system (2.1.3) gives the following additional equations:

$$\begin{aligned} \tau_{xxy}+v\tau_x+\zeta_t+u\zeta_x=0, \tau_{uxx}=0,\\ \zeta-v\xi_x+v\tau_u+v\eta^2_t-v\zeta_v=0,\\ \tau_u-\zeta_v-2\xi_x+\eta^2_t-\eta^1_y=0,\\ \tau-u\xi_x-\xi_t+u\eta^2_t=0,\\ 2\tau_{ux}-\xi_{xx}=0. \end{aligned} \tag{2.2.3}$$

Upon simplifying the above two sets, we get the following list of equations

$$\begin{aligned} \eta^1=\eta^1(y), \eta^2=\eta^2(t), \xi=\xi(x,t),\\ \tau=f(x,t)u+g(x,t), \zeta=p(x,y,t)v+q(x,y,t),\\ \tau_{yt}+u\tau_{xy}+\zeta_{xx}=0, \tau_{ut}+u\tau_{ux}+\tau_x=0,\\ 2\tau_u+\xi_x-\eta^1_y-\zeta_v=0, 2\zeta_{xv}-\xi_{xx}=0,\\ \tau+u\tau_u-\xi_t-u\eta^1_y+u\xi_x-u\zeta_v=0,\\ \zeta_t+u\zeta_x+v\tau_x=0, 2\tau_{ux}-\xi_{xx}=0,\\ \zeta-v\xi_x+v\tau_u-v\zeta_v+v\eta^2_t=0,\\ \tau_u-\zeta_v-2\xi_x-\eta^1_y+\eta^2_t=0,\\ \tau_u-\eta^2_t-\eta^1_y+2\xi_x-\zeta_v=0,\\ \tau-u\xi_x-\xi_t+u\eta^2_t=0. \end{aligned} \tag{2.2.4}$$

From equations (2.2.4), we can derive the group infinitesimals $\xi, \eta^1, \eta^2, \zeta$ and τ for the system (2.1.3) as follows :

$$\begin{aligned} \xi(x,t)&=-x\dot{p}(t)-k_1x+\sigma(t),\\ \eta^2(t)&=-2p(t)-2k_1t+k_3,\\ \eta^1(y)&=k_1y+k_2,\ \zeta=\dot{p}(t)v,\\ \tau&=(\dot{p}(t)+k_1)u-x\ddot{p}(t)+\dot{\sigma}(t), \end{aligned} \tag{2.2.5}$$

where $(\cdot)$ dot represents the differentiation w.r.t. t and $\sigma(t)$, $p(t)$ are the arbitrary functions of their arguments and k_1, k_2, k_3 are the arbitrary constants.

The arbitrary functions $\sigma(t)$ and $p(t)$ give rise to an infinite-dimensional Lie algebra of symmetries. Let us write the general element of Lie algebra of infinitesimal generators as follows [69] :

$$H = H_1(p) + H_2(\sigma) + H_3 + H_4 + H_5, \tag{2.2.6}$$

where

$$\begin{aligned} H_1(p(t)) &= -x\dot{p}(t)\frac{\partial}{\partial x} - 2p(t)\frac{\partial}{\partial t} + \dot{p}(t)v\frac{\partial}{\partial v} + (\dot{p}(t)u - x\ddot{p}(t))\frac{\partial}{\partial u}, \\ H_2(\sigma(t)) &= \sigma(t)\frac{\partial}{\partial x} + \dot{\sigma}(t)\frac{\partial}{\partial u}, \\ H_3 &= -x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - 2t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}, \\ H_4 &= \frac{\partial}{\partial y},\ H_5 = \frac{\partial}{\partial t}. \end{aligned} \tag{2.2.7}$$

Next, the associated commutator relations between the vector fields (2.2.7) are given in the Table 2.1.

Table 2.1: Commutator table

	H_1	H_2	H_3	H_4	H_5
H_1	0	$-H_2(2p\dot{\sigma}) + \sigma\dot{p}\frac{\partial}{\partial x} - \dot{\sigma}\dot{p}\frac{\partial}{\partial u} + \sigma\ddot{p}\frac{\partial}{\partial u}$	$H_1(2t\dot{p}) + 4p\frac{\partial}{\partial t} - 2x\ddot{p}\frac{\partial}{\partial u}$	0	$-H_1(\dot{p})$
H_2	$H_2(2p\dot{\sigma}) - \sigma\dot{p}\frac{\partial}{\partial x} + \dot{\sigma}\dot{p}\frac{\partial}{\partial u} - \sigma\ddot{p}\frac{\partial}{\partial u}$	0	$-\sigma\frac{\partial}{\partial x} + \dot{\sigma}\frac{\partial}{\partial u} + H_2(2t\dot{\sigma})$	0	$-H_2(\dot{\sigma})$
H_3	$-H_1(2t\dot{p}) - 4p\frac{\partial}{\partial t} + 2x\ddot{p}\frac{\partial}{\partial u}$	$-H_2(2t\dot{\sigma}) + \sigma\frac{\partial}{\partial x} - \dot{\sigma}\frac{\partial}{\partial u}$	0	$-H_4$	$2H_5$
H_4	0	0	H_4	0	0
H_5	$H_1(\dot{p})$	$H_2(\dot{\sigma})$	$-2H_5$	0	0

Another commutation relations turn out to be

$$\begin{aligned} [H_1(p_1(t)), H_1(p_2(t))] &= H_1(2p_1(t)\dot{p_2}(t)) + H_1(2p_2(t)\dot{p_1}(t)) + 2x(\dot{p_1}(t)\ddot{p_2}(t) \\ &\qquad - \dot{p_2}(t)\ddot{p_1}(t))\frac{\partial}{\partial u}, \\ [H_2(\sigma_1(t)), H_2(\sigma_2(t))] &= 0, \end{aligned} \tag{2.2.8}$$

which is again an infinite-dimensional symmetry algebra and is of the form Virasoro type subalgebra [69] which commonly exists in most of the integrable equations. Now, by restricting the arbitrary functions $p(t)$ and $\sigma(t)$ to Laurent polynomials in variable

t, a base for this subalgebra is given by

$$H_1(t^n) = -nxt^{n-1}\frac{\partial}{\partial x} - 2t^n\frac{\partial}{\partial t} + nt^{n-1}v\frac{\partial}{\partial v} + (nt^{n-1}u - n(n-1)xt^{n-2})\frac{\partial}{\partial u},$$
$$H_2(t^n) = t^n\frac{\partial}{\partial x} + nt^{n-1}\frac{\partial}{\partial u}, \ n \in \mathbb{Z}. \tag{2.2.9}$$

The commutation relations between these subalgebras are

$$[H_1(t^n), H_1(t^m)] = 2(n-m)H_1(t^{m+n-1}), \ [H_2(t^n), H_2(t^m)] = 0,$$
$$[H_1(t^n), H_2(t^m)] = (n-2m)H_2(t^{m+n-1}), \tag{2.2.10}$$

which are of the form of Kac-Moody-Virasoro type subalgebra [67], [68]. This subalgebra is similar to the Kac-Moody-Virasoro type subalgebra as shown in [34] for the system (2.1.2). This type of algebra also exists in other integrable equations such as [18], [67], [69], etc.

2.3 Similarity reductions and Exact solutions

In this section, we propose the similarity reductions and exact solutions for the corresponding infinitesimal symmetries (2.2.5) by solving the following characteristic equations [79]

$$\frac{dx}{\xi} = \frac{dy}{\eta^1} = \frac{dt}{\eta^2} = \frac{du}{\tau} = \frac{dv}{\zeta}, \tag{2.3.1}$$

where ξ, η^1, η^2, τ and ζ are given by equations (2.2.5). Next, we analyze the following particular cases:

Case 1: For vector field $H_1(p(t))$, the characteristic equations (2.3.1) become

$$\frac{dx}{-x\dot{p}(t)} = \frac{dy}{0} = \frac{dt}{-2p(t)} = \frac{du}{(\dot{p}(t)u - x\ddot{p}(t))} = \frac{dv}{\dot{p}(t)v}. \tag{2.3.2}$$

On solving the first equality, first and third terms, first and fifth terms of equations (2.3.2), we have

$$\alpha = y, \beta = \frac{p(t)}{x^2}, v = \frac{F(\alpha, \beta)}{x} \tag{2.3.3}$$

respectively. For u, we solve the first and fourth terms of equations (2.3.2), which give

$$d(xu) = \frac{\ddot{p}(t)}{\dot{p}(t)}xdx. \tag{2.3.4}$$

Next, we have

$$p(t) = x^2\beta \Longrightarrow dp = 2x\beta dx. \tag{2.3.5}$$

Replacing xdx by $\frac{dp}{2\beta}$ in equation (2.3.4) and then integrating, we get

$$u = \frac{\dot{p}(t)}{2p(t)}x + \frac{G(\alpha,\beta)}{x}, \tag{2.3.6}$$

where $\alpha,\ \beta$ are similarity variables and $F(\alpha,\beta)$ and$G(\alpha,\beta)$ are new dependent variables.

Using equations (2.3.3), (2.3.6) in equations (2.1.3), the system of partial differential equations (PDEs) reduces to

$$\begin{aligned} &2\beta^2 F_{\beta\beta} + 5\beta F_\beta + F - GG_\alpha - \beta G_\alpha G_\beta - \beta GG_{\alpha\beta} = 0,\\ &2\beta^2 G_{\alpha\beta\beta} + 5\beta G_{\alpha\beta} - \beta FG_\beta + \beta GF_\beta - FG + G_\alpha = 0. \end{aligned} \tag{2.3.7}$$

Observe that the first equation of (2.3.7) can be written as

$$\frac{\partial}{\partial\beta}(2\beta^2 F_\beta) + \frac{\partial}{\partial\beta}(\beta F) - \frac{\partial}{\partial\beta}(\beta GG_\alpha) = 0, \tag{2.3.8}$$

which readily integrates to give

$$2\beta^2 F_\beta + \beta F - \beta GG_\alpha = \mu'(\alpha), \tag{2.3.9}$$

where $\mu(\alpha)$ is an arbitrary function of α and prime ($'$) represents the differentiation. Next, by taking the transformation $\beta = \exp(\nu)$, the second equation of equations (2.3.7) and equation (2.3.9) can be written respectively, as

$$\begin{aligned} &2\frac{\partial^2}{\partial\nu^2}G_\alpha + 3\frac{\partial}{\partial\nu}G_\alpha + G_\alpha + G\frac{\partial}{\partial\nu}F - F\frac{\partial}{\partial\nu}G - FG = 0,\\ &2\frac{\partial F}{\partial\nu} + F - GG_\alpha = e^{-\nu}\mu^{'}(\alpha). \end{aligned} \tag{2.3.10}$$

Since the system (2.3.10) is difficult to handle in its generality, therefore, we make an assumption. Let

$$2\frac{\partial^2}{\partial\nu^2}G_\alpha + 3\frac{\partial}{\partial\nu}G_\alpha + G_\alpha = 0. \tag{2.3.11}$$

Equation (2.3.11) yields to give

$$G_\alpha = A^{'}(\alpha)e^{-\nu} + B^{'}(\alpha)e^{\frac{-\nu}{2}}, \tag{2.3.12}$$

where $A(\alpha)$and$B(\beta)$ are arbitrary functions of their argument. On further integration, it gives

$$G(\alpha,\nu) = A(\alpha)e^{-\nu} + B(\alpha)e^{\frac{-\nu}{2}} + \psi(\nu), \tag{2.3.13}$$

where $\psi(\nu)$ is an arbitrary function of integration. On ignoring the arbitrary function $\psi(\nu)$, and by the compatibility of the remaining equations (2.3.10), we get

$$3FG - G^2G_\alpha + 2F\frac{\partial G}{\partial \nu} = e^{-\nu}\mu^{'}(\alpha)G. \tag{2.3.14}$$

Substituting equation (2.3.13) into equation (2.3.14), we obtain

$$F(\alpha,\nu) = \frac{1}{A(\alpha)e^{-\nu} + 2B(\alpha)e^{\frac{-\nu}{2}}}\left[\begin{array}{r} A(\alpha)\mu^{'}(\alpha)e^{-2\nu} + B(\alpha)\mu^{'}(\alpha)e^{\frac{-3\nu}{2}} + A^{'}(\alpha)A(\alpha)^2e^{-3\nu} + \\ +B^{'}(\alpha)A(\alpha)^2e^{\frac{-5\nu}{2}} + A^{'}(\alpha)B(\alpha)^2e^{-2\nu} + B^{'}(\alpha)B(\alpha)^2e^{\frac{-3\nu}{2}} + \\ +2A(\alpha)B(\alpha)B^{'}(\alpha)e^{-2\nu} + 2A(\alpha)B(\alpha)A^{'}(\alpha)e^{\frac{-5\nu}{2}} \end{array}\right]. \tag{2.3.15}$$

Now, substituting the equations (2.3.13) and (2.3.15) into equation (2.3.10) and comparing the coefficients of various exponential terms and equating them to zero, we get two sets of nonlinear algebraic equations as

$$\begin{aligned}
-10A(\alpha)^2B(\alpha)\mu^{'}(\alpha) + 29A(\alpha)^2B(\alpha)^2B^{'}(\alpha) + 29A(\alpha)B(\alpha)^3A^{'}(\alpha) &= 0,\\
8A(\alpha)^3B(\alpha)B^{'}(\alpha) + 19A(\alpha)^2B(\alpha)^2A^{'}(\alpha) - A(\alpha)^3\mu^{'}(\alpha) &= 0,\\
11A(\alpha)B(\alpha)^3B^{'}(\alpha) + 4B(\alpha)^4A^{'}(\alpha) - 7A(\alpha)B(\alpha)^2\mu^{'}(\alpha) &= 0,\\
A(\alpha)^4B^{'}(\alpha) + 7A(\alpha)^3B(\alpha)A^{'}(\alpha) &= 0,\\
B(\alpha)^4B^{'}(\alpha) - B(\alpha)^3\mu^{'}(\alpha) &= 0,\\
A(\alpha)^4A^{'}(\alpha) &= 0,
\end{aligned} \tag{2.3.16}$$

and

$$\begin{aligned}
16A(\alpha)B(\alpha)^2B^{'}(\alpha) + 12B(\alpha)^3A^{'}(\alpha) - 8A(\alpha)B(\alpha)\mu^{'}(\alpha) &= 0,\\
21A(\alpha)B(\alpha)^2A^{'}(\alpha) + 13A(\alpha)^2B(\alpha)B^{'}(\alpha) - 2A(\alpha)^2\mu^{'}(\alpha) &= 0,\\
17A(\alpha)^2B(\alpha)A^{'}(\alpha) - 3A(\alpha)^3B^{'}(\alpha) &= 0,\\
B(\alpha)^3B^{'}(\alpha) - B(\alpha)^2\mu^{'}(\alpha) &= 0,\\
A(\alpha)^3A^{'}(\alpha) &= 0.
\end{aligned} \tag{2.3.17}$$

On combining equations (2.3.16) and (2.3.17) and solving, the following subcases arise:

Subcase 1: $A(\alpha) = 0$. In this case, we obtain

$$B(\alpha) = \pm\sqrt{2\mu(\alpha) + a_1}. \tag{2.3.18}$$

Therefore, equations (2.3.13) and (2.3.15) become

$$G(\alpha, \beta) = \pm\frac{\sqrt{2\mu(\alpha) + a_1}}{\sqrt{\beta}},\ F(\alpha, \beta) = 0. \tag{2.3.19}$$

Using equation (2.3.19) in equations (2.3.3), (2.3.6), we have

$$u(x, y, t) = \frac{\dot{p}(t)x}{2p(t)} \pm \sqrt{\frac{2\mu(y) + a_1}{p(t)}},\ v(x, y, t) = 0, \tag{2.3.20}$$

where $\mu(y)$ is an arbitrary function and a_1 is an arbitrary constant.

Subcase 2: $A(\alpha) = a_1 \neq 0$. In this case, we get

$$B(\alpha) = b_1,\ \mu(\alpha) = c_1. \tag{2.3.21}$$

where b_1 and c_1 are arbitrary constants.

Using equation (2.3.21) into (2.3.13) and (2.3.15), we have

$$G(\alpha, \beta) = a_1\beta^{-1} + b_1\beta^{\frac{-1}{2}},\ F(\alpha, \beta) = 0. \tag{2.3.22}$$

Substituting equations (2.3.22) into equations (2.3.3), (2.3.6), we get

$$u(x, y, t) = \frac{\dot{p}(t)x}{2p(t)} + \frac{a_1x + b_1\sqrt{p(t)}}{p(t)}, v(x, y, t) = 0, \tag{2.3.23}$$

where a_1 and b_1 are arbitrary constants and $p(t)$ is an arbitrary function of t.

Case 2: Similarity variables corresponding to vector field $H_2(\sigma(t))$, are as follows:

$$\alpha = t,\ \beta = y,\ v = G(\alpha, \beta),\ u = x\frac{\dot{\sigma}(t)}{\sigma(t)} + F(\alpha, \beta). \tag{2.3.24}$$

Using equations (2.3.24) in equation (2.1.3), the reduced system of PDEs take the form

$$\begin{aligned} F_{\alpha\beta} + \frac{\dot{\sigma}(\alpha)}{\sigma(\alpha)}F_\beta &= 0, \\ G_\alpha + \frac{\dot{\sigma}(\alpha)}{\sigma(\alpha)}G &= 0. \end{aligned} \tag{2.3.25}$$

Integrating these equations, we obtain

$$F(\alpha,\beta)=\frac{A(\beta)}{\sigma(\alpha)}+B(\alpha), \tag{2.3.26}$$

and

$$G(\alpha,\beta)=\frac{D(\beta)}{\sigma(\alpha)},\ \ \sigma(\alpha)\neq 0. \tag{2.3.27}$$

where $A(\beta)$, $B(\alpha)$ and $D(\beta)$ are arbitrary functions of their arguments.

Substituting equations (2.3.26) and (2.3.27) into equations (2.3.24), we get

$$\begin{aligned}
u(x,y,t) &= \frac{\dot{\sigma(t)}}{\sigma(t)}x+\frac{A(y)}{\sigma(t)}+B(t),\\
v(x,y,t) &= \frac{D(y)}{\sigma(t)},\ \ \sigma(t)\neq 0.
\end{aligned} \tag{2.3.28}$$

Remark: The solution (2.3.28) coincides with the one as reported by Ma et al. in [111].

Case 3: If $k_3\neq 0$ and all other parameters in the given symmetries (2.2.5) are zero, then the characteristic equations (2.3.1) yield

$$x=\alpha, y=\beta, u=F(\alpha,\beta), v=G(\alpha,\beta). \tag{2.3.29}$$

Using equations (2.3.29) into equation (2.1.3), the reduced system of PDEs is

$$\begin{aligned}
G_{\alpha\alpha}+F_\alpha F_\beta+FF_{\alpha\beta}&=0,\\
GF_\alpha+FG_\alpha+F_{\alpha\alpha\beta}&=0.
\end{aligned} \tag{2.3.30}$$

After solving equations (2.3.30) using MAPLE software, we obtain

$$\begin{aligned}
&F(\alpha,\beta)=0, G(\alpha,\beta)=f_1(\beta)\alpha+f_2(\beta),\\
&F(\alpha,\beta)=\frac{-1}{c_1\alpha+c_2}, G(\alpha,\beta)=f_1(\beta)(c_1\alpha+c_2),\\
&F(\alpha,\beta)=f_1(\beta), G(\alpha,\beta)=f_2(\beta),\\
&F(\alpha,\beta)=\frac{2}{2f_1(\beta)-\alpha}, G(\alpha,\beta)=\left(\frac{4\frac{d}{d\beta}f_1(\beta)}{(2f_1(\beta)-\alpha)^3}+f_2(\beta)\right)(2f_1(\beta)-\alpha),\\
&F(\alpha,\beta)=\frac{2}{2f_1(\beta)+\alpha}, G(\alpha,\beta)=\left(\frac{-4\frac{d}{d\beta}f_1(\beta)}{(2f_1(\beta)+\alpha)^3}+f_2(\beta)\right)(2f_1(\beta)+\alpha),
\end{aligned} \tag{2.3.31}$$

where $f_1,\ f_2$ are arbitrary functions of β and c_1, c_2 are the arbitrary constants. Substituting equations (2.3.31) in equations (2.3.29), we find $u(x, y, t)$ and $v(x, y, t)$ respectively, as follows:

$$\begin{aligned}
&u(x, y, t) = 0, v(x, y, t) = f_1(y)x + f_2(y),\\
&u(x, y, t) = \frac{-1}{c_1x + c_2}, v(x, y, t) = f_1(y)(c_1x + c_2),\\
&u(x, y, t) = f_1(y), (x, y, t) = f_2(y),\\
&u(x, y, t) = \frac{2}{2f_1(y) - x}, v(x, y, t) = \left(\frac{4\frac{d}{dy}f_1(y)}{(2f_1(y) - x)^3} + f_2(y)\right)(2f_1(y) - x),\\
&u(x, y, t) = \frac{2}{2f_1(y) + x}, v(x, y, t) = \left(\frac{-4\frac{d}{dy}f_1(y)}{(2f_1(y) + x)^3} + f_2(y)\right)(2f_1(y) + x).
\end{aligned} \tag{2.3.32}$$

Case 4: If $k_2 \neq 0$ and all other parameters in the symmetries (2.2.5) are zero, then the equations (2.3.1) yield

$$x = \alpha, t = \beta, u = F(\alpha, \beta), v = G(\alpha, \beta), \tag{2.3.33}$$

Next, using equations (2.3.33) into equation (2.1.3), the reduced system of PDEs is

$$F_{\alpha\alpha} = 0,\ G_\beta + GF_\alpha + FG_\alpha = 0. \tag{2.3.34}$$

Further, on solving equations (2.3.34) with the aid of software MAPLE, we get

$$\begin{aligned}
&F(\alpha, \beta) = \left(\frac{d}{d\beta}(c_1(\beta))\right)\alpha + c_2(\beta),\\
&G(\alpha, \beta) = c_3\left(\alpha e^{-c_1(\beta)} - \int c_2(\beta)e^{-c_1(\beta)}d\beta\right)e^{-\int\left(\frac{d}{d\beta}c_1(\beta)\right)d\beta},
\end{aligned} \tag{2.3.35}$$

where $c_1(\beta),\ c_2(\beta)$ are arbitrary functions and c_3 is an arbitrary constant. Substituting equations (2.3.35) into equations (2.3.33), we obtain $u(x, y, t)$ and $v(x, y, t)$ respectively as

$$\begin{aligned}
&u(x, y, t) = \left(\frac{d}{dt}(c_1(t))\right)x + c_2(t),\\
&v(x, y, t) = c_3\left(xe^{-c_1(t)} - \int c_2(t)e^{-c_1(t)}dt\right)e^{-\int\left(\frac{d}{dt}c_1(t)\right)dt}.
\end{aligned} \tag{2.3.36}$$

Case 5: If k_1, k_2 and k_3 are non-zero and all other parameters in the symmetries (2.2.5) are zero, then the equations (2.3.1) yield the similarity variables as follows:

$$\alpha = x\left(y + \frac{k_2}{k_1}\right), \beta = \frac{x^2}{t - \frac{k_3}{2k_1}}, u = \frac{F(\alpha, \beta)}{x},\ v = G(\alpha, \beta),\ k_1 \neq 0. \tag{2.3.37}$$

On substituting equations (2.3.37) into equation (2.1.3), the reduced PDEs take the form

$$\alpha(GF)_\alpha - \beta^2 G_\beta + 2\beta(GF)_\beta - GF + \alpha^2 F_{\alpha\alpha\alpha} + 4\alpha\beta F_{\alpha\alpha\beta} + 2\beta F_{\alpha\beta} + 4\beta^2 F_{\alpha\beta\beta} = 0,$$
$$\alpha^2 G_{\alpha\alpha} - \beta^2 F_{\alpha\beta} + 4\alpha\beta G_{\alpha\beta} + 2\beta G_\beta + 4\beta^2 G_{\beta\beta} + \alpha(F_\alpha)^2 - FF_\alpha + 2\beta F_\alpha F_\beta$$
$$+ \alpha F F_{\alpha\alpha} + 2\beta F F_{\beta\alpha} = 0. \quad (2.3.38)$$

This system is further analyzed for its invariance properties in section 2.4.

Case 6: If k_2 and k_3 are non-zero and all other parameters in the given symmetries (2.2.5) are zero, then the characteristic equations (2.3.1) gives

$$x = \alpha, k_3 y - k_2 t = \beta, u = F(\alpha, \beta), v = G(\alpha, \beta), \quad (2.3.39)$$

and the system (2.1.3) reduces to

$$G_{\alpha\alpha} + k_3 F_\alpha F_\beta + k_3 F F_{\alpha\beta} - k_2 k_3 F_{\beta\beta} = 0,$$
$$GF_\alpha + FG_\alpha + k_3 F_{\alpha\alpha\beta} - k_2 G_\beta = 0, \quad (2.3.40)$$

where k_2 and k_3 are arbitrary constants. The above system is further analyzed for its invariance analysis in section 2.4.

2.4 Symmetry analysis and Exact solutions of reduced PDEs

The reduced PDEs (2.3.38) in two independent variables can be further analyzed for their invariance properties with the help of the Lie symmetry approach. Consider the one-parameter Lie group of point transformations under which equations (2.3.38) remain invariant of the form [44]

$$\alpha_1 = \alpha + \epsilon\xi^1(\alpha, \beta, F, G) + O(\epsilon^2),$$
$$\beta_1 = \beta + \epsilon\xi^2(\alpha, \beta, F, G) + O(\epsilon^2),$$
$$F_1 = F + \eta^1(\alpha, \beta, F, G) + O(\epsilon^2), \quad (2.4.1)$$
$$G_1 = G + \eta^2(\alpha, \beta, F, G) + O(\epsilon^2).$$

On applying the group of point transformations (2.4.1) on equations (2.3.38), we obtain the symmetries as follows:

$$\xi^1 = c_1\alpha,\ \xi^2 = 0,\ \eta^1 = 0,\ \eta^2 = -c_1 G, \tag{2.4.2}$$

where c_1 is an arbitrary constant.

Exact solutions:

The similarity variables associated with the symmetries (2.4.2) can be obtained by solving the following characteristic equations

$$\frac{d\alpha}{\xi^1} = \frac{d\beta}{\xi^2} = \frac{dF}{\eta^1} = \frac{dG}{\eta^2}, \tag{2.4.3}$$

where $\xi^1,\ \xi^2,\ \eta^1,\ \eta^2$ are given by (2.4.2).

Integrating equations (2.4.3), we obtain the similarity variable as follows

$$\beta = \lambda, F = L(\lambda), G = \frac{M(\lambda)}{\alpha}. \tag{2.4.4}$$

On using equations (2.4.4) in equations (2.3.38), the system of PDEs reduces to system of ordinary differential equations (ODEs) as follows:

$$\begin{aligned} 2\lambda ML^{'} + 2\lambda LM^{'} - \lambda^2 M^{'} - 2LM = 0,\\ 2\lambda^2 M^{''} - \lambda M^{'} + M = 0. \end{aligned} \tag{2.4.5}$$

Further, on solving the system (2.4.5) yields

$$M = a\lambda^{\frac{1}{2}} + b\lambda,\ L = \frac{\lambda}{2} + \frac{c}{2a\lambda^{\frac{-1}{2}} + 2b}, \tag{2.4.6}$$

where a, b, c are arbitrary constants.

Consequently, we have

$$\begin{aligned} F(\alpha,\beta) &= \frac{\beta}{2} + \frac{c}{2a\beta^{\frac{-1}{2}} + 2b},\\ G(\alpha,\beta) &= \frac{a\beta^{\frac{1}{2}} + b\beta}{\alpha}. \end{aligned} \tag{2.4.7}$$

On substituting equations (2.4.7) into equations (2.3.37), we find the exact solution as follows

$$u(x,y,t)=\frac{1}{2}\left(\frac{x}{t-\frac{k_3}{2k_1}}\right)+c\left[2a\left(\frac{x^2}{t-\frac{k_3}{2k_1}}\right)^{\frac{-1}{2}}x+2bx\right]^{-1},$$

$$v(x,y,t)=\frac{a\left(\frac{x^2}{t-\frac{k_3}{2k_1}}\right)^{\frac{1}{2}}+\frac{bx^2}{t-\frac{k_3}{2k_1}}}{x\left(y+\frac{k_2}{k_1}\right)},\ k_1\neq 0. \tag{2.4.8}$$

Next, applying the group of point transformations (2.4.1) on equations (2.3.40), we obtain the following symmetries

$$\xi^1=\frac{1}{2}c_1\alpha+c_3,\ \xi^2=c_1\beta+c_2,\ \eta^1=\frac{-c_1F}{2},\ \eta^2=\frac{-3c_1G}{2}, \tag{2.4.9}$$

where c_1, c_2 and c_3 are arbitrary constants.

Exact solutions:

The similarity variables associated with the point symmetries (2.4.9) can be obtained by solving the characteristic equations (2.4.3). We now present the following particular cases:

Case 1: If $c_1\neq 0$ and $c_2,\ c_3$ are zero, then the characteristic equations (2.4.3) give

$$\frac{\alpha^2}{\beta}=\lambda, F=\frac{L(\lambda)}{\alpha}, G=\frac{M(\lambda)}{\alpha^3}. \tag{2.4.10}$$

Using equations (2.4.10) in equations (2.3.40), the system of PDEs reduces to

$$\begin{aligned}4\alpha^2M^{''}+10\alpha M^{'}+12M-k_2k_3(2\alpha^3L^{'}+\alpha^4L^{''})-k_3(2\alpha^3(L^{'})^2+2\alpha^3LL^{''})=0,\\ k_2\alpha^2M^{'}-4LM+2\alpha ML^{'}+2\alpha LM^{'}-k_3(6\alpha^3L^{''}+4\alpha^4L^{'''})=0.\end{aligned} \tag{2.4.11}$$

Since the above system is difficult to deal in its generality. So, we make an assumption. Let

$$4\alpha^2M^{''}+10\alpha M^{'}+12M=0, \tag{2.4.12}$$

which can easily give

$$M(\alpha)=\alpha^{\frac{-3}{4}}\left[c_1\cos\left(\frac{\sqrt{39}\log\alpha}{4}\right)+c_2\sin\left(\frac{\sqrt{39}\log\alpha}{4}\right)\right]. \tag{2.4.13}$$

Now, from the remaining part of the first equation of equations (2.4.11), we can deduce that

$$L'' = \frac{-2(k_2L' + (L')^2)}{k_2\alpha + 2L}. \tag{2.4.14}$$

Differentiating it once, we get

$$L''' = \frac{6(k_2)^2L' + 18k_2(L')^2 + 12(L')^3}{(k_2\alpha + 2L)^2}. \tag{2.4.15}$$

Next, substituting equation (2.4.14) and (2.4.15) into second equation of equations (2.4.11), we have

$$\begin{aligned}
&k_2^3\alpha^4M' - 12k_2^2k_3\alpha^4L' + 24k_2k_3\alpha^3LL' + 24k_3\alpha^3L(L')^2 - 60k_2k_3\alpha^4(L')^2 - 48k_3\alpha^4(L')^3 \\
&+ 8\alpha L^3M' + 6k_2^2\alpha^3LM' + 12k_2\alpha^2L^2M' + 8\alpha L^2L'M + 2k_2^2\alpha^3ML' + 8k_2\alpha^2LL'M \\
&- 16L^3M - 4k_2^2\alpha^2LM - 16k_2\alpha L^2M = 0,
\end{aligned} \tag{2.4.16}$$

where $M(\alpha) = \alpha^{\frac{-3}{4}}\left(c_1\cos\left(\frac{\sqrt{39}\log\alpha}{4}\right) + c_2\sin\left(\frac{\sqrt{39}\log\alpha}{4}\right)\right)$.

Since the equation (2.4.16) is of the type [33], therefore, it can be written as

$$\begin{aligned}
&- 48k_3x^4p^3 + (24k_3x^3y - 60k_2k_3x^4)p^2 + \big(24k_2k_3x^3y + 8xy^2M - 12k_2^2k_3x^4 \\
&+ 2k_2^2x^2M + 8k_2x^2yM\big)p + 8xy^3M' + 6k_2^2x^3yM' + 12k_2x^2y^2M' + k_2^3x^4M' \\
&- 16y^3M - 4k_2^2x^2yM - 16k_2xy^2M = 0,
\end{aligned} \tag{2.4.17}$$

where $L' = \frac{dy}{dx} = p$, $L = y, \alpha = x$.

Now, differentiating equation (2.4.17) partially with respect to p, we get

$$\begin{aligned}
-144x^4k_3p^2 + (48k_3x^3y - 120k_2k_3x^4)p + 24k_2k_3x^3y + 8xy^2M - 12k_2^2k_3x^4 \\
+2k_2^2x^2M + 8k_2x^2yM = 0.
\end{aligned} \tag{2.4.18}$$

Here the p discriminant of equation (2.4.18) becomes

$$p = \frac{1}{12k_3x^2}\left[-5k_2k_3x^2 + 2k_3xy \pm \sqrt{\begin{matrix} 13k_2^2k_3^2x^4 + 4k_2k_3^2x^3y + 2k_2^2k_3x^2M + \\ +8k_2k_3x^2yM + 4k_3^2x^2y^2 + 8k_3xy^2M \end{matrix}}\right], \tag{2.4.19}$$

On integration of (2.4.19), we can obtain y. Refer to [33] for more details.

Case 2: If $c_3 \neq 0$ and $c_1,\ c_2$ are zero, then the characteristic equations (2.4.3) give

$$\beta = \lambda, F = L(\lambda), G = M(\lambda). \tag{2.4.20}$$

Using equations (2.4.20) in equations (2.3.40), the system of PDEs reduces to

$$L_{\lambda\lambda} = 0,\ \ M_{\lambda} = 0. \tag{2.4.21}$$

On integrating equations (2.4.21), we obtain

$$L = a\lambda + b, M = c. \tag{2.4.22}$$

Now, using equations (2.4.22) into equations (2.4.20), we get

$$F(\alpha, \beta) = a\beta + b, G(\alpha, \beta) = c, \tag{2.4.23}$$

where $a,\ b$ and c are arbitrary constants.
After substituting the equations (2.4.23) into equations (2.3.39), we have

$$u(x, y, t) = a(k_3 y - k_2 t) + b, v(x, y, t) = c. \tag{2.4.24}$$

Case 3: If $c_2 \neq 0$ and $c_1,\ c_3$ are zero, then the characteristic equations (2.4.3) yield

$$\alpha = \lambda, F = L(\lambda), G = M(\lambda). \tag{2.4.25}$$

Using equation (2.4.25) in equation (2.3.40), system of PDEs reduces to system of ODEs

$$\begin{aligned} M_{\lambda\lambda} &= 0, \\ LM_{\lambda} + ML_{\lambda} &= 0. \end{aligned} \tag{2.4.26}$$

On integrating the first equation of (2.4.26), we obtain

$$M = a\lambda + b. \tag{2.4.27}$$

Now, using equation (2.4.27) into the second equation of (2.4.26), we have

$$L = (a\lambda + b)^{-1} c. \tag{2.4.28}$$

Using (2.4.27) and (2.4.28) into equation (2.4.25), we have

$$\begin{aligned} F(\alpha, \beta) &= (a\alpha + b)^{-1} c, \\ G(\alpha, \beta) &= a\alpha + b, \end{aligned} \tag{2.4.29}$$

where a, b, c are arbitrary constants. After substituting (2.4.29) into equation (2.3.39), we have

$$u(x, y, t) = (ax + b)^{-1} c, , v(x, y, t) = ax + b. \tag{2.4.30}$$

2.5 Discussion

In this chapter, an attempt has been made to illustrate the application of the classical Lie symmetry technique to study the (2+1) dimensional dispersive long wave equation. The Lie symmetries, similarity reductions, and closed-form solutions for the DLW system are presented. We have pointed out that the underlying symmetry algebra of this equation is infinite-dimensional, and it exhibits the Kac-Moody-Virasoro type subalgebra, which is possible in various other integrable equations too. Using Lie symmetries, we obtain the similarity reductions and presented the various particular solutions. Some similarity reductions are further analyzed by the method of Lie group of infinitesimal transformations and obtained the exact invariant solutions. Also, we recover a solution given by Ma et al. [111] wherein the authors had used the Clarkson Kruskal direct method to study its symmetry reductions (refer to equation (2.3.28)).

Chapter 3

Symmetry analysis of the Schrödinger equation with time dependent coefficients

3.1 Introduction

Triki and Biswas [38] introduced the generalization of the Kaup-Newell (KN) model, called as new derivative nonlinear Schrödinger model

$$iq_t + aq_{xx} + ib(|q|^{2n}q)_x = 0, \tag{3.1.1}$$

where a, and, b are nonzero real arbitrary constants and q is complex variable. For $n > 2$, equation (3.1.1) includes the non-Kerr dispersion term $(|q|^{2n}q)_x$. For $n = 2$ equation (3.1.1) becomes the Kaup-Newell equation. The presence of derivative nonlinear term of arbitrary order in the equation (3.1.1) appears to be a natural way to further enhance this model for describing the propagation of sufficiently short pulses [38]. Many mathematicians investigated this equation by using different techniques. Triki and Biswas [38] investigated the exact chirped solitons, bright solutions, kink solutions, and conservation laws in the single-mode optical fiber of equation (3.1.1). Some exact chirped singular solutions were presented by Zhau et al. in [85]. Yildirim

[108] has proposed some optical singular and bright solutions using Trial equation architecture. Some optical solutions of equation (3.1.1) were furnished by using Sine-Gordon expansion and the Riccatti Bernoulli sub-ODE methods [3].

In view of equation (3.1.1), we generalize the constant coefficients to complex variable coefficients and consider the nonlinear Schrödinger equation of the form

$$iq_t + (a_1(t) + ia_2(t))q_{xx} + (b_1(t) + ib_2(t))(|q|^{2n}q)_x = 0, \tag{3.1.2}$$

where $a_1(t), a_2(t), b_1(t)$, and $b_2(t)$ are arbitrary real functions and $q(x,t)$ is a complex valued function that expresses wave envelope. In equation (3.1.2), first term represents "the evolution term", second term represents "the group velocity dispersion" while the third term denotes "the non-Kerr dispersion term". The time dependent variable coefficients of group velocity dispersion and non-Kerr dispersion term are, $a_1(t), a_2(t)$ and $b_1(t), b_2(t)$ respectively, and are arbitrary smooth functions of the variable t.

The chapter is structured as follows: In section 3.2, we obtain the symmetries of the equation (3.1.2) by utilizing the Lie symmetry method. The similarity variables and similarity reductions for the obtained symmetries are being presented in section 3.3. It is noticed that the reduced ODEs are complex in nature and lack Lie symmetries. Therefore, the power series method has been used to find the solutions of ODEs [36], in section 3.4. Finally, the chapter is concluded in section 3.5.

3.2 Lie symmetry analysis

In order to explore the symmetries of the equation (3.1.2), we adopt the classical Lie symmetry method [79].

Let us first separate the real and imaginary parts of equation (3.1.2) by taking

$$q(x,t) = u(x,t) + iv(x,t). \tag{3.2.1}$$

Using equation (3.2.1) into equation (3.1.2), we get

$$
\begin{aligned}
& -v_t + a_1(t)u_{xx} - a_2(t)v_{xx} + b_1(t)(2n(u^2u_x + uvv_x)(u^2+v^2)^{n-1}) + b_1(t)(u^2+v^2)^n u_x \\
& \qquad - 2nb_2(t)(u^2+v^2)^{n-1}(uvu_x + v^2v_x) - b_2(t)(u^2+v^2)^n v_x = 0, \\
& u_t + a_2(t)u_{xx} + a_1(t)v_{xx} + b_2(t)(u^2+v^2)^n u_x + b_2(t)(2n(u^2u_x + uvv_x)(u^2+v^2)^{n-1}) \\
& \qquad + b_1(t)(u^2+v^2)^n v_x + 2nb_1(t)(u^2+v^2)^{n-1}(uvu_x + v^2v_x) = 0. \qquad (3.2.2)
\end{aligned}
$$

Next, consider a one-parameter Lie group of point transformations under which equations (3.2.2) remain invariant of the form [79]

$$
\begin{aligned}
x^* &= x + \epsilon\xi(x,t,u,v) + O(\epsilon^2), \\
t^* &= t + \epsilon\eta(x,t,u,v) + O(\epsilon^2), \\
u^* &= u + \epsilon\tau(x,t,u,v) + O(\epsilon^2), \qquad (3.2.3) \\
v^* &= v + \epsilon\zeta(x,t,u,v) + O(\epsilon^2),
\end{aligned}
$$

where ϵ is a group parameter and ξ, η, τ, ζ are the group infinitesimal of the Lie point transformations. Applying the group of point transformations (3.2.3) on equations (3.2.2) and on comparing the coefficients of various derivatives of u and v with respect to x and t and equating them to zero, the first equation of (3.2.2) gives the following list of equations:

(i) $\xi_u = 0, \xi_v = 0,$

(ii) $\eta_v = 0, \eta_x = 0, \eta_u = 0,$

(iii) $-2a_1(t)\tau_{uv} + 2a_2(t)\zeta_{uv} = 0,$

(iv) $-2a_1(t)\tau_{vv} + 2a_2(t)\zeta_{vv} = 0,$

(v) $-2a_1(t)\tau_{uu} + 2a_2(t)\zeta_{uu} = 0,$

(vi) $-a_1(t)\tau_v + a_2'(t)\eta - 2a_2(t)\xi_x + a_2(t)\eta_t - a_1(t)\zeta_u = 0,$

(vii) $-a_1'(t)\eta - a_1(t)\tau_u + 2a_1(t)\xi_x + a_1(t)\zeta_v - a_1(t)\eta_t = 0,$

(viii) $\zeta_t - a_1(t)\tau_{xx} + a_2(t)\zeta_{xx} + (u^2+v^2)^n(b_2(t)\zeta_x - b_1(t)\tau_x) + 2n(u^2+v^2)^{n-1}(v^2b_2(t)\zeta_x - u^2b_1(t)\tau_x) + 2nuv(u^2+v^2)^{n-1}(b_2(t)\tau_x - b_1(t)\zeta_x) = 0,$

(ix) $\xi_t - 2a_1(t)\tau_{vx} + 2a_2(t)\zeta_{xv} - a_2(t)\xi_{xx} - (u^2+v^2)^n(b_1(t)\zeta_u + b_1(t)\tau_v - b_2(t)\eta_t + b_2(t)\xi_x - b_2'(t)\eta) - 2nu^2b_1(t)(uu^2+v^2)^{n-1}\tau_v + nv(u^2+v^2)^{n-1}(6b_2(t)\zeta - 2b_1(t)\tau) + 2nu(u^2+v^2)^{n-1}(b_2(t)\tau - b_1(t)\zeta)2nuv(u^2+v^2)^{n-1}(b_1(t)\xi_x - b_1'(t)\eta + b_2(t)\tau_v - b_1(t)\eta_t - b_2(t)\zeta_u) + 2nv^2(u^2+v^2)^{n-1}(b_2'(t)\eta - b_2(t)\xi_x + b_2(t)\eta_t + b_1(t)\zeta_u) - 4n(n-1)b_1(t)(u^2+v^2)^{n-2}(v\zeta + u\tau) + 4n(n-1)b_2(t)(u^2+v^2)^{n-2}(uv^2\tau + v^3\zeta) = 0,$

(x) $-2a_1(t)\tau_{xu}+a_1(t)\xi_{xx}+2a_2(t)\zeta_{xu}+(u^2+v^2)^n(-b_1(t)\eta-b_1(t)\tau_u+b_1(t)\xi_x+b_1(t)\zeta_v-b_1(t)\eta_t)+nb_1(t)(u^2+v^2)^{n-1}(-4u\tau+2ub_2(t)\zeta+2vb_2(t)\tau-2b_1(t)v\zeta)+2nu^2(u^2+v^2)^{n-1}(-b_1^{'}(t)\eta-b_1(t)\tau_u+b_1(t)\xi_x+b_1(t)\zeta_v-b_1(t)\eta_t-b_2(t)\zeta_u)+2nuv(u^2+v^2)^{n-1}(b_2^{'}(t)\eta+b_1(t)\tau_u-b_2(t)\xi_x-b_2(t)\zeta_v+b_2(t)\eta_t)+4n(n-1)(u^2+v^2)^{n-2}(b_2(t)u^2v\tau-b_1(t)u^3\tau-b_1(t)u^2v\zeta+b_2(t)uv^2\zeta)=0,$

Similarly, the second equation of (3.2.2) gives the following additional equations

(i) $a_2(t)\tau_{uu}+a_1(t)\zeta_{uu}=0,$

(ii) $2a_2(t)\tau_{uv}+2a_1(t)\zeta_{uv}=0,$

(iii) $2a_2(t)\tau_{vv}+2a_1(t)\zeta_{vv}=0,$

(iv) $a_1(t)\tau_v+a_2^{'}(t)\eta-2a_2(t)\xi_x+a_2(t)\eta_t+a_1(t)\zeta_u=0,$

(v) $a_1^{'}(t)\eta-a_1(t)\tau_u-2a_1(t)\xi_x+a_1(t)\zeta_v+a_1(t)\eta_t=0,$

(vi) $\tau_t+a_2(t)\tau_{xx}+a_1(t)\zeta_{xx}+(u^2+v^2)^n(-b_2(t)\zeta_x+b_1(t)\tau_x)+2n(u^2+v^2)^{n-1}(v^2b_1(t)\zeta_x+u^2b_2(t)\tau_x)+2nuv(u^2+v^2)^{n-1}(b_1(t)\tau_x+b_2(t)\zeta_x)=0,$

(vii) $\xi_t+2a_2(t)\tau_{ux}+2a_1(t)\zeta_{xu}-a_2(t)\xi_{xx}+(u^2+v^2)^n(b_2^{'}(t)\eta-b_2(t)\xi_x+b_1(t)\zeta_u+b_2(t)\eta_t+b_1(t)\tau_v)n(v^2+u^2)^{n-1}(6ub_2(t)\tau+2b_2(t)v\zeta+2b_1(t)v\tau+2b_1(t)u\zeta+2nv^2b_1(t)\zeta u)+2nu^2(u^2+v^2)^{n-1}(b_2^{'}(t)\eta-b_2(t)\xi_x+b_2(t)\eta_t+b_1(t)\tau_v)+2nuv(u^2+v^2)^{n-1}(b_2(t)\zeta_u+b_1^{'}(t)\eta-b_1(t)\xi_x+b_1(t)\eta_t-b_2(t)\tau_v)+4n(n-1)(u^2+v^2)^{n-2}(b_2(t)u^3\tau+b_2(t)u^2v\zeta+b_1(t)u^2v\tau+b_1(t)uv^2\zeta)=0,$

(viii) $2a_2(t)\tau_{xv}-a_1(t)\xi_{xx}+2a_1(t)\zeta_{xv}+(u^2+v^2)^n(b_1^{'}(t)\eta-b_1(t)\xi_x+b_1(t)\zeta_v-b_1(t)\tau_u+b_1(t)\eta_t)+2n(u^2+v^2)^{n-1}(b_2(t)u^2\tau_v-v^2b_1(t)\xi_x+b_1(t)v^2\zeta_v-b_1(t)v^2\tau_u+b_1(t)v^2\eta_t-b_2(t)v^2\tau_v+v^2b_1^{'}(t)\eta)+2n(u^2+v^2)^{n-1}(vb_2(t)\tau+ub_2(t)\zeta+ub_1(t)\tau+b_1(t)\zeta-2b_1(t)v\zeta)+2nuv(u^2+v^2)^{n-1}(b_2^{'}(t)\eta+b_2(t)\zeta_v+b_2(t)\xi_x+b_1(t)\tau_v-b_2(t)\tau_u+b_2(t)\eta_t+b_1(t)\tau_v)+4n(n-1)(u^2+v^2)^{n-2}(u^2vb_2(t)\tau+b_2(t)uv^2\zeta+b_1(t)uv^2\tau+b_1(t)v^3\zeta)=0.$

On simplifying and trying to solve the above sets of equations, we come across the three different cases for the determination of infinitesimals ξ,η,τ,ζ as follows:

Case 1: For $n=1$, $a_1(t),a_2(t),b_1(t)$ and $b_2(t)$ as non-zero, we have the following subcases to consider:

Subcase 1: For $\eta\neq 0$, the symmetries of the equation (3.1.2) obtained are as follows:

$$\begin{aligned}
\xi &= \alpha x+\beta,\\
\eta &= \frac{2\alpha}{a_1(t)}\int a_1(t)dt+\frac{\sigma}{a_1(t)},\\
\tau &= -\gamma v+\delta u,\\
\zeta &= \gamma u+\delta v,
\end{aligned}\tag{3.2.4}$$

where $\alpha, \beta, \gamma, \delta$ and σ are arbitrary constants and the coefficient functions $a_1(t), a_2(t)$, $b_1(t), b_2(t)$ are determined by the following governing equations

$$\alpha b_1(t) - \frac{2\alpha}{a_1^2(t)} b_1(t) a_1^{'}(t) \int a_1(t)dt - \frac{\sigma}{a_1^2(t)} b_1(t) a_1^{'}(t) + \frac{2\alpha}{a_1(t)} b_1^{'}(t) \int a_1(t)dt \\ + \frac{\sigma b_1^{'}(t)}{a_1(t)} + 2\delta b_1(t) = 0, \tag{3.2.5}$$

$$a_2(t) = k_1 a_1(t), \tag{3.2.6}$$

$$b_2(t) = k_2 b_1(t). \tag{3.2.7}$$

where k_1 and k_2 are nonzero arbitrary constants.

Next, the associated Lie algebra of symmetries (3.2.4) consists of following vector fields [79]

$$\begin{aligned} H_1 &= x\frac{\partial}{\partial x} + \left(\frac{2}{a_1(t)} \int a_1(t)dt\right)\frac{\partial}{\partial t}, \\ H_2 &= \frac{1}{a_1(t)}\frac{\partial}{\partial t}, \\ H_3 &= -v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}, \\ H_4 &= u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \\ H_5 &= \frac{\partial}{\partial x}. \end{aligned} \tag{3.2.8}$$

On solving the equation (3.2.5), we obtain the forms of the variable coefficients as follows:

$$\begin{aligned} b_1(t) &= a_1(t)\Big(2\alpha \int a_1(t)dt + \sigma\Big)^{-\frac{\delta}{\alpha} - \frac{1}{2}}, \\ a_2(t) &= k_1 a_1(t), \\ b_2(t) &= k_2 a_1(t)\Big(2\alpha \int a_1(t)dt + \sigma\Big)^{-\frac{\delta}{\alpha} - \frac{1}{2}}, \end{aligned}$$

where $a_1(t)$ is an arbitrary constant.

The associated commutator table for the vector fields is given as

Table 3.1: Commutator table

	H_1	H_2	H_3	H_4	H_5
H_1	0	$2H_2$	0	0	$-H_5$
H_2	$-2H_2$	0	0	0	0
H_3	0	0	0	0	0
H_4	0	0	0	0	0
H_5	H_5	0	0	0	0

Subcase 2: For $\eta = 0$, the symmetries of equation (3.1.2) are given by

$$\begin{aligned} \xi &= \beta, \\ \eta &= 0, \\ \tau &= -\gamma v, \\ \zeta &= \gamma u. \end{aligned} \tag{3.2.9}$$

It may be mentioned here that in this subcase, the equation (3.1.2) remains invariant for arbitrary forms of variable coefficients.

Case 2: For $n \neq 1, b_1(t) = 0$ and $a_1(t),\ a_2(t),\ b_2(t)$ as non-zero, the symmetries of equation (3.1.2) are as follows

$$\begin{aligned} \xi &= \alpha x + \beta, \\ \eta &= \frac{2\alpha}{a_1(t)} \int a_1(t)dt + \frac{\sigma}{a_1(t)}, \\ \tau &= -\gamma v + \delta u, \\ \zeta &= \gamma u + \delta v, \end{aligned} \tag{3.2.10}$$

where $\alpha, \sigma, \beta, \gamma, \delta$ are arbitrary constants and $a_1(t), a_2(t), b_2(t)$ are coefficient functions which are determined from the following equations:

$$\begin{aligned} &\alpha b_2(t) + \frac{2\alpha}{a_1(t)} b_2'(t) \int a_1(t)dt + \frac{\sigma}{a_1(t)} b_2'(t) - \frac{2\alpha}{a_1^2(t)} b_2(t) a_1'(t) \int a_1(t)dt \\ &\quad + 2n\delta b_2(t) - \frac{\sigma}{a_1^2(t)} b_2(t) a_1'(t) = 0, \end{aligned} \tag{3.2.11}$$

$$a_2(t) = k_1 a_1(t). \tag{3.2.12}$$

Now, the vector fields for the corresponding Lie algebra of infinitesimal symmetries (3.2.10) are as follows:

$$\begin{aligned}
H_1 &= x\frac{\partial}{\partial x} + \left(\frac{2}{a_1(t)}\int a_1(t)dt\right)\frac{\partial}{\partial t},\\
H_2 &= \frac{1}{a_1(t)}\frac{\partial}{\partial t},\\
H_3 &= -v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v},\\
H_4 &= u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v},\\
H_5 &= \frac{\partial}{\partial x}.
\end{aligned} \tag{3.2.13}$$

On solving the equation (3.2.11), we get

$$\begin{aligned}
a_2(t) &= k_1 t,\\
b_2(t) &= a_1(t)\Big(2\alpha\int a_1(t)dt + \sigma\Big)^{-\frac{n\delta}{\alpha}-\frac{1}{2}},
\end{aligned}$$

where $a_1(t)$ is an arbitrary function of t.

Case 3: For $n \neq 1, a_1(t) = 0,\ b_1(t) = 0$ and $a_2(t), b_2(t)$ as non-zero, the symmetries of equation (3.1.2) are as follows:

$$\begin{aligned}
\xi &= 0,\\
\eta &= \frac{\sigma}{a_2(t)},\\
\tau &= -\gamma v + \delta u,\\
\zeta &= \gamma u + \delta v,
\end{aligned} \tag{3.2.14}$$

where γ, σ, δ are arbitrary constants and $a_2(t), b_2(t)$ are coefficient functions which are governed by the following relation

$$\sigma\frac{b_2'(t)}{a_2(t)} - \frac{\sigma}{a_2^2(t)}b_2(t)a_2'(t) + 2n\delta b_2(t) = 0. \tag{3.2.15}$$

Now, the vector fields for the associated Lie algebra (3.2.14) are as follows:

$$\begin{aligned}
H_1 &= \frac{\sigma}{a_2(t)}\frac{\partial}{\partial t},\ a_2(t) \neq 0\\
H_2 &= -v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v},\\
H_3 &= u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}.
\end{aligned} \tag{3.2.16}$$

On simplifying equation (3.2.15), it gives

$$b_2(t) = a_2(t)e^{-\frac{2n\delta}{\sigma}\int a_2(t)dt}, \sigma \neq 0, \tag{3.2.17}$$

where $a_2(t)$ is an arbitrary function of argument t.

3.3 Similarity variables and Similarity reductions

In this section, we present the similarity variables and similarity reductions for the associated symmetries constructed in section 3.2.

Case 1: For $n = 1,\ a_1(t), , a_2(t),\ \ b_1(t), b_2(t)$ as nonzero.

Subcase 1: For $\eta \neq 0$, the similarity variables associated with the symmetries (3.2.4) can be obtained by solving the characteristic equations [79]

$$\frac{dx}{\xi} = \frac{dt}{\eta} = \frac{du}{\tau} = \frac{dv}{\zeta}, \tag{3.3.1}$$

where $\xi,\ \eta,\ \tau,\ \zeta$ are given by equation (3.2.4), and equations (3.3.1) become

$$\frac{dx}{\alpha x + \beta} = \frac{dt}{\frac{2\alpha}{a_1(t)}\int a_1(t)dt + \frac{\sigma}{a_1(t)}} = \frac{du}{-\gamma v + \delta u} = \frac{dv}{\gamma u + \delta v}, \tag{3.3.2}$$

which is equivalent to

$$\frac{dx}{\alpha x + \beta} = \frac{dt}{\frac{2\alpha}{a_1(t)}\int a_1(t)dt + \frac{\sigma}{a_1(t)}} = \frac{dq}{(\delta + i\gamma)q} \tag{3.3.3}$$

On integrating equations (3.3.3), we get the similarity variable and the new dependent variable as follows:

$$\theta = (\alpha x + \beta)\Big(2\alpha\int a_1(t)dt + \sigma\Big)^{\frac{-1}{2}},$$
$$q = \Big(2\alpha\int a_1(t)dt + \sigma\Big)^{\frac{\delta}{2\alpha} + \frac{i\gamma}{2\alpha}} F(\theta). \tag{3.3.4}$$

Using equations (3.3.4) in equation (3.1.2), the reduced ordinary differential equation (ODE) is given by

$$(i\delta - \gamma)F - i\alpha\theta F_\theta + (1 + ik_1)\alpha^2 F_{\theta\theta} + \alpha(i - k_2)(|F|^2_\theta F + F_\theta|F|^2) = 0, \tag{3.3.5}$$

where $\delta, \gamma, \alpha, k_1, k_2$ are arbitrary constants. Further, the ODE (3.3.5) is converted into the system of ordinary differential equations by substituting$F(\theta) = U(\theta)e^{iV(\theta)}$ and on separating the real and imaginary parts, we have

$$\alpha^2 U'' - \alpha^2 U (V')^2 - \gamma U - 2k_1\alpha^2 U'V' - k_1\alpha^2 UV'' - \alpha U^3 V' - 3k_2\alpha U^2 U' + \alpha\theta UV' = 0,$$
$$\alpha^2 UV'' + 2\alpha^2 U'V' + \delta U + k_1\alpha^2 U'' - k_1\alpha^2 U(V')^2 + 3\alpha U^2 U' - \alpha k_2 U^3 V' - \alpha\theta U' = 0. \tag{3.3.6}$$

Subcase 2: For $\eta = 0$, the similarity variables corresponding to symmetries (3.2.9) are

$$\theta = t, q = e^{\frac{i\gamma}{\beta}x}F(\theta). \tag{3.3.7}$$

On using equation (3.3.7) in equation (3.1.2), we have

$$iF' - \frac{\gamma^2}{\beta^2}(a_1(\theta) + ia_2(\theta))F + i\frac{\gamma}{\beta}(b_1(\theta) + ib_2(\theta))F|F|^2 = 0, \beta \neq 0. \tag{3.3.8}$$

Using $F(\theta) = U(\theta)e^{iV(\theta)}$in the reduced ODE (3.3.8) and on separating the real and imaginary parts, we get the system of ODEs as

$$U(\theta)V'(\theta) + \frac{\gamma^2}{\beta^2}a_1(\theta)U(\theta) + \frac{\gamma}{\beta}b_2(\theta)U^3 = 0,$$
$$U'(\theta) - \frac{\gamma^2}{\beta^2}a_2(\theta)U(\theta) + \frac{\gamma}{\beta}b_1(\theta)U^3 = 0, \beta \neq 0. \tag{3.3.9}$$

On solving the second equation of equations (3.3.9), with the aid of software MAPLE, we get

$$U(\theta) = \pm\beta^{\frac{1}{2}} e^{\frac{\gamma^2 \int a_2(\theta)d\theta}{\beta^2}} \left[\left(\rho\beta + 2\gamma \int \left(e^{\frac{\gamma^2 \int a_2(\theta)d\theta}{\beta^2}}\right)^2 b_1(\theta)d\theta\right)\right]^{-\frac{1}{2}}. \tag{3.3.10}$$

As a result first equation in (3.3.9) yields

$$V(\theta) = -\frac{\gamma}{\beta^2}\int \Big(\beta b_2(\theta)U(\theta)^2 + \gamma a_1(\theta)\Big)d\theta + \phi, \beta \neq 0, \tag{3.3.11}$$

where $\beta, \gamma, \rho, \phi$ are arbitrary constants. Substituting (3.3.10) into (3.3.11) and integrating, we can obtain $V(\theta)$. Hence, the exact solution of equation (3.1.2) can be obtained by substituting the$F(\theta) = U(\theta)e^{iV(\theta)}$into equation (3.3.7).

Case 2: For $n \neq 1$, $b_1(t) = 0$ and $a_1(t)$, $a_2(t)$, $b_2(t)$ as nonzero, the similarity variables corresponding to the symmetries (3.2.10) are given by

$$\theta = (\alpha x + \beta)\Big(2\alpha \int a_1(t)dt + \sigma\Big)^{\frac{-1}{2}},$$
$$q = \Big(2\alpha \int a_1(t)dt + \sigma\Big)^{\frac{\delta}{2\alpha} + \frac{i\gamma}{2\alpha}} F(\theta), \tag{3.3.12}$$

Using equations (3.3.12) into equation (3.1.2), the reduced ODE is

$$(i\delta - \gamma)F - i\alpha\theta F_\theta + (1 + ik_1)\alpha^2 F_{\theta\theta} + i\alpha(|F|_\theta^{2n} F + F_\theta |F|^{2n}) = 0, \tag{3.3.13}$$

where $\delta, \gamma, \alpha, k_1$ are arbitrary constants.
Next, on substituting $F(\theta) = U(\theta)e^{iV(\theta)}$ into equation (3.3.13) and splitting the real and imaginary parts, we get a system of nonlinear ODEs as follows:

$$\alpha^2 U'' - \alpha^2 U(V')^2 - \gamma U - 2k_1\alpha^2 U'V' - k_1\alpha^2 UV'' - \alpha U U^{2n} V' + \alpha\theta U V' = 0,$$
$$\alpha^2 UV'' + 2\alpha^2 U'V' + \delta U + k_1\alpha^2 U'' - k_1\alpha^2 U(V')^2 + 2n\alpha U^{2n} U' - \alpha\theta U' + \alpha U^{2n} U' = 0. \tag{3.3.14}$$

Case 3: For $n \neq 1$, $a_1(t) = 0$, $b_1(t) = 0$ and $a_2(t)$, $b_2(t)$ as nonzero, the similarity variable and new dependent variable for symmetries (3.2.14) areas follows

$$\theta = x, q = e^{(\delta + i\gamma)\int \frac{a_2(t)}{\sigma} dt} F(\theta). \tag{3.3.15}$$

Using equations (3.3.15) into equation (3.1.2), the reduced ODE takes the form

$$\frac{(\delta + i\gamma)}{\sigma} F + F_{\theta\theta} + (|F|_\theta^{2n} F + F_\theta |F|^{2n}) = 0, \tag{3.3.16}$$

where γ, δ are arbitrary constants and $\sigma \neq 0$.
On substituting$F(\theta) = U(\theta)e^{iV(\theta)}$into (3.3.16) and on separating the real and imaginary parts, we get

$$\frac{\delta}{\sigma} U + U'' - U(V')^2 + U'U^{2n} + 2nU'U^{2n} = 0,$$
$$\frac{\gamma}{\sigma} U + 2U'V' + UV'' + UU^{2n}V' = 0. \tag{3.3.17}$$

Due to lack of symmetries of reduced ODEs (3.3.6), (3.3.14), and (3.3.17), we resort to construct the power series solutions to these ODEs. Some researchers have already applied the power series method to obtain the solutions of the nonlinear ODEs in [36]. Therefore, in the next section, we present the application of the power series method and find the series solutions of the reduced ordinary differential equations. We have restricted ourselves to the series solution of a system of ODEs (3.3.6) given in Case 1 only, but one can easily obtain the power series solution for the reduced ordinary differential equations given in Case 2 and Case 3, similarly.

3.4 Power Series solutions

In this section, we will apply the power series method to the system of ODEs (3.3.6). Furthermore, we will also show the convergence of the power series solutions of (3.3.6). Consider the solution of equations (3.3.6) in terms of power series of the form

$$U(\theta) = \sum_{k=0}^{\infty} p_k\theta^k, \; V(\theta) = \sum_{k=0}^{\infty} q_k\theta^k, \tag{3.4.1}$$

where $p_n, \; q_n$ are real constants and are to be determined.
Substituting equations (3.4.1) into system (3.3.6), we have

$$\begin{aligned}
&-\gamma p_0 + 2\alpha^2 p_2 - \alpha^2 p_0 q_1^2 - 2k_1\alpha^2 p_1 q_1 - 2k_1\alpha^2 p_0 q_2 - \alpha p_0^3 q_1 - 3k_2\alpha p_0^2 p_1 + \alpha \sum_{k=1}^{\infty}\sum_{j=0}^{k-1}(k-j)p_j q_{k-j}\theta^k \\
&-\gamma \sum_{k=1}^{\infty} p_k\theta^k + \alpha^2 \sum_{k=1}^{\infty}(k+1)(k+2)p_{k+2}\theta^k - \alpha^2 \sum_{k=1}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{j}(i+1)(j-i+1)q_{i+1}q_{j-i+1}p_{k-j}\theta^k \\
&-2k_1\alpha^2 \sum_{k=1}^{\infty}\sum_{j=0}^{k}(j+1)(k-j+1)p_{j+1}q_{k-j+1}\theta^k - k_1\alpha^2 \sum_{k=1}^{\infty}\sum_{j=0}^{k}(k-j+1)(k-j+2)q_{k-j+2}p_j\theta^k \\
&-\alpha \sum_{k=1}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{j}\sum_{l=0}^{i}(k-j+1)p_l p_{i-l}p_{j-i}q_{k-j+1}\theta^k - 3k_2\alpha \sum_{k=1}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{j}(k-j+1)p_i p_{j-i}q_{k-j+1}\theta^k = 0,
\end{aligned} \tag{3.4.2}$$

$$\begin{aligned}
&2k_1\alpha^2 p_2 + \delta p_0 + 2\alpha^2 p_1 q_1 + 2\alpha^2 p_0 q_2 - k_1\alpha^2 p_0 q_1^2 + 3\alpha p_0^2 p_1 - \alpha k_2 p_0^3 q_1 - \alpha \sum_{k=1}^{\infty} k p_k\theta^k \\
&+ k_1\alpha^2 \sum_{k=1}^{\infty}(k+1)(k+2)p_{k+2}\theta^k + \delta \sum_{k=1}^{\infty} p_k\theta^k + 2\alpha^2 \sum_{k=1}^{\infty}\sum_{j=0}^{k}(j+1)(k-j+1)p_{j+1}q_{k-j+1}\theta^k \\
&+ \alpha^2 \sum_{k=1}^{\infty}\sum_{j=0}^{k}(k-j+1)(k-j+2)p_j q_{k-j+2}\theta^k - k_1\alpha^2 \sum_{k=1}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{j}(i+1)(j-i+1)p_{k-j}q_{i+1}q_{j-i+1}\theta^k \\
&+ 3\alpha \sum_{k=1}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{j}(k-j+1)p_i p_{j-i}p_{k-j+1}\theta^k - \alpha k_2 \sum_{k=1}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{j}\sum_{l=0}^{i}(k-j+1)p_l p_{i-l}p_{j-i}q_{k-j+1}\theta^k = 0.
\end{aligned} \tag{3.4.3}$$

On comparing the coefficients for $k = 0$ from equations (3.4.2) and (3.4.3), we get

$$\begin{aligned}
-\gamma p_0 + 2\alpha^2 p_2 - \alpha^2 p_0 q_1^2 - 2k_1\alpha^2 p_1 q_1 - 2k_1\alpha^2 p_0 q_2 - \alpha p_0^3 q_1 - 3k_2\alpha p_0^2 p_1 = 0, \\
2k_1\alpha^2 p_2 + \delta p_0 + 2\alpha^2 p_1 q_1 + 2\alpha^2 p_0 q_2 - k_1\alpha^2 p_0 q_1^2 + 3\alpha p_0^2 p_1 - \alpha k_2 p_0^3 q_1 = 0,
\end{aligned} \tag{3.4.4}$$

which on solving yields

$$p_2 = -\frac{1}{2\alpha^2}\Bigg[-\gamma p_0 - \alpha^2 q_1^2 p_0 - \alpha q_1 p_0^3 - 3k_2\alpha p_0^2 p_1 + \frac{k_1}{k_1^2+1}\Big((\gamma k_1 + \delta)p_0$$
$$+ 3\alpha(k_1 k_2 + 1)p_0^2 p_1 + \alpha k_1 p_0^3 q_1 - \alpha k_2 q_1 p_0^3\Big)\Bigg],$$
$$q_2 = \frac{1}{2\alpha^2 p_0(k_1^2+1)}\Bigg[-2\alpha^2(k_1^2+1)p_1 q_1 - (\gamma k_1 + \delta)p_0 - 3\alpha(k_1 k_2 + 1)p_0^2 p_1$$
$$- \alpha k_1 q_1 p_0^3 + \alpha k_2 q_1 p_0^3\Bigg]. \qquad (3.4.5)$$

Next, on comparing the coefficients for $k = 1$ from equations (3.4.2) and (3.4.3), we get

$$q_3 = \frac{1}{12p_0\alpha^2(k_1-1)}\Big[-\gamma p_1 - \alpha^2(q_1^2 p_1 + 4q_1 q_2 p_0) + \alpha p_0 q_1 - 2k_1\alpha^2 p_1 q_2 - 2k_1\alpha^2(2p_1 q_2 + 2p_2 q_1)$$
$$- \alpha(2p_0^3 q_2 + 3p_0^2 p_1 q_1) - 3\alpha k_2(2p_0^2 p_2 + 2p_0 p_1^2)\Big] - \frac{1}{12p_0\alpha^2(k_1+1)}\Big[-k_1\alpha^2(q_1^2 p_1 + 4q_1 q_2 p_0)$$
$$+ \delta p_1 + 2\alpha^2 p_1 q_2 - \alpha p_1 + 4\alpha^2(p_1 q_2 + p_2 q_1) + 3\alpha(2p_0^2 q_2 + 2p_0 p_1^2) - \alpha k_2(2p_0^3 q_2 + 3p_0^2 p_1 q_1)\Big]. \qquad (3.4.6)$$

$$p_3 = -\frac{1}{12\alpha^2(k_1+1)}\Big[-k_1\alpha^2(q_1^2 p_1 + 4q_1 q_2 p_0) + \delta p_1 + 2\alpha^2 p_1 q_2 - \alpha p_1 + 4\alpha^2(p_1 q_2 + p_2 q_1)$$
$$+ 3\alpha(2p_0^2 q_2 + 2p_0 p_1^2) - \alpha k_2(2p_0^3 q_2 + 3p_0^2 p_1 q_1)\Big] - \frac{1}{12\alpha^2(k_1-1)}\Big[-\alpha^2(q_1^2 p_1 + 4q_1 q_2 p_0)$$
$$- \gamma p_1 + \alpha p_0 q_1 - 2k_1\alpha^2 p_1 q_2 - 2k_1\alpha^2(2p_1 q_2 + 2p_2 q_1) - \alpha(2p_0^3 q_2 + 3p_0^2 p_1 q_1)$$
$$- 3\alpha k_2(2p_0^2 p_2 + 2p_0 p_1^2)\Big], \qquad (3.4.7)$$

where $p_2,\ q_2$ are given by equations (3.4.5). For $k > 1$, we have the general relations as follows:

$$p_{k+2} = -\frac{1}{2p_0\alpha^2(k_1+1)(k+1)(k+2)}\Big[\delta p_k - \alpha k p_k \quad + 2\alpha^2\sum_{j=0}^{k}(j+1)(k-j+1)p_{j+1}q_{k-j+1}$$
$$+ \alpha^2\sum_{j=1}^{k}(k-j+1)(k-j+2)p_j q_{k-j+2} + \alpha^2\sum_{j=1}^{k}(k-j+1)(k-j+2)p_j q_{k-j+2}$$
$$- k_1\alpha^2\sum_{j=0}^{k}\sum_{i=0}^{j}(i+1)(j-i+1)q_{i+1}q_{j-i+1}p_{k-j} + 3\alpha\sum_{j=0}^{k}\sum_{i=0}^{j}(k-j+1)p_i p_{j-i}p_{k-j+1}$$
$$- \alpha k_2\sum_{j=0}^{k}\sum_{i=0}^{j}\sum_{l=0}^{i}(k-j+1)p_l p_{i-l}p_{j-i}q_{k-j+1}\Big] - \frac{1}{2p_0\alpha^2(k_1-1)(k+1)(k+2)}\Big[-\gamma p_k$$

$$
\begin{aligned}
&+\alpha\sum_{j=0}^{k-1}(k-j)p_jq_{k-j}-\alpha^2\sum_{j=0}^{k}\sum_{i=0}^{j}(i+1)(j-i+1)q_{i+1}q_{j-i+1}p_{k-j}\\
&-2k_1\alpha^2\sum_{j=0}^{k}(j+1)(k-j+1)p_{j+1}q_{k-j+1}-k_1\alpha^2\sum_{j=1}^{k}(k-j+1)(k-j+2)q_{k-j+2}p_j\\
&-\alpha\sum_{j=0}^{k}\sum_{i=0}^{j}\sum_{l=0}^{i}(k-j+1)p_lp_{j-i}p_{i-l}q_{k-j+1}-3\alpha k_2\sum_{j=0}^{k}\sum_{i=0}^{j}(k-j+1)p_ip_{j-i}p_{k-j+1}\Big],
\end{aligned}
\tag{3.4.8}
$$

$$
\begin{aligned}
q_{k+2}=&\frac{1}{2p_0\alpha^2(k_1-1)(k+1)(k+2)}\Big[-\gamma p_k+\alpha\sum_{j=0}^{k-1}(k-j)p_jq_{k-j}\\
&-\alpha^2\sum_{j=0}^{k}\sum_{i=0}^{j}(i+1)(j-i+1)q_{i+1}q_{j-i+1}p_{k-j}-2k_1\alpha^2\sum_{j=0}^{k}(j+1)(k-j+1)p_{j+1}q_{k-j+1}\\
&-k_1\alpha^2\sum_{j=1}^{k}(k-j+1)(k-j+2)q_{k-j+2}p_j-\alpha\sum_{j=0}^{k}\sum_{i=0}^{j}\sum_{l=0}^{i}(k-j+1)p_lp_{i-l}p_{j-i}q_{k-j+1}\\
&-3\alpha k_2\sum_{j=0}^{k}\sum_{i=0}^{j}(k-j+1)p_ip_{j-i}p_{k-j+1}\Big]-\frac{1}{2p_0\alpha^2(k_1+1)(k+1)(k+2)}\Big[\delta p_k-\alpha kp_k\\
&+2\alpha^2\sum_{j=0}^{k}(j+1)(k-j+1)p_{j+1}q_{k-j+1}+\alpha^2\sum_{j=1}^{k}(k-j+1)(k-j+2)p_jq_{k-j+2}\\
&+\alpha^2\sum_{j=1}^{k}(k-j+1)(k-j+2)p_jq_{k-j+2}-k_1\alpha^2\sum_{j=0}^{k}\sum_{i=0}^{j}(i+1)(j-i+1)q_{i+1}q_{j-i+1}p_{k-j}\\
&+3\alpha\sum_{j=0}^{k}\sum_{i=0}^{j}(k-j+1)p_ip_{j-i}p_{k-j+1}-\alpha k_2\sum_{j=0}^{k}\sum_{i=0}^{j}\sum_{l=0}^{i}(k-j+1)p_lp_{i-l}p_{j-i}q_{k-j+1}\Big].
\end{aligned}
\tag{3.4.9}
$$

From these relations one can determine the other terms of two sequences $\{p_k\}_{k=0}^{\infty}$ and $\{q_k\}_{k=0}^{\infty}$. Therefore, it implies that for system (3.3.6), there exists a power series solutions (3.4.1) with the coefficients given by equations (3.4.8), (3.4.9). Moreover, we can show the convergence of the power series solution (3.4.1) to the equation (3.3.6). From equations (3.4.8), we have

$$
\begin{aligned}
|p_{k+2}|\le M\Big[&|p_k|+\sum_{j=0}^{k}|p_{j+1}||q_{k-j+1}|+\sum_{j=1}^{k}|p_j||q_{k-j+2}|+\sum_{j=0}^{k}\sum_{i=0}^{j}|p_{k-j}||q_{i+1}||q_{j-i+1}|\\
&+\sum_{j=0}^{k}\sum_{i=0}^{j}|p_i||p_{j-i}||q_{k-j+1}|+\sum_{j=0}^{k}\sum_{i=0}^{j}|p_i||p_{j-i}||p_{k-j+1}|+\sum_{j=0}^{k-1}|p_j||q_{k-j}|\\
&+\sum_{j=0}^{k}\sum_{i=0}^{j}\sum_{l=0}^{i}|p_l||p_{i-l}||p_{j-i}||q_{k-j+1}|\Big],k=1,2,\ldots
\end{aligned}
\tag{3.4.10}
$$

where $M = \max\Big\{\frac{|\alpha-\delta|}{|2\alpha^2(k_1+1)|}+\frac{|\gamma|}{|2\alpha^2(k_1-1)|}, \frac{|k_1|}{|k_1-1|}-\frac{1}{|k_1-1|}, \frac{1}{|2\alpha(k_1-1)|}, \frac{|3k_2|}{|2\alpha(k_1-1)|}-\frac{3}{|2\alpha(k_1+1)|}, \frac{|k_1|}{|2(k_1-1)|}+\frac{1}{|2(k_1+1)|}, \frac{|k_2|}{|2(k_1+1)|}+\frac{1}{|2(k_1-1)|}\Big\}$. Similarly, from equation (3.4.9), we get

$$\begin{aligned}|q_{k+2}| \leq N\Big[&|p_k| + \sum_{j=0}^{k}|p_{j+1}||q_{k-j+1}| + \sum_{j=1}^{k}|p_j||q_{k-j+2}| + \sum_{j=0}^{k}\sum_{i=0}^{j}|p_{k-j}||q_{i+1}||q_{j-i+1}| \\ &+ \sum_{j=0}^{k}\sum_{i=0}^{j}|p_i||p_{j-i}||q_{k-j+1}| + \sum_{j=0}^{k}\sum_{i=0}^{j}\sum_{l=0}^{i}|p_l||p_{i-l}||p_{j-i}||q_{k-j+1}| + \sum_{j=0}^{k-1}|p_j||q_{k-j}| \\ &+ \sum_{j=0}^{k}|p_i||p_{j-i}||p_{k-j+1}|\Big], k = 1, 2, \ldots\end{aligned} \tag{3.4.11}$$

where $N = \max\Big\{\frac{|\alpha-\delta|}{|p_0||2\alpha^2(k_1+1)|}+\frac{|\gamma|}{|p_0||2\alpha^2(k_1-1)|}, \frac{|k_1|}{|p_0||2(k_1+1)|}+\frac{1}{|p_0||2(k_1-1)|} \frac{|3k_2|}{|p_0||2\alpha(k_1-1)|}+\frac{3}{|p_0||2\alpha(k_1+1)|}, \frac{|k_1|}{|p_0||2(k_1-1)|}+\frac{1}{|p_0||2(k_1+1)|}, \frac{1}{|p_0||2\alpha(k_1-1)|}, \frac{|k_1|}{|p_0||k_1-1|}+\frac{1}{|p_0||k_1+1|}, \frac{1}{|p_0||2\alpha(k_1-1)|}+ + \frac{|k_2|}{|p_0||2\alpha(k_1+1)|}\Big\}$.

We now consider the two power series as follows:

$$R = R(\theta) = \sum_{k=0}^{\infty} r_k\theta^k,\ S = S(\theta) = \sum_{k=0}^{\infty} s_k\theta^k, \tag{3.4.12}$$

with $r_i = |p_i|,\ \ s_i = |q_i|$, for $k = 1, 2, \ldots$. Therefore,

$$\begin{aligned}r_{n+2} = M\Big[&r_k + \sum_{j=0}^{k} r_{j+1}s_{k-j+1} + \sum_{j=1}^{k} r_j s_{k-j+2} + \sum_{j=0}^{k}\sum_{i=0}^{j} s_{i+1}s_{j-i+1}r_{k-j} + \sum_{j=0}^{k}\sum_{i=0}^{j} r_i r_{j-i} s_{k-j+1} \\ &+ \sum_{j=0}^{k-1} r_j s_{k-j} + \sum_{j=0}^{k}\sum_{i=0}^{j} r_i r_{j-i} r_{k-j+1} + \sum_{j=0}^{k}\sum_{i=0}^{j}\sum_{l=0}^{i} r_l r_{i-l} r_{j-i} s_{k-j+1}\Big], \\ s_{n+2} = N\Big[&r_k + \sum_{j=0}^{k} r_{j+1}s_{k-j+1} + \sum_{j=1}^{k} r_j s_{k-j+2} + \sum_{j=0}^{k}\sum_{i=0}^{j} s_{i+1}s_{j-i+1}r_{k-j} + \sum_{j=0}^{k}\sum_{i=0}^{j} r_i r_{j-i} s_{k-j+1} \\ &+ \sum_{j=0}^{k-1} r_j s_{k-j} + \sum_{j=0}^{k}\sum_{i=0}^{j} r_i r_{j-i} r_{k-j+1} + \sum_{j=0}^{k}\sum_{i=0}^{j}\sum_{l=0}^{i} r_l r_{i-l} r_{j-i} s_{k-j+1}\Big],\end{aligned} \tag{3.4.13}$$

We can see that $|p_k| \leq r_k,\ \ |q_k| \leq s_k$, where $k = 0, 1, 2, \ldots$. That is, the two series (3.4.12) are the majorant series of (3.4.1). Next, we deduce that the two series (3.4.12), have the positive radius of convergence. Therefore, by some mathematical calculation, we get

$$R(\theta) = r_0 + r_1\theta + r_2\theta^2 + \sum_{k=1}^{\infty} r_{k+2}\theta^{k+2}$$
$$= r_0 + r_1\theta + r_2\theta^2 + M\Bigg[\sum_{k=1}^{\infty} r_k\theta^{k+2} + \sum_{k=1}^{\infty}\sum_{j=0}^{k} r_{j+1}s_{k-j+1}\theta^{k+2} + \sum_{k=1}^{\infty}\sum_{j=1}^{k} r_j s_{k-j+2}\theta^{k+2}$$
$$+ \sum_{k=1}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{j} s_{i+1}s_{j-i+1}r_{k-j}\theta^{k+2} + \sum_{k=1}^{\infty}\sum_{j=0}^{k-1} r_j s_{k-j}\theta^{k+2} + \sum_{k=1}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{j} r_i r_{j-i} s_{k-j+1}\theta^{k+2}$$
$$+ \sum_{k=1}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{j} r_i r_{j-i} r_{k-j+1}\theta^{k+2} + \sum_{k=1}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{j}\sum_{l=0}^{i} r_l r_{i-l} r_{j-i} s_{k-j+1}\theta^{k+2}\Bigg], \quad (3.4.14)$$

which is further equal to

$$R(\theta) = r_0 + r_1\theta + r_2\theta^2 + M\Big[2RS + \theta^2 SR - 2r_0 S - 2s_0 R - \theta s_1 R - \theta^2 s_0 R + S^2 R + s_0^2 R + R^3\theta$$
$$+ \theta^3 S + s_0\theta R^2 - 2s_0\theta R^2 - r_0 r_1\theta^2 R - 2r_0\theta R^2 + \theta R^3 S - s_0\theta R^3 + \theta^2 R + 2r_0 s_0 - 2s_0 RS$$
$$+ (r_0 s_1 - r_0^3)\theta + (2r_0^2 r_1 - r_0 - r_1 s_1 - s_1^2 r_0 - r_0^2 s_1 - r_0^3 s_1)\theta^2\Big]. \quad (3.4.15)$$

Similarly,

$$S(\theta) = s_0 + s_1\theta + s_2\theta^2 + \sum_{k=1}^{\infty} s_{k+2}\theta^{k+2}$$
$$= s_0 + s_1\theta + s_2\theta^2 + N\Bigg[\sum_{k=1}^{\infty} r_k\theta^{k+2} + \sum_{k=1}^{\infty}\sum_{j=0}^{k} r_{j+1}s_{k-j+1}\theta^{k+2} + \sum_{k=1}^{\infty}\sum_{j=1}^{k} r_j s_{k-j+2}\theta^{k+2}$$
$$+ \sum_{k=1}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{j} s_{i+1}s_{j-i+1}r_{k-j}\theta^{k+2} + \sum_{k=1}^{\infty}\sum_{j=0}^{k-1} r_j s_{k-j}\theta^{k+2} + \sum_{k=1}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{j} r_i r_{j-i} s_{k-j+1}\theta^{k+2}$$
$$+ \sum_{k=1}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{j} r_i r_{j-i} r_{k-j+1}\theta^{k+2} + \sum_{k=1}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{j}\sum_{l=0}^{i} r_l r_{i-l} r_{j-i} s_{k-j+1}\theta^{k+2}\Bigg], \quad (3.4.16)$$

which is further equal to

$$S(\theta) = s_0 + s_1\theta + s_2\theta^2 + N\Big[2RS + \theta^2 SR - 2r_0 S - 2s_0 R - \theta s_1 R - \theta^2 s_0 R + S^2 R + s_0^2 R - 2s_0 RS$$
$$+ \theta^3 S + s_0\theta R^2 - 2s_0\theta R^2 - r_0 r_1\theta^2 R - 2r_0\theta R^2 + \theta R^3 S - s_0\theta R^3 + \theta^2 R + 2r_0 s_0 + R^3\theta$$
$$+ (r_0 s_1 - r_0^3)\theta + (2r_0^2 r_1 - r_0 - r_1 s_1 - s_1^2 r_0 - r_0^2 s_1 - r_0^3 s_1)\theta^2\Big]. \quad (3.4.17)$$

Now consider the implicit functional system concerning the independent variable θ:

$$F(\theta, R, S) = R - r_0 - r_1\theta - r_2\theta^2 - M\Big[2RS + \theta^2 SR - 2r_0 S + (s_0^2 - 2s_0 - \theta s_1)R - \theta^2 s_0 R + S^2 R$$
$$+ R^3\theta + \theta^3 S + s_0\theta R^2 - 2s_0\theta R^2 - r_0 r_1\theta^2 R - 2r_0\theta R^2 + \theta R^3 S - s_0\theta R^3 + \theta^2 R + 2r_0 s_0$$
$$- 2s_0 RS + (r_0 s_1 - r_0^3)\theta + (2r_0^2 r_1 - r_0 - r_1 s_1 - s_1^2 r_0 - r_0^2 s_1 - r_0^3 s_1)\theta^2\Big] = 0,$$
$$G(\theta, R, S) = S - s_0 - s_1\theta - s_2\theta^2 - N\Big[2RS + \theta^2 SR - 2r_0 S + (s_0^2 - 2s_0 - \theta s_1)R - \theta^2 s_0 R + S^2 R$$
$$+ R^3\theta + \theta^3 S + s_0\theta R^2 - 2s_0\theta R^2 - r_0 r_1\theta^2 R - 2r_0\theta R^2 + \theta R^3 S - s_0\theta R^3 + \theta^2 R + 2r_0 s_0$$
$$- 2s_0 RS + (r_0 s_1 - r_0^3)\theta + (2r_0^2 r_1 - r_0 - r_1 s_1 - s_1^2 r_0 - r_0^2 s_1 - r_0^3 s_1)\theta^2\Big] = 0. \quad (3.4.18)$$

Since F and G are analytic in the neighborhood of $(0, r_0, s_0)$ and $F(0, r_0, s_0) = 0$, $G(0, r_0, s_0) = 0$. Therefore, the Jacobian is given by

$$J = \frac{\partial(F,G)}{\partial(R,S)} = 1 \neq 0. \tag{3.4.19}$$

If we choose the parameters $r_0 = |p_0|$ and $s_0 = |q_0|$ accordingly, then by Implicit function theorem [99], we find that $R = R(\theta)$ and $S = S(\theta)$ are analytic in the neighbourhood of the point $(0, r_0, s_0)$ and have the positive radius of convergence. This implies that the two power series (3.4.1) converge in a neighborhood of $(0, r_0, s_0)$. Hence the proof is completed. Thus, we find that the power series solutions (3.4.1) are the exact analytic solutions of the equations (3.3.6) given as follows:

$$\begin{aligned} U(\theta) &= p_0 + p_1\theta - \frac{1}{2\alpha^2}\Big[-\gamma p_0 - \alpha^2 q_1^2 p_0 - \alpha q_1 p_0^3 - 3k_2\alpha p_0^2 p_1 + \frac{k_1}{k_1^2+1}\Big((\gamma k_1 + \delta)p_0 \\ &\quad + 3\alpha(k_1k_2+1)p_0^2 p_1 + \alpha k_1 p_0^3 q_1 - \alpha k_2 q_1 p_0^3\Big)\Big]\theta^2 + \sum_{k=1}^{\infty} p_{k+2}\theta^{k+2}, \\ V(\theta) &= q_0 + q_1\theta + \frac{1}{2\alpha^2 p_0(k_1^2+1)}\Big[-2\alpha^2(k_1^2+1)p_1q_1 - (\gamma k_1+\delta)p_0 - 3\alpha(k_1k_2+1)p_0^2 p_1 \\ &\quad - \alpha k_1 q_1 p_0^3 + \alpha k_2 q_1 p_0^3\Big]\theta^2 + \sum_{k=1}^{\infty} q_{k+2}\theta^{k+2}, \end{aligned} \tag{3.4.20}$$

where $p_i, q_i (i = 0, 1)$ are arbitrary constants and $p_{k+2}, q_{k+2} (k = 1, 2, \ldots)$ are given by equations (3.4.8) and (3.4.9).
From equations (3.4.20), we get

$$\begin{aligned} F(\theta) &= \Bigg(p_0 + p_1\theta - \frac{1}{2\alpha^2}\Big[-\gamma p_0 - \alpha^2 q_1^2 p_0 - \alpha q_1 p_0^3 - 3k_2\alpha p_0^2 p_1 + \frac{k_1}{k_1^2+1}\Big((\gamma k_1+\delta)p_0 \\ &\quad + 3\alpha(k_1k_2+1)p_0^2 p_1 + \alpha k_1 p_0^3 q_1 - \alpha k_2 q_1 p_0^3\Big)\Big]\theta^2 + \sum_{k=1}^{\infty} p_{k+2}\theta^{k+2}\Bigg) \times \exp\Bigg(q_0 + q_1\theta \\ &\quad + \frac{1}{2\alpha^2 p_0(k_1^2+1)}\Big[-2\alpha^2(k_1^2+1)p_1q_1 - (\gamma k_1+\delta)p_0 - 3\alpha(k_1k_2+1)p_0^2 p_1 \\ &\quad - \alpha k_1 q_1 p_0^3 + \alpha k_2 q_1 p_0^3\Big]\theta^2 + \sum_{k=1}^{\infty} q_{k+2}\theta^{k+2}\Bigg). \end{aligned} \tag{3.4.21}$$

The exact solution of the equation (3.1.2) is given by

$$q = \Big(2\alpha\int a_1(t)dt+\sigma\Big)^{\frac{\delta}{2\alpha}+\frac{i\gamma}{2\alpha}}\Bigg[p_0+p_1(\alpha x+\beta)\Big(2\alpha\int a_1(t)dt+\sigma\Big)^{\frac{-1}{2}}-\frac{1}{2\alpha^2}\Big[-\gamma p_0-\alpha^2 q_1^2 p_0$$
$$-\alpha q_1 p_0^3-3k_2\alpha p_0^2 p_1+\frac{k_1}{k_1^2+1}\Big((\gamma k_1+\delta)p_0+3\alpha(k_1k_2+1)p_0^2p_1+\alpha k_1 p_0^3 q_1$$
$$-\alpha k_2 q_1 p_0^3\Big)\Big](\alpha x+\beta)^2\Big(2\alpha\int a_1(t)dt+\sigma\Big)^{-1}+\sum_{k=1}^{\infty}p_{k+2}\Big((\alpha x+\beta)\Big(2\alpha\int a_1(t)dt+\sigma\Big)^{\frac{-1}{2}}\Big)^{k+2}\Bigg]$$
$$\times\exp\Bigg[q_0+q_1(\alpha x+\beta)\Big(2\alpha\int a_1(t)dt+\sigma\Big)^{\frac{-1}{2}}+\frac{1}{2\alpha^2 p_0(k_1^2+1)}\Big[-2\alpha^2(k_1^2+1)p_1q_1-(\gamma k_1+\delta)p_0$$
$$-3\alpha(k_1k_2+1)p_0^2p_1-\alpha k_1 q_1 p_0^3+\alpha k_2 q_1 p_0^3\Big](\alpha x+\beta)^2\Big(2\alpha\int a_1(t)dt+\sigma\Big)^{-1}$$
$$+\sum_{k=1}^{\infty}q_{k+2}\Big((\alpha x+\beta)\Big(2\alpha\int a_1(t)dt+\sigma\Big)^{\frac{-1}{2}}\Big)^{k+2}\Bigg]. \qquad (3.4.22)$$

3.5 Discussion

The application of Lie group method has been performed on a nonlinear Schrödinger equation with time dependent variable coefficients. This method is feasible and efficient for the analysis of nonlinear PDEs. We have obtained the Lie group of infinitesimals for the arbitrary time-dependent variable coefficients. The similarity variables, reduce the equation (3.1.2) to nonlinear ODEs. Due to the lack of Lie symmetries of the obtained reduced system of second-order nonlinear ODEs, we take recourse to the power series method on the reduced ODEs (3.3.6) only and obtained the exact power series solutions. The convergence of the power series solution is also established. The results confirm the belief that the power series method is applicable to explore the exact explicit solution of the nonlinear ordinary differential equations. The software Maple has been utilized in solving some reduced ordinary differential equations.

Chapter 4

Symmetry reduction and exact solutions of the KP equation with variable coefficients

4.1 Introduction

The KP equation arises in the various physical phenomena, including plasma physics and shallow-water waves. This equation describes the evolution of small amplitude surface waves, weak dispersion, weak nonlinearity, and propagation along the x-axis with the waves weakly perturbed in the y-direction. KP equation is completely integrable and has an infinite number of conservation laws and symmetries, a Lax pair, Painlève property and the Hirota bilinear representations [74] etc. The generalized (2+1) dimensional KP equation in normalized form is [91]

$$(u_t + \beta(k(u))_x + \alpha u_{xxx})_x + \gamma u_{yy} = 0, \tag{4.1.1}$$

where $\alpha,\ \beta,\ \gamma$ are arbitrary constants and $k(u)$ is an arbitrary nonlinear term. Exact solutions and symmetries of equation (4.1.1) with different classes of $k(u)$ were studied by Elwakil et al. [91]. KP equations have many versions according to the different forms of $k(u)$, which are related to the physical phenomena. If $k(u) = u^n$, then the corresponding KP equation with power-law nonlinearity has been used to describe the flow of shallow water waves [24]. Biswas and Ranasinghe [6] investigated this type of equation and constructed one soliton solution using solitary wave ansatz. Some exact solutions of the KP equation (4.1.1) were classified by Pandir et al. through the Trial expansion method [107].

Liu and Zeng [49] investigated the (3+1) dimensional Kadomtsev-Petviashvilli (KP) equation with variable coefficients of the form

$$(u_t + \lambda(t)uu_x + \mu(t)u_{xxx})_x + \gamma(t)u_{yy} + \delta(t)u_{zz} = 0, \tag{4.1.2}$$

where $\mu(t), \gamma(t), \delta(t), \lambda(t)$ are arbitrary functions. Various researchers have worked on this equation by using different techniques to obtain the exact solutions. Zhao and Bai [40] found the exact explicit solutions of equation (4.1.2). Zhang et al. [51] investigated the exact solutions and an auto B$\ddot{a}$cklund transformation of equation (4.1.2) using the homogeneous balance principle. Liu and Zeng [49] applied an extended variable coefficient homogeneous balance method to find an auto B$\ddot{a}$cklund transformation of equation (4.1.2). Khalique and Adem [16] investigated the symmetries of equation (4.1.2) with constant coefficients. Some new exact solutions were obtained by Zhao and Han [112]. Symmetry analysis of equation (4.1.2) with constant coefficients and with power law nonlinearity was studied by Adem et al. [1].

In view of equations (4.1.1) and (4.1.2), we propose to study a new generalized (3+1) dimensional KP equation with variable coefficients and an arbitrary nonlinear term of the form

$$(u_t + \lambda(t)(k(u))_x + \mu(t)u_{xxx})_x + \gamma(t)u_{yy} + \delta(t)u_{zz} = 0, \tag{4.1.3}$$

where $\mu(t), \gamma(t), \delta(t), \lambda(t)$ are arbitrary functions and $k(u)$ is an arbitrary nonlinear term. The objective of the work is to find the Lie symmetries and deduce the exact solutions of equation (4.1.3).

This chapter is organized as follows: Infinitesimals of KP equation (4.1.3) are constructed in section 4.2. In section 4.3, we illustrate a case in which the arbitrary coefficients of the equation (4.1.3) are taken as power functions of 't' and give the similarity reductions and some new exact solutions alongwith some nonlinear ordinary differential equations. Concluding remarks are presented in section 4.4.

4.2 Symmetry analysis

Herein, we investigate the Lie symmetries of the KP equation (4.1.3). Consider the one-parameter group of point transformations under which equation (4.1.3) remains invariant of the form [44]

$$\begin{aligned} x^* &= x + \epsilon\xi(t,x,y,z,u) + O(\epsilon^2), \\ y^* &= y + \epsilon\eta(t,x,y,z,u) + O(\epsilon^2), \\ z^* &= z + \epsilon\zeta(t,x,y,z,u) + O(\epsilon^2), \\ t^* &= t + \epsilon\tau(t,x,y,z,u) + O(\epsilon^2), \\ u^* &= u + \epsilon\Theta(t,x,y,z,u) + O(\epsilon^2), \end{aligned} \tag{4.2.1}$$

where ϵ is the group parameter and $\xi, \eta, \zeta, \tau, \Theta$ are the infinitesimals of the Lie group of point transformations. Applying the group of point transformations (4.2.1) on equation (4.1.3), we obtain a set of determining equations as follows:

(i) $\xi_u = \eta_u = \zeta_u = \tau_u = \tau_x = \zeta_x = \eta_x = \tau_z = \tau_y = \Theta_{uu} = 0,$

(ii) $4\mu(t)\frac{\partial^2\Theta}{\partial u\partial x} - 6\mu(t)\frac{\partial^2\xi}{\partial x^2} = 0,$

(iii) $-2\delta(t)\frac{\partial\zeta}{\partial z} + 4\delta(t)\frac{\partial\xi}{\partial x} + \delta'(t)\tau - \tau\delta(t)\frac{\mu'(t)}{\mu(t)} = 0,$

(iv) $2\gamma(t)\frac{\partial\zeta}{\partial y} + 2\delta(t)\frac{\partial\eta}{\partial z} = 0,$

(v) $-2\gamma(t)\frac{\partial\eta}{\partial y} + \gamma'(t)\tau + 4\gamma(t)\frac{\partial\xi}{\partial x} - \gamma(t)\frac{\mu'(t)}{\mu(t)}\tau = 0,$

(vi) $\frac{\partial^2\Theta}{\partial x\partial u} = 0, \ -\frac{\partial\tau}{\partial t} + 3\frac{\partial\xi}{\partial x} - \tau\frac{\mu'(t)}{\mu(t)} = 0,$

(vii) $\frac{\partial\zeta}{\partial t} + 2\delta(t)\frac{\partial\xi}{\partial z} = 0,$

(viii) $-\gamma(t)\frac{\partial^2\zeta}{\partial y^2} + 2\delta(t)\frac{\partial^2\Theta}{\partial z\partial u} - \delta(t)\frac{\partial^2\zeta}{\partial z^2} = 0,$

(ix) $\frac{\partial\eta}{\partial t} + 2\gamma(t)\frac{\partial\xi}{\partial y} = 0,$

(x) $2\gamma(t)\frac{\partial^2\Theta}{\partial u\partial y} - \delta(t)\frac{\partial^2\eta}{\partial z^2} - \gamma(t)\frac{\partial^2\eta}{\partial y^2} = 0,$

(xi) $\lambda(t)k'(u)\frac{\partial^2\Theta}{\partial x^2} + \frac{\partial^2\Theta}{\partial x\partial t} + \gamma(t)\frac{\partial^2\Theta}{\partial y^2} + \delta(t)\frac{\partial^2\Theta}{\partial z^2} + \mu(t)\frac{\partial^4\Theta}{\partial x^4} = 0,$

(xii) $\tau\lambda^{'}(t)k^{'}(u) + \Theta\lambda(t)k^{''}(u) + 2\lambda(t)k^{'}(u)\frac{\partial\xi}{\partial x} - \frac{\partial\xi}{\partial t} + 6\mu(t)\frac{\partial^3\Theta}{\partial x^2\partial u} - 4\mu(t)\frac{\partial^3\xi}{\partial x^3}$
$- \lambda(t)k^{'}(u)\frac{\mu^{'}(t)}{\mu(t)}\tau = 0,$

(xiii) $2\lambda(t)k^{''}(u)\frac{\partial\Theta}{\partial x} + 2\lambda(t)k^{'}(u)\frac{\partial^2\Theta}{\partial x\partial u} - \lambda(t)k^{'}(u)\frac{\partial^2\xi}{\partial x^2} + \frac{\partial^2\Theta}{\partial u\partial t} - \gamma(t)\frac{\partial^2\xi}{\partial y^2} - \delta(t)\frac{\partial^2\xi}{\partial z^2}$
$- \frac{\partial^2\xi}{\partial x\partial t} - \mu(t)\frac{\partial^4\xi}{\partial x^4} + 4\mu(t)\frac{\partial^4\Theta}{\partial x^3\partial u} = 0,$

(xiv) $\lambda^{'}(t)k^{''}(u)\tau + \lambda(t)k^{'''}(u)\Theta + 2\lambda(t)k^{''}(u)\frac{\partial\Theta}{\partial u} - 2\lambda(t)k^{''}(u)\frac{\partial\xi}{\partial x} - \lambda(t)k^{''}(u)\frac{\partial\Theta}{\partial u},$
$+ 4\lambda(t)k^{''}(u)\frac{\partial\xi}{\partial x} - \lambda(t)k^{''}(u)\frac{\mu^{'}(t)}{\mu(t)}\tau = 0.$

On solving the above set of equations, we obtain the following infinitesimals of the invariant group of equation (4.1.3):

$$\begin{aligned}
\tau(t) &= \frac{3}{\mu(t)}\int f(t)\mu(t)dt + \frac{c_1}{\mu(t)},\\
\eta(y,z,t) &= \Bigg(2f(t) + \frac{3\delta^{'}(t)}{2\delta(t)\mu(t)}\int f(t)\mu(t)dt + \frac{c_1\delta^{'}(t)}{2\delta(t)\mu(t)}\\
&\quad - \frac{3\mu^{'}(t)}{2\mu(t)^2}\int f(t)\mu(t)dt - \frac{c_1\mu^{'}(t)}{2\mu(t)^2}\Bigg)y - k_1c_2z + \psi(t),\\
\zeta(y,z,t) &= \Bigg(2f(t) + \frac{3\delta^{'}(t)}{2\delta(t)\mu(t)}\int f(t)\mu(t)dt + \frac{c_1\delta^{'}(t)}{2\delta(t)\mu(t)}\\
&\quad - \frac{3\mu^{'}(t)}{2\mu(t)^2}\int f(t)\mu(t)dt - \frac{c_1\mu^{'}(t)}{2\mu(t)^2}\Bigg)z + c_2y + \rho(t),\\
\xi(x,y,z,t) &= f(t)x - \frac{\rho^{'}(t)}{2\delta(t)}z - \frac{\psi^{'}(t)}{2\delta(t)}y + \alpha(t)\\
&\quad - \frac{1}{4\delta(t)}\left(z^2 + \frac{y^2}{k_1}\right)\Bigg[\frac{d}{dt}\Bigg(2f(t) + \frac{3\delta^{'}(t)}{2\delta(t)\mu(t)}\int f(t)\mu(t)dt\\
&\quad + \frac{c_1\delta^{'}(t)}{2\delta(t)\mu(t)} - \frac{3\mu^{'}(t)}{2\mu(t)^2}\int f(t)\mu(t)dt - \frac{c_1\mu^{'}(t)}{2\mu(t)^2}\Bigg)\Bigg],\\
\Theta(x,y,z,t,u) &= p(t)u + q_1(y,z,t)x + q_2(y,z,t),
\end{aligned} \tag{4.2.2}$$

where c_1, c_2 are arbitrary constants and $\alpha(t),\ \rho(t),\ \psi(t)$ are arbitrary functions. The presence of arbitrary functions in the above group infinitesimals indicates that the underlying Lie algebra of KP equation (4.1.3) is infinite-dimensional.

Further, the invariance of equation (4.1.3) under the one-parameter group (4.2.1) yields the following admissible form of the nonlinear term:

$$k(u) = \frac{c_3u^2}{2} + c_4u + c_5 \tag{4.2.3}$$

where $c_3,\ c_4,\ c_5$ are arbitrary constants.
It is worth mentioning here that the functions $f(t),\ p(t),\ q_1(y,z,t)$ and $q_2(y,z,t)$ which appear in the expression of group infinitesimals (4.2.2) are restricted by the following governing equations

$$\begin{aligned} &2\lambda(t)c_3q_1(y,z,t)+p^{'}(t)+f^{'}(t)+\frac{d}{dt}\left(\frac{3\delta^{'}(t)}{2\delta(t)\mu(t)}\int f(t)\mu(t)dt+\frac{c_1\delta^{'}(t)}{2\delta(t)\mu(t)}\right.\\ &\left.-\frac{3\mu^{'}(t)}{2\mu(t)^2}\int f(t)\mu(t)dt-\frac{c_1\mu^{'}(t)}{2\mu(t)^2}\right)=0, \end{aligned} \tag{4.2.4}$$

$$\begin{aligned} &-p(t)\lambda(t)c_4+c_3\lambda(t)q_2(y,z,t)+\frac{1}{4}\left(z^2+\frac{y^2}{k_1}\right)\frac{d}{dt}\left(\frac{1}{\delta(t)}\frac{d}{dt}\left(2f(t)+\frac{3\delta^{'}(t)}{2\delta(t)\mu(t)}\int f(t)\mu(t)dt\right.\right.\\ &\left.\left.+\frac{c_1\delta^{'}(t)}{2\delta(t)\mu(t)}-\frac{3\mu^{'}(t)}{2\mu(t)^2}\int f(t)\mu(t)dt-\frac{c_1\mu^{'}(t)}{2\mu(t)^2}\right)\right)=0, \end{aligned} \tag{4.2.5}$$

$$k_1\delta(t)\frac{\partial^2 q_1}{\partial y^2}+\delta(t)\frac{\partial^2 q_1}{\partial z^2}=0, \tag{4.2.6}$$

$$\frac{\partial q_1}{\partial t}+k_1\delta(t)\frac{\partial^2 q_2}{\partial y^2}+\delta(t)\frac{\partial^2 q_2}{\partial z^2}=0, \tag{4.2.7}$$

$$f^{'}(t)-c_3\lambda(t)q_1(y,z,t)=0, \tag{4.2.8}$$

$$\begin{aligned} &\lambda(t)p(t)+2\lambda(t)f(t)+\frac{3\lambda^{'}(t)}{\mu(t)}\int f(t)\mu(t)dt-\frac{3\lambda(t)\mu^{'}(t)}{\mu(t)^2}\int f(t)\mu(t)dt+\frac{c_1\lambda^{'}(t)}{\mu(t)}\\ &-\frac{c_1\lambda(t)\mu^{'}(t)}{\mu(t)^2}=0, \end{aligned} \tag{4.2.9}$$

$$\gamma(t)=k_1\delta(t).$$

It may be noted here that these governing equations involve the variable coefficients $\lambda(t),\mu(t),\ \gamma(t)$ and $\delta(t)$ too in them. The specific choice of these variable coefficients will help us in determining the forms of unknown functions $f(t),\ p(t),\ q_1(y,z,t)$ and $q_2(y,z,t)$ and vice versa from equations (4.2.4)-(4.2.9). In the next section, we choose the variable coefficients of specific nature and work out the admissible forms of the unknown functions $f(t),\ p(t),\ q_i(y,z,t), i=1,2.$ with the help of equations (4.2.4)-(4.2.9). To demonstrate this we have preferred to take the variable coefficients as power functions of 't'.

Table 4.1: Generators of the (3+1) dimensional KP equation

Forms of Variable Coefficients	Restrictions on Parameters	Cases	Subcases	Generators
$\lambda(t)=t^r$ $\mu(t)=t^m$ $\gamma(t)=k_1t^n$ $\delta(t)=t^n$	$c_3=2, c_4=c_5=0,$ $\psi(t)=\alpha(t)=\sigma(t)=0,$ $k(u)=u^2,$	$(i) m+n-2r=0$	$(i)\ r=-1$	$(a.1)\quad H_1=x\frac{\partial}{\partial x}+\frac{3t}{m+1}\frac{\partial}{\partial t}-z\frac{\partial}{\partial z}-y\frac{\partial}{\partial y}+u\frac{\partial}{\partial u},$ $(a.2)\quad H_2=-\frac{(m+1)^2}{4}(z^2+\frac{y^2}{k_1})\frac{\partial}{\partial x}+t^{-m}\frac{\partial}{\partial t}-(m+1)t^{-m-1}y\frac{\partial}{\partial y}-(m+1)t^{-m-1}z\frac{\partial}{\partial z}+(m+1)t^{-m-1}u\frac{\partial}{\partial u},$ $(a.3)\quad H_3=-k_1z\frac{\partial}{\partial y}+y\frac{\partial}{\partial z}$
			$(ii)\ m=n=r\neq -1$	$(b.1)\quad H_1=x\frac{\partial}{\partial x}+2y\frac{\partial}{\partial y}+\frac{3t}{m+1}\frac{\partial}{\partial t}+2z\frac{\partial}{\partial z}-2u\frac{\partial}{\partial u},$ $(b.2)\quad H_3=-k_1z\frac{\partial}{\partial y}+y\frac{\partial}{\partial z},$ $(b.3)\quad H_2=t^{-m}\frac{\partial}{\partial t},$
$\lambda(t)=t^r$ $\mu(t)=t^m$ $\gamma(t)=k_1t^n$ $\delta(t)=t^n$	$c_3=2, c_4=c_5=0,$ $\psi(t)=\alpha(t)=\sigma(t)=0,$ $k(u)=u^2$	$(ii)\ m+1=0$		$(c.1)\quad H_1=t\frac{\partial}{\partial t}+\frac{n+1}{2}y\frac{\partial}{\partial y}+\frac{n+1}{2}z\frac{\partial}{\partial z}-(r+1)u\frac{\partial}{\partial u},$ $(c.2)\quad H_2=-k_1z\frac{\partial}{\partial y}+y\frac{\partial}{\partial z},$ $(c.3)\quad H_3=\Big(\frac{(4r-n+3)}{2r-n+1}\Big)yt^{\frac{6r-3n+3}{2}}\frac{\partial}{\partial y}+\Big(\frac{4r-n+3}{2r-n+1}\Big)zt^{\frac{6r-3n+3}{2}}\frac{\partial}{\partial z}+\Big(xt^{\frac{6r-3n+3}{2}}-\frac{12r-3n+9}{8}\Big(z^2+\frac{y^2}{k_1}\Big)t^{\frac{6r-3n+1}{2}}\Big)\frac{\partial}{\partial x}+\Big[\frac{3x}{4}t^{\frac{4r-3n+1}{2}}-\Big(\frac{6r-2n+4}{2r-n+1}\Big)t^{\frac{6r-5n+1}{2}}u-\frac{3(6r-5n+1)(4r-n+3)}{32}t^{\frac{4r-5n-1}{2}}(z^2+\frac{y^2}{k_1})\Big]\frac{\partial}{\partial u}+\frac{2}{2r-n+1}t^{\frac{6r-3n+5}{2}}\frac{\partial}{\partial t}.$
$\lambda(t)=t^r$ $\mu(t)=t^m$ $\gamma(t)=k_1t^n$ $\delta(t)=t^n$	$c_3=2, c_4=c_5=0,$ $\psi(t)=\alpha(t)=\sigma(t)=0,$ $k(u)=u^2$	$(iii)\ c_1=0$	$B=0$	$(d.1)\quad H_1=x\frac{\partial}{\partial x}+\frac{t}{m+1}\frac{\partial}{\partial t}+\frac{(3n+m+4)}{2(m+1)}y\frac{\partial}{\partial y}+\frac{(3n+m+4)}{2(m+1)}z\frac{\partial}{\partial z}+\frac{(m-3r-2)}{m+1}u\frac{\partial}{\partial u},$ $(d.2)\quad H_2=-k_1z\frac{\partial}{\partial y}+y\frac{\partial}{\partial z},$

4.3 Variable coefficients as power functions of 't'

Let

$$\mu(t) = t^m$$
$$\lambda(t) = t^r$$
$$\delta(t) = t^n,$$

so that $\gamma(t) = k_1 t^n$.

For $c_3 = 2, c_4 = c_5 = 0$, $\psi(t) = \alpha(t) = \sigma(t) = 0$ and $k(u) = u^2$, on solving equations (4.2.4), (4.2.8) *and* (4.2.9), we have

$$f'(t) - \frac{3}{2}(2r - m - n)t^{-m-2}\int t^{m+1} f'(t)dt + \frac{c_1}{2}(m+1)(2r-m-n)t^{-m-2} = 0. \tag{4.3.1}$$

Now, by observing the equation (4.3.1), we can consider the following special cases:

- **Case 1:** $2r - m - n = 0, m \neq -1$
- **Case 2:** $m + 1 = 0$
- **Case 3:** $c_1 = 0$

Case 1: In this case equation (4.3.1) readily yields

$$f(t) = c_6 \tag{4.3.2}$$

Next, on using equation (4.3.2) in equations (4.2.4)-(4.2.6), (4.2.8), (4.2.9), we get

$$\begin{aligned} q_1 = 0,\ p(t) &= -2c_6 + \frac{3c_6(m-r)}{m+1} + c_1(m-r)t^{-m-1}, \\ q_2(t,y,z) &= \frac{c_1(m-r)(m+1)(r+1)t^{-3(1+r)}}{4}\left(z^2 + \frac{y^2}{k_1}\right). \end{aligned} \tag{4.3.3}$$

Further, using equation (4.3.3) into equation (4.2.7), we arrive at the following condition

$$c_1(m+1)(m-r)(r+1) = 0,$$

which leads to two subcases

- **Subcase 1:** $r = -1$
- **Subcase 2:** $n = m = r, r \neq -1$

Subcase 1: In this case we get $n = -(m+2)$ from $2r - m - n = 0$ and equations (4.3.3) reduce to

$$q_1 = 0,\ q_2 = 0,\ p(t) = c_6 + c_1(m+1)t^{-m-1}.$$

Consequently, with the help of equations (4.2.2), we get symmetry generators $(a.1) - (a.3)$ as listed in Table 4.1.

Now, we present the similarity reduction and exact solutions for each of the generators $(a.1) - (a.3)$ as follows:

(i) For generator $(a.1)$ the similarity variables are given by

$$a = yz^{-1},\ b = xy,\ c = yt^{\frac{m+1}{3}}. \tag{4.3.4}$$

Using equation (4.3.4) in equation (4.1.3), the reduced PDE is

$$\begin{aligned}&3c^6\phi_{bbbb} + 6c^3\phi_b^2 + 6c^3\phi\phi_{bb} + (m+1)c^4\phi_{bc} + 6k_1\phi - 6k_1a\phi_a\\&- 6k_1b\phi_b - 6k_1c\phi_c + 3k_1a^2\phi_{aa} + 3k_1c^2\phi_{cc} + 3k_1b^2\phi_{bb} + 6k_1ab\phi_{ab}\\&+ 6k_1ac\phi_{ac} + 6k_1bc\phi_{bc} + 6a^3\phi_a + 3a^4\phi_{aa} = 0.\end{aligned} \tag{4.3.5}$$

On subjecting the equation (4.3.5) to invariance under one parameter group leads to the following symmetry generators

$$\begin{aligned}F_1 &= \Big(\frac{a^2}{k_1} + 1\Big)\frac{\partial}{\partial a} + \frac{b}{a}\frac{\partial}{\partial b} + \frac{c}{a}\frac{\partial}{\partial c} + \frac{\phi}{a}\frac{\partial}{\partial \phi},\\F_2 &= \frac{a}{c}\frac{\partial}{\partial a} + \Big(\frac{b}{c} + \frac{(m+1)c^2}{6k_1}\Big)\frac{\partial}{\partial b} + \frac{\partial}{\partial c} + \Big(\frac{\phi}{c} + \frac{(m+1)^2c^2}{18k_1}\Big)\frac{\partial}{\partial \phi},\\F_3 &= c\frac{\partial}{\partial b} + \frac{(m+1)c}{6}\frac{\partial}{\partial \phi},\\F_4 &= -\frac{6a^2c^{-1}}{(m+1)}\frac{\partial}{\partial a} + \frac{c^2}{a}\frac{\partial}{\partial b} + \frac{(m+1)c^2}{3a}\frac{\partial}{\partial \phi}.\end{aligned} \tag{4.3.6}$$

As the process of further reduction of PDE (4.3.5) and obtaining the exact solution is similar for all four generators listed in equation (4.3.6), we therefore, restricted ourselves to the last two generators only, namely F_3 and F_4. The similarity variables for generator F_3 are

$$\alpha = a,\ \beta = c,\ \phi = \frac{(m+1)b}{6} + \psi(\alpha,\beta). \tag{4.3.7}$$

Using equation (4.3.7) into equation (4.3.5), the PDE reduces to

$$\begin{aligned} &k_1\beta^2\psi_{\beta\beta} + k_1\alpha^2\psi_{\alpha\alpha} + 2k_1\alpha\beta\psi_{\alpha\beta} - 2k_1\alpha\psi_\alpha - 2k_1\beta\psi_\beta + 2k_1\psi \\ &+ 2\alpha^3\psi_\alpha + \alpha^4\psi_{\alpha\alpha} + \frac{\beta^3(m+1)^2}{18} = 0. \end{aligned} \tag{4.3.8}$$

On solving the equation (4.3.8) using software MAPLE, we get

$$\begin{aligned} \psi = &\frac{C_2\alpha}{\alpha^2+k_1} + \frac{C_1\alpha^2}{\alpha^2+k_1} + \frac{C_3}{2\alpha^2+2k_1} + C_4\beta + C_5\beta^2 - \frac{m^2\beta^3}{36k_1} \\ &- \frac{2m\beta^3}{36k_1} - \frac{\beta^3}{36k_1} - \frac{18C_3}{36k_1}, \end{aligned}$$

where C_1, C_2, C_3, C_4, C_5 are arbitrary constants and $k_1 \neq 0$.
Reverting back to original variables $x, y, z,\ t$, the exact solution of equation (4.1.3) is given by

$$\begin{aligned} u(x,y,z,t) = &\frac{(m+1)x}{6} + \frac{C_2 z}{y^2+k_1z^2} + \frac{C_1 y}{y^2+k_1z^2} + \frac{C_3z^2}{2y(y^2+k_1z^2)} + C_4t^{\frac{m+1}{3}} \\ &+ C_5yt^{\frac{2m+2}{3}} - \frac{m^2y^2t^{m+1}}{36k_1} - \frac{my^2t^{m+1}}{18k_1} - \frac{y^2t^{m+1}}{36k_1} - \frac{C_3}{2k_1y}. \end{aligned} \tag{4.3.9}$$

For generator F_4, the similarity variables are as follows:

$$\alpha = c,\ \beta = \frac{(m+1)c^3a^{-2}}{12} - b,\ \phi = \frac{(m+1)^2c^3a^{-2}}{36} + \psi(\alpha,\beta). \tag{4.3.10}$$

Using equations (4.3.10) in equation (4.3.5), the PDE reduces to

$$\begin{aligned} &18\alpha^6\psi_{\beta\beta\beta\beta} + 36\alpha^3\psi_\beta^2 + 36\alpha^3\psi\psi_{\beta\beta} - 6(m+1)\alpha^4\psi_{\alpha\beta} + 36k_1\psi \\ &+ 36k_1\alpha\beta\psi_{\alpha\beta} - 36k_1\alpha\psi_\alpha + 18k_1\alpha^2\psi_{\alpha\alpha} + (m+1)^2\alpha^3 \\ &+ 3(m+1)\alpha^3\psi_\beta - 36k_1\beta\psi_\beta + 18k_1\beta^2\psi_\beta = 0. \end{aligned} \tag{4.3.11}$$

Further applying the group method on the above equation, the symmetry generator is given by

$$G = \alpha\frac{\partial}{\partial\beta} - \frac{(m+1)\alpha}{6}\frac{\partial}{\partial\psi},$$

which leads to the similarity variables as

$$r = \alpha,\ \psi = -\frac{(m+1)\beta}{6} + \theta(r),$$

and the PDE (4.3.11) reduces to an ODE :

$$36k_1r^2\theta'' - 72k_1r\theta' + 72k_1\theta + 3(m+1)^2r^3 = 0. \tag{4.3.12}$$

After solving the above equation, we get

$$\theta = C_1r + C_2r^2 - \frac{(m+1)^2r^3}{24k_1},$$

where C_1, C_2 are constants and $k_1 \neq 0$.

Reverting back to original variables x, y, z, t, we get the exact solution of equation (4.1.3) given by

$$\begin{aligned} u(x,y,z,t) = {} & \frac{(m+1)^2z^2t^{m+1}}{36} - \frac{(m+1)^2t^{m+1}z^2}{72} + \frac{(m+1)x}{6} \\ & + C_2yt^{\frac{2m+2}{3}} + C_1t^{\frac{m+1}{3}} - \frac{(m+1)^2y^2t^{m+1}}{24k_1}. \end{aligned} \tag{4.3.13}$$

(ii) For generator $(a.2)$, the similarity variables are given by

$$a = yz^{-1},\ b = yt^{m+1},\ c = -x + \frac{m+1}{4}\big(z^2 + \frac{y^2}{k_1}\big)t^{m+1},\ u = \frac{\phi(a,b,c)}{y}. \tag{4.3.14}$$

Using equation (4.3.14) into equation (4.1.3), the reduced PDE is

$$\begin{aligned} & b^2\phi_{cccc} + 2b\phi_c^2 + 2a^3\phi_a - 2k_1a\phi_a + a^4\phi_{aa} + k_1a^2\phi_{aa} + 2k_1\phi \\ & - 2k_1b\phi_b + 2b\phi\phi_{cc} + 2k_1ab\phi_{ab} + k_1b^2\phi_{bb} = 0. \end{aligned} \tag{4.3.15}$$

Further, investigating the above equation for its invariance properties, the symmetry generators are

$$\begin{aligned} F_1 &= \frac{a^2}{b}\frac{\partial}{\partial a}, \\ F_2 &= \frac{\partial}{\partial c}. \end{aligned}$$

We consider the similarity variables for generator $F_1 + F_2$ as follows:

$$\alpha = b,\ \beta = cb^{-1} + a^{-1},\ \phi = \psi(\alpha,\beta). \tag{4.3.16}$$

Using equations (4.3.16) into equation (4.3.15), the reduced PDE is

$$\psi_{\beta\beta\beta\beta} + 2\alpha\psi_\beta^2 + 4k_1\alpha^2\beta\psi_\beta + 2k_1\alpha^2\psi - 2k_1\alpha^3\psi_\alpha + k_1\alpha^4\psi_{\alpha\alpha}$$
$$- 2k_1\alpha^3\beta\psi_{\alpha\beta} + (\alpha^2 + 2\alpha\psi + k_1\alpha^2\beta^2)\psi_{\beta\beta} = 0. \tag{4.3.17}$$

On applying the group method on the above equation, the symmetry generators are

$$G_1 = -2\alpha\frac{\partial}{\partial\alpha} + \beta\frac{\partial}{\partial\beta} + \alpha\frac{\partial}{\partial\psi},$$
$$G_2 = \frac{1}{\alpha}\frac{\partial}{\partial\beta}.$$

Now, the similarity variables for the generator $G_1 + G_2$ are obtained as

$$r = \beta\alpha^{1/2} - \alpha^{-1/2},\ \psi = -\frac{\alpha}{2} + \theta(r).$$

In view of these similarity variables, the equation (4.3.17) reduces to an ODE

$$4\theta^{''''} + 8(\theta^{'})^2 + 7k_1r\theta^{'} + 8k_1\theta + 8\theta\theta^{''} + k_1r^2\theta^{''} = 0. \tag{4.3.18}$$

Further analysis of equation (4.3.18) for Lie group invariance leads to the trivial symmetries.

(iii) For generator $(a.3)$, the similarity variables are as follows

$$\alpha = x,\ \beta = t,\ \gamma = y^2 + k_1z^2,\ u = \phi(\alpha, \beta, \gamma). \tag{4.3.19}$$

Using (4.3.19) in equation (4.1.3), the reduced PDE is

$$\phi_{\alpha\beta} + 2\beta^r\phi_\alpha^2 + \beta^m\phi_{\alpha\alpha\alpha\alpha} + 4k_1\beta^n\phi_\gamma + 4k_1\gamma\beta^n\phi_{\gamma\gamma} + 2\beta^r\phi\phi_{\alpha\alpha} = 0. \tag{4.3.20}$$

Analyzing the above PDE for its invariance properties, the symmetry generators are as follows:

$$L_1 = \frac{3\beta}{m+1}\frac{\partial}{\partial\beta} + \alpha\frac{\partial}{\partial\alpha} - 2\gamma\frac{\partial}{\partial\gamma} + \phi\frac{\partial}{\partial\phi},$$
$$L_2 = f(\beta)\frac{\partial}{\partial\alpha} + \frac{\beta f^{'}(\beta)}{2}\frac{\partial}{\partial\phi},$$
$$L_3 = \beta^{-m}\frac{\partial}{\partial\beta} - \frac{(m+1)^2\gamma}{4k_1}\frac{\partial}{\partial\alpha} - 2(m+1)\gamma\beta^{n+1}\frac{\partial}{\partial\gamma} + (m+1)\beta^{n+1}\phi\frac{\partial}{\partial\phi}.$$

For generator L_1, the similarity variables are

$$a = \alpha\beta^{-\frac{m+1}{3}},\ b = \gamma^{1/2}\beta^{\frac{m+1}{3}},\ \phi = \beta^{\frac{m+1}{3}}\psi(a,b),$$

and PDE (4.3.20) reduces to

$$\begin{aligned} &3b\psi_{aaaa} + 6b\psi\psi_{aa} + 3k_1 b\psi_{bb} + 3k_1\psi_b + 6b\psi_a^2 + (m+1)b^2\psi_{ab} \\ &\quad - (m+1)ab\psi_{aa} = 0. \end{aligned} \tag{4.3.21}$$

Symmetry generator of equation (4.3.21) is

$$G_1 = \frac{\partial}{\partial a} + \frac{m+1}{6}\frac{\partial}{\partial \psi},$$

and the similarity variables for generator G_1 are as under

$$b = r,\ \psi = \frac{(m+1)a}{6} + \theta(r),$$

which lead to an ODE

$$k_1 r\theta^{''} + k_1\theta^{'} + \frac{r(m+1)^2}{18} = 0. \tag{4.3.22}$$

The solution to equation (4.3.22) can be expressed as

$$\theta(r) = -\frac{(m+1)^2 r^2}{72k_1} + C_2 \log r + C_1.$$

where C_1, C_2 are arbitrary constants and $k_1 \neq 0$.

Reverting back to original variables x, y, z, t, we get the exact solution of equation (4.1.3) given by

$$\begin{aligned} u(x,y,z,t) = &\frac{(m+1)x}{6} + C_1 t^{\frac{m+1}{3}} + \frac{C_2}{2}t^{\frac{m+1}{3}}\log(y^2 + k_1 z^2) \\ &+ \frac{(m+1)C_2}{3}t^{\frac{m+1}{3}}\log t - \frac{(m+1)^2}{72k_1}t^{m+1}(y^2 + k_1 z^2). \end{aligned} \tag{4.3.23}$$

For generator L_2, the similarity variables are

$$a = \beta,\ b = \gamma,\ \phi = \frac{\beta f^{'}(\beta)}{2f(\beta)}\alpha + \psi(a,b). \tag{4.3.24}$$

Using (4.3.24) in equation (4.3.20), the reduced PDE is

$$4k_1 b a^n\psi_{bb} + 4k_1 a^n\psi_b + \frac{f^{'}(a)}{2f(b)} + a\frac{f^{''}(a)}{2f(b)} = 0. \tag{4.3.25}$$

After solving the equation (4.3.25) with the use of software MAPLE, we obtain

$$\psi = \frac{ba^{-n}}{8k_1 f(a)}\big(af^{''}(a) + f^{'}(a)\big) + g_1(a)\log b + g_2(a),$$

where $g_1(a), g_2(a)$ and $f(a)$ are arbitrary functions and $k_1 \neq 0$.

The exact solution of equation (4.1.3) is given by

$$\begin{aligned} u(x,y,z,t) = &\frac{xtf^{'}(t)}{2f(t)} + g_1(t)\log(y^2+k_1z^2) + g_2(t) \\ &+ \frac{(y^2+k_1z^2)t^{-n}}{8k_1 f(t)}\Big(tf^{''}(t) + f^{'}(t)\Big). \end{aligned} \tag{4.3.26}$$

For generator L_3 the similarity variables are

$$a = \gamma\beta^{2(m+1)},\ b = -\frac{(m+1)}{4k_1}\gamma\beta^{m+1} + \alpha,\ \phi = \beta^{m+1}\psi(a,b),$$

and equation (4.3.20) reduces to

$$\psi_{bbbb} + 2\psi_b^2 + 2\psi\psi_{bb} + 4k_1\psi_a + 4k_1 a\psi_{aa} = 0. \tag{4.3.27}$$

The symmetry generators of equation (4.3.27) are

$$\begin{aligned} G_1 &= 4a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b} - 2\frac{\partial}{\partial \psi}, \\ G_2 &= \frac{\partial}{\partial b}. \end{aligned}$$

Next, we consider the similarity variables for generator $G_1 + G_2$ as

$$r = ba^{-1/4},\ \psi = a^{-1/2}\theta(r). \tag{4.3.28}$$

Using equations (4.3.28) in equation (4.3.27), the PDE reduces to an ODE

$$4\theta_{rrrr} + 8\theta\theta_{rr} + 8\theta_r^2 + 4k_1\theta - 5k_1 r\theta_r + k_1 r^2\theta_{rr} = 0. \tag{4.3.29}$$

Further analysis of the above ODE for its invariance properties, we find that it admits only the trivial symmetries.

Subcase 2: For this subcase, the equations (4.3.3) reduce to

$$q_1 = 0,\ q_2 = 0,\ p(t) = -2c_6.$$

Using now the group infinitesimals as described in equations (4.2.2) we arrive at the symmetry generators $(b.1)-(b.3)$ as mentioned in Table 4.1.

We now present the similarity reductions and exact solutions for generators $(b.1)-(b.3)$ as follows:

(i) Similarity variables for the generator $(b.1)$ are

$$a = xy^{-1/2},\ b = yz^{-1}\ c = t^{-1}y^{\frac{3}{2}(m+1)},\ u = \frac{\phi}{y}. \tag{4.3.30}$$

Using equation (4.3.30) in equation (4.1.3), the reduced PDE is given by

$$\begin{aligned}
&-4(m+1)c^4\phi_{ac} + 12\phi_{aaaa} + 24\phi\phi_{aa} + 24\phi_a^2 + 24b^3\phi_b + 12b^4\phi_{bb} \\
&+ 24k_1\phi + 21k_1a\phi_a - 24k_1b\phi_b - 15k_1c\phi_c + 3k_1a^2\phi_{aa} - 12k_1ab\phi_{ab} \\
&- 6k_1ac\phi_{ac} + 12k_1b^2\phi_{bb} + 12k_1bc\phi_{bc} + 3k_1c^2\phi_{cc} = 0.
\end{aligned} \tag{4.3.31}$$

On further, analyzing the invariance properties of the above equation, the symmetry generators obtained as follows:

$$\begin{aligned}
F_1 &= \frac{2b}{c^2}\frac{\partial}{\partial b} - \left(\frac{a}{c^2} + \frac{2(m+1)c}{3k_1}\right)\frac{\partial}{\partial a} + \frac{1}{c}\frac{\partial}{\partial c} + \left(\frac{2\phi}{c^2} + \frac{(m+1)^2c^4}{9k_1}\right)\frac{\partial}{\partial \phi}, \\
F_2 &= \left(2 + \frac{2a^2}{k_1}\right)\frac{\partial}{\partial b} - \frac{a}{b}\frac{\partial}{\partial a} + \frac{c}{b}\frac{\partial}{\partial c} + \frac{2\phi}{b}\frac{\partial}{\partial \phi}, \\
F_3 &= \frac{1}{c}\frac{\partial}{\partial a} + \frac{(m+1)c^2}{6}\frac{\partial}{\partial \phi}, \\
F_4 &= \frac{b^2}{c^2}\frac{\partial}{\partial b} + \frac{(m+1)c}{3b}\frac{\partial}{\partial a} - \frac{(m+1)^2c^4}{18b}\frac{\partial}{\partial \phi}.
\end{aligned} \tag{4.3.32}$$

For further analysis from the point of view of reduction and exact solutions, we consider only two cases of generators F_3 and F_4.

The similarity variables for generator F_3 can be easily obtained as follows:

$$\alpha = b,\ \beta = c,\ \phi = \frac{(m+1)c^3a}{6} + \psi(\alpha, \beta). \tag{4.3.33}$$

Using (4.3.33) into (4.3.31), the PDE reduces to

$$\begin{aligned}
&36\alpha^4\psi_{\alpha\alpha} + 72\alpha^3\psi_\alpha + 72k_1\psi - 4(m+1)^2\beta^6 - 72k_1\alpha\psi_\alpha - 45k_1\beta\psi_\beta \\
&+ 36k_1\alpha^2\psi_{\alpha\alpha} + 36k_1\alpha\beta\psi_{\alpha\beta} + 9k_1\beta^2\psi_{\beta\beta} = 0.
\end{aligned} \tag{4.3.34}$$

After solving this equation by using software MAPLE, we get

$$\psi = \frac{C_1\alpha^2}{\alpha^2+k_1} + \frac{C_2\alpha}{\alpha^2+k_1} + \frac{C_3}{24\alpha^2+24k_1} + C_4\beta^2 + C_5\beta^4 + \frac{(m+1)^2\beta^6}{18k_1} - \frac{C_3}{24k_1},$$

where C_1, C_2, C_3, C_4, C_5 are arbitrary constants and $k_1 \neq 0$.

Reverting back to original variables, the exact solution of (4.1.3) is given by

$$\begin{aligned} u(x,y,z,t) = {} & \frac{(m+1)}{6}xt^{-m-1} + \frac{C_1 y}{y^2+k_1z^2} + \frac{C_2 z}{y^2+k_1z^2} + \frac{C_3z^2}{24y(y^2+k_1z^2)} \\ & + C_4t^{\frac{-2m-2}{3}} + C_5yt^{\frac{-4m-4}{3}} + \frac{(m+1)^2}{18k_1}y^2t^{-2m-2} - \frac{C_3}{24k_1y}. \end{aligned} \tag{4.3.35}$$

Similarity variables for generator F_4 are

$$\alpha = c,\ \beta = \frac{(m+1)c^3b^{-2}}{2} + 3a,\ \phi = \frac{(m+1)^2c^6b^{-2}}{36} + \psi(\alpha,\beta). \tag{4.3.36}$$

Using above equation in equation (4.3.31), the reduced PDE is

$$\begin{aligned} & 9k_1\alpha^2\psi_{\alpha\alpha} - 45k_1\alpha\psi_\alpha + 2(m+1)^2\alpha^6 + 2916\psi_{\beta\beta\beta\beta} + 648\psi_\beta^2 + 648\psi\psi_{\beta\beta} \\ & + 72k_1\beta - 36(m+1)\alpha^4\psi_{\beta\alpha} - 18k_1\beta\alpha\psi_{\beta\alpha} + 9k_1\beta^2\psi_{\beta\beta} \\ & + 36(m+1)\alpha^3\psi_\beta + 63k_1\beta\psi_\beta = 0. \end{aligned} \tag{4.3.37}$$

The symmetry generator of equation (4.3.37) is

$$G = \frac{1}{\alpha}\frac{\partial}{\partial\beta} + \frac{m+1}{18}\alpha^2\frac{\partial}{\partial\psi}.$$

Similarity variables for generator G are

$$r = \alpha,\ \psi = \frac{(m+1)\alpha^3\beta}{18} + \theta(r).$$

Using these similarity variables in equation (4.3.37), the reduced ODE is

$$k_1r^2\theta_{rr} - 5k_1r\theta_r + 8k_1\theta = 0. \tag{4.3.38}$$

On solving it, we can easily obtain

$$\theta = C_1r^2 + C_2r^4,$$

where C_1, C_2 are arbitrary constants.

On reverting back the exact solution of equation (4.1.3) is given by

$$u(x,y,z,t) = \frac{(m+1)^2}{18}z^2t^{-2m-2} + \frac{(m+1)}{6}xt^{-m-1} + C_1t^{\frac{-2m-2}{3}} + C_2yt^{\frac{-4m-4}{3}}. \tag{4.3.39}$$

(ii) Similarity variables for the generator ($b.2$) are

$$a = x,\ b = t,\ c = y^2 + k_1 z^2,\ u = \phi(a, b, c).$$

Using above equations in equation (4.1.3), the reduced PDE is given by

$$b^m \phi_{aaaa} + \phi_{ab} + 2b^r \phi_a^2 + 4k_1 b^n \phi_c + 4k_1 c b^n \phi_{cc} + 2b^r \phi \phi_{aa} = 0. \tag{4.3.40}$$

The symmetry generators of equation (4.3.40) are

$$F_1 = b^{-m} \frac{\partial}{\partial b},$$
$$F_2 = f_2(b) \frac{\partial}{\partial a} + \frac{b^{-m} f_2'(b)}{2} \frac{\partial}{\partial \phi}.$$

Now the similarity variables for the generator F_2 are

$$\alpha = b,\ \beta = c,\ \phi = \frac{x^{-m}}{2} \frac{f_2'(b)}{f_2(b)} a + \psi(\alpha, \beta) = 0. \tag{4.3.41}$$

Using equation (4.3.41) in equation (4.3.40), the reduced PDE is

$$4k_1 \beta \alpha^n \psi_{\beta\beta} + 4k_1 \alpha^n \psi_\beta + \frac{\alpha^{-m}}{2} \frac{f_2''(\alpha)}{f_2(\alpha)} - \frac{m\alpha^{-m-1}}{2} \frac{f_2'(\alpha)}{f_2(\alpha)} = 0. \tag{4.3.42}$$

After solving the above PDE with the aid of software MAPLE, we get

$$\psi = -\frac{1}{8k_1 f_2(\alpha)} \Big(- m\alpha^{-m-n-1} f_2'(\alpha) + \alpha^{-m-n} f_2''(\alpha) \Big) \beta + g_1(\alpha) \log \beta + g_2(\alpha),$$

where $g_1(\alpha),\ g_2(\alpha),\ f_2(\alpha)$ are arbitrary functions of α.

Reverting back to the original variables, the exact solution of equation (4.1.3) is given by

$$\begin{aligned} u(x, y, z, t) = & \frac{t^{-m}}{2} \frac{f_2'(t)}{f_2(t)} x + g_1(t) \log(y^2 + k_1 z^2) + g_2(t) \\ & - \frac{(y^2 + k_1 z^2)}{8k_1 f_2(t)} \Big(- m t^{-m-n-1} f_2'(t) + t^{-m-n} f_2''(t) \Big). \end{aligned} \tag{4.3.43}$$

(iii) Similarity variables for the generator ($b.3$) are

$$a = x,\ b = y,\ c = z,\ u = \phi(a, b, c). \tag{4.3.44}$$

Using equations (4.3.44) into equation (4.1.3), the PDE reduces to

$$\phi_{aaaa} + 2\phi_a^2 + k_1 \phi_{bb} + \phi_{cc} + 2\phi \phi_{aa} = 0, \tag{4.3.45}$$

which on solving by using software MAPLE gives

$$\phi(a,b,c) = -6C_2^2\tanh(C_1 + C_2a + C_3b + C_4c)^2 - \frac{1}{2C_2^2}(C_3^2k_1 - 8C_2^4 + C_4^2),$$

where C_1, C_2, C_3, C_4 are arbitrary constants and $C_2 \neq 0$. On back substitution the exact solution of equation (4.1.3) is given by

$$u(x,y,z,t) = -6C_2^2\tanh(C_1 + C_2x + C_3y + C_4z)^2 - \frac{1}{2C_2^2}(C_3^2k_1 - 8C_2^4 + C_4^2). \tag{4.3.46}$$

Case 2: In this case the equation (4.3.1) gives

$$f(t) = At^{\frac{3(2r-n+1)}{2}} \tag{4.3.47}$$

On using equation (4.3.47) in equations (4.2.4)-(4.2.6), (4.2.8), (4.2.9), we get

$$\begin{aligned} q_1(t) &= \frac{3A}{4}(2r-n+1)t^{\frac{4r-3n+1}{2}},\\ q_2(t) &= -\frac{3A}{32}(6r-5n+1)(4r-n+3)t^{\frac{4r-5n-1}{2}}(z^2+\frac{y^2}{k_1}),\\ p(t) &= -\frac{2A}{2r-n+1}(r+1)t^{\frac{3}{2}(2r-n+1)} - (r+1)c_1 - 2At^{\frac{3}{2}(2r-n+1)}, \end{aligned} \tag{4.3.48}$$

where r, n, c_1, A are arbitrary constants and $A \neq 0$.
Further, using equations (4.3.48) into equation (4.2.7), we arrive at the following condition

$$A(n^2 - (8r+6)n + 8r^2 + 8r + 1) = 0,$$

which leads to one subcase
Subcase : $n = 4r \pm 2(\sqrt{2}r + \sqrt{2}) + 3,\ A \neq 0$
Using the equations (4.3.48) and (4.3.47) into equations (4.2.2), we obtain the symmetry generators $(c.1) - (c.3)$ presented in Table 4.1.
Next, we give the similarity reductions and exact solutions for the generators $(c.1) - (c.3)$.

(i) For generator $(c.1)$ the similarity variables are

$$a = yz^{-1},\ b = x,\ c = yt^{\frac{-n-1}{2}},\ u = y^{\frac{-2r-2}{n+1}}\phi(a,b,c). \tag{4.3.49}$$

Using equations (4.3.49) in equation (4.1.3), the PDE reduces to

$$\begin{aligned}
&c^2\phi_{bbbb} + k_1c^2\phi_{cc} + 2k_1ac\phi_{ac} + k_1a^2\phi_{aa} + 2a^3\phi_a + a^4\phi_{aa} + 2c^{\frac{2n-2r}{n+1}}\phi_b^2 \\
&\quad + 2c^{\frac{2n-2r}{n+1}}\phi\phi_{bb} - \frac{4(r+1)}{n+1}k_1c\phi_c + \frac{2k_1(r+1)}{n+1}\Big(\frac{2(r+1)}{n+1}+1\Big)\phi \\
&\quad - \frac{4(r+1)}{n+1}k_1a\phi_a - \frac{n+1}{2}c^3\phi_{bc} = 0. \qquad (4.3.50)
\end{aligned}$$

Further, symmetry generators of this equation are

$$\begin{aligned}
F_1 &= \Big(\frac{a^2}{k_1}+1\Big)\frac{\partial}{\partial a} + \frac{c}{a}\frac{\partial}{\partial c} + \Big(\frac{2+2r}{a(n+1)}\Big)\phi\frac{\partial}{\partial \phi}, \\
F_2 &= \frac{a}{c}\frac{\partial}{\partial a} - \frac{(n+1)c}{4k_1}\frac{\partial}{\partial b} + \frac{\partial}{\partial c} + \Big(\frac{2+2r}{c(n+1)}\phi + \frac{(n+1)^2c^{\frac{2r+n+3}{n+1}}}{16k_1}\Big)\frac{\partial}{\partial \phi}, \qquad (4.3.51) \\
F_3 &= \frac{\partial}{\partial b}.
\end{aligned}$$

For generator F_1, the similarity variables are

$$\alpha = b,\ \beta = \Big(1+\frac{k_1}{a^2}\Big)^{\frac{-1}{2}}c^{-1},\ \phi = c^{\frac{2r+2}{n+1}}\psi(\alpha,\beta).$$

Using the above equations in equation (4.3.50), the reduced PDE is

$$\psi_{\alpha\alpha\alpha\alpha} + 2\psi_\alpha^2 + 2\psi\psi_{\alpha\alpha} + k_1\beta^4\psi_{\beta\beta} + k_1\beta^3\psi_\beta - (r+1)\psi_\alpha - \frac{(n+1)\beta}{2}\psi_{\alpha\beta} = 0. \qquad (4.3.52)$$

The symmetry generator for this PDE is

$$G_1 = \frac{\partial}{\partial \alpha},$$

and the similarity variable is

$$r = \beta,\ \psi = \theta(r).$$

As a consequence, the equation (4.3.52) reduces to an ODE

$$k_1r^4\theta''(r) + k_1r^3\theta'(r) = 0,$$

which on solving, leads to

$$\theta(r) = C_1\log(r) + C_2,$$

where C_1, C_2 are arbitrary constants.

Once again, reverting back to original variables, the exact solution of equation (4.1.3) is given by

$$u(x,y,z,t) = t^{-r-1}\Bigg[-\frac{C_1}{2}\log\Big(1+\frac{k_1z^2}{y^2}\Big) - C_1\log(y) + \frac{(n+1)}{2}C_1\log(t) + C_2\Bigg]. \quad (4.3.53)$$

(ii) For generator $(c.2)$, the similarity variables are

$$a = x,\ b = y^2 + k_1z^2,\ c = t,\ u = \phi(a,b,c).$$

Using the above equation in equation (4.1.3), the reduced PDE is

$$\phi_{aaaa} + 2c^{r+1}\phi_a^2 + 2c^{r+1}\phi\phi_{aa} + c\phi_{ac} + 4k_1c^{n+1}\phi_b + 4k_1c^{n+1}b\phi_{bb} = 0. \quad (4.3.54)$$

Further analyzing the invariance properties of the above PDE, the symmetry generators are

$$F_1 = f(c)\frac{\partial}{\partial a} + \frac{t^{-r}}{2}f'(c)\frac{\partial}{\partial \phi},$$
$$F_2 = b\frac{\partial}{\partial b} + \frac{c}{n+1}\frac{\partial}{\partial c} - \Big(\frac{r+1}{n+1}\Big)\phi\frac{\partial}{\partial \phi}.$$

The similarity variables for generator F_1 are given by

$$\alpha = b,\ \beta = c,\ \phi = \frac{c^{-r}a}{2}\frac{f'(c)}{f(c)} + \psi(\alpha,\beta).$$

Using the above equations into equation (4.3.54), the reduced PDE is

$$4k_1\beta^{n+1}\psi_\alpha + 4k_1\alpha\beta^{n+1}\psi_{\alpha\alpha} - \frac{r\beta^{-r}}{2}\frac{f'(\beta)}{f(\beta)} + \frac{\beta^{-r+1}}{2}\frac{f''(\beta)}{f(\beta)} = 0. \quad (4.3.55)$$

After solving the above equation using software MAPLE, we get

$$\psi(\alpha,\beta) = g(\beta)\ln(\alpha) + h(\beta) + \frac{\alpha}{8k_1f(\beta)}\Big(r\beta^{-n-r-1}f'(\beta) - \beta^{-n-r}f''(\beta)\Big),$$

where $g(\beta),\ h(\beta),\ f(\beta)$ are arbitrary functions.

Reverting back to original variables the exact solutions of equation (4.1.3) is given by

$$u(x,y,z,t) = \frac{xt^{-r}}{2}\frac{f'(t)}{f(t)} + g(t)\ln(y^2+k_1z^2) + h(t)$$
$$+ \frac{y^2+k_1z^2}{8k_1f(t)}\Big(rt^{-n-r-1}f'(t) - t^{-n-r}f''(t)\Big). \quad (4.3.56)$$

For generator F_2 the similarity variables are given by

$$\alpha = a,\ \beta = bc^{-n-1},\ \phi = c^{-r-1}\psi(\alpha,\beta).$$

Using this equation in equation (4.3.54), the PDE reduces to

$$\psi_{\alpha\alpha\alpha\alpha} + 2\psi\psi_{\alpha\alpha} + 2\psi_{\alpha}^2 - (n+1)\beta\psi_{\alpha\beta} - (r+1)\psi_{\alpha} + 4k_1\psi_{\beta} + 4k_1\beta\psi_{\beta\beta} = 0. \tag{4.3.57}$$

Investigating the invariance properties of the reduced PDE (4.3.57), the symmetry generator is

$$G_1 = \frac{\partial}{\partial\alpha},$$

and similarity variable is

$$r = \beta,\ \psi = \theta(r).$$

The PDE (4.3.57) reduces to

$$4k_1 r\theta^{''}(r) + 4k_1\theta^{'}(r) = 0.$$

On solving it, we get

$$\theta(r) = C_1 + C_2\ln(r),$$

where $C_1,\ C_2$ are arbitrary constants.
Reverting back to original variables, the exact solutions of equation (4.1.3) is given by

$$u(x,y,z,t) = C_1 t^{-r-1} + C_2 t^{-r-1}\log(y^2 + k_1 z^2) - (n-1)C_2 t^{-r-1}\log(t). \tag{4.3.58}$$

Case 3: In this case ($c_1 = 0$) equation (4.3.1) gives

$$f(t) = A + Bt^{-\frac{5m}{2}-\frac{3n}{2}+3r-1}, \tag{4.3.59}$$

where A, B are arbitrary constants.
On using equation (4.3.59) in equations (4.2.4)-(4.2.6), (4.2.8), (4.2.9), we get

$$\begin{aligned} q_2(y,z,t) &= \frac{B(4r-3m-n)}{32(2r-m-n)}(-5m-2+6r-3n)(6r-4-5m-5n)t^{-\frac{5m}{2}-\frac{5n}{2}+2r-3}\Big(z^2+\frac{y^2}{k_1}\Big), \\ q_1(t) &= \frac{B}{4}(-5m-3n+6r-2)t^{\frac{5m}{2}+\frac{3n}{2}-2r+2}, \\ p(t) &= \frac{(m-3r-2)A}{m+1}+\Big(\frac{2(m-r)B}{2r-m-n}-2B\Big)t^{-\frac{5m}{2}-\frac{3n}{2}+3r-1}, \end{aligned} \tag{4.3.60}$$

where m, n, r are arbitrary constants.
Further, using equation (4.3.59) into equation (4.2.7), we arrive at the following condition

$$\begin{aligned} &B(6r-2-5m-3n)(32r^2-24r-52mr-36nr+16m+20m^2 \\ &+28mn+8n+8n^2)=0. \end{aligned}$$

From above equation we take up the special subcase $B=0$.
For $B=0$, equation (4.3.59) and (4.3.60) reduce to

$$f(t)=A,\ q_1=0\ q_2=0,\ p(t)=\frac{(m-3r-2)A}{m+1}. \tag{4.3.61}$$

Using the equations (4.3.61) into equations (4.2.2), we obtain the symmetry generators $(d.1)-(d.2)$ given in Table 4.1.
Now, we present the similarity reductions and particular solutions for the generators $(d.1)-(d.2)$.

(i) Similarity variables for generator $(d.1)$ are

$$a=xy^{\frac{-2(m+1)}{3n+m+4}},\ b=yz^{-1},\ c=ty^{\frac{-6}{3n+m+4}},\ u=y^{\frac{2(m-3r-2)}{3n+m+4}}\phi(a,b,c). \tag{4.3.62}$$

Using equation (4.3.62) in equation (4.1.3), the PDE (4.1.3) reduces to

$$\begin{aligned}
&\phi_{ac} + 2c^r\phi_a^2 + 2c^r\phi\phi_{aa} + c^m\phi_{aaaa} + 2b^3c^n\phi_b + b^4c^n\phi_{bb} \\
&+ k_1\Bigg[\frac{2m-6r-4}{3n+m+4}\Big(\frac{2m-6r-4}{3n+m+4} - 1\Big)c^n\phi + 2bc^n\frac{2m-6r-4}{3n+m+4}\phi_b \\
&- ac^n\Big(\frac{4(m+1)}{3n+m+4}\Big)\Big(\frac{2m-6r-4}{3n+m+4}\Big)\phi_a - \frac{12(2m-6r-4)}{3n+m+4}c^{n+1}\phi_c + b^2c^n\phi_{bb} \\
&- \frac{4(m+1)}{3n+m+4}abc^n\phi_{ab} - \frac{6bc^{n+1}}{3n+m+4}\phi_{cb} + \frac{2(m+1)}{3n+m+4}\Big(\frac{2m+2}{3n+m+4} + 1\Big)ac^n\phi_a \\
&+ \frac{4(m+1)^2}{(3n+m+4)^2}b^2c^n\phi_{aa} + \frac{6}{3n+m+4}\Big(\frac{6}{3n+m+4} + 1\Big)c^{n+1}\phi_c \\
&- \frac{6bc^{n+1}}{3n+m+4}\phi_{bc} + \frac{24(m+1)}{(3n+m+4)^2}bc^{n+1}\phi ac + \frac{36}{(3n+m+4)^2}c^{n+2}\phi_{cc}\Bigg] = 0.
\end{aligned} \tag{4.3.63}$$

Due to highly complex nature of nonlinear PDE (4.3.63), we leave it as it is, though, theoretically, one may seek the invariance of equation (4.3.63) under one-parameter Lie group of transformations. More specifically, the tedious calculations involved have kept us from its further analysis.

(ii) Similarity variables for generator ($d.2$) are

$$\alpha = x,\ \beta = y^2 + k_1z^2,\ \gamma = t,\ u = \phi. \tag{4.3.64}$$

On using (4.3.64) in equation (4.1.3), the reduced PDE is given by

$$\phi_{\alpha\gamma} + 2t^{r\gamma}\phi_\alpha^2 + 2t^{r\gamma}\phi\phi_{\alpha\alpha} + t^{m\gamma}\phi_{\alpha\alpha\alpha\alpha} + 4k_1t^{n\gamma}\phi_\beta + 4k_1\beta t^{n\gamma}\phi_{\beta\beta} = 0. \tag{4.3.65}$$

Symmetry generators for equation (4.3.65) are

$$\begin{aligned}
F_1 &= \beta\frac{\partial}{\partial\beta} + \frac{m}{m+3n}\frac{\partial}{\partial\alpha} + \frac{3\gamma}{m+3n}\frac{\partial}{\partial\gamma} + \frac{m-3r}{m+3n}\phi\frac{\partial}{\partial\phi}, \\
F_2 &= f(\gamma)\frac{\partial}{\partial\alpha} + \frac{\gamma^{-r}}{2}f'(\gamma)\frac{\partial}{\partial\phi}.
\end{aligned} \tag{4.3.66}$$

As the method of reduction of PDE (4.3.65) is similar for both generators given in equation (4.3.66). So, we restrict ourselves to generator F_2 only.
Similarity variables for generators F_2 are

$$a = \beta,\ b = \gamma,\ \phi = \frac{\gamma^{-r}f'(\gamma)}{2f(\gamma)}\alpha + \psi(a,b). \tag{4.3.67}$$

Using equations (4.3.67) in equation (4.3.65), we have

$$-\frac{rb^{-r-1}}{2}\frac{f'(b)}{f(b)}+\frac{b^{-r}}{2}\frac{f''(b)}{f(b)}+4k_1b^n\psi_a+4k_1ab^n\psi_{aa}=0. \tag{4.3.68}$$

On solving equation (4.3.68), we get

$$\psi(a,b)=h_1(b)\log(a)+h_2(b)+\frac{a}{8k_1f(b)}\Big(rb^{-n-r-1}f'(b)-b^{-n-r}f''(b)\Big),$$

where $f(b),\ h_1(b),\ h_2(b)$ are arbitrary functions and $k_1\neq 0, f(b)\neq 0$.

The exact solution of equation (4.1.3) is given by

$$\begin{aligned}u(x,y,z,t)=&\frac{t^{-r}x}{2}\frac{f'(t)}{f(t)}+h_1(t)\log(y^2+k_1z^2)\\&+\frac{(y^2+k_1z^2)t^{-n-r}}{8k_1f(t)}\Big(rt^{-1}f'(t)-f''(t)\Big)+h_2(t).\end{aligned} \tag{4.3.69}$$

4.4 Discussion

The invariance properties of the (3+1) dimensional Kadomtsev-Petviashvilli (KP) equation with variable coefficients and an arbitrary nonlinear term have been successfully analyzed by using the classical Lie symmetry method. The obtained symmetry algebra of equation (4.1.3) is infinite-dimensional. For the admissible forms of the variable coefficients, a number of governing equations have been derived, in general. For the sake of illustration, from the point of view of dimensional reductions of KP equation and derivation of exact solutions, we have taken, for example, the variable coefficients as power functions of 't', although there is tremendous scope of experimenting with various other physically relevant forms of the variable coefficients. The symmetry generators for all the possible considered cases, subcases have been shown in Table 4.1. In the process of dimensional reductions of the KP equation, we have exploited the invariance of reduced nonlinear partial differential equations and have reported various Lie group generators. However, for the purpose of demonstration and exact solutions, we have confined ourselves to few generators only, while leaving the scope of further reductions/exact solutions in the future study.

Chapter 5

Lie symmetries of the Gilson-Pickering equation with time dependent coefficients

5.1 Introduction

In 1995, Claire Gilson and Andrew Pickering proposed a model, named Gilson-Pickering equation [15] of the form

$$u_t - \epsilon u_{xxt} + 2\kappa u_x - u u_{xxx} - \alpha u u_x - \beta u_x u_{xx} = 0, \tag{5.1.1}$$

where $\epsilon, \alpha, \beta, \kappa$ are real arbitrary constants. They investigated this equation for its Painlève analysis and provided the traveling wave solutions. This equation includes many other nonlinear equations for special values of parameters, viz.,

(i) Fornberg-Whitham equation [12], for $\epsilon = 1, \alpha = -1, \beta = 3, \kappa = 1/2$,

(ii) Rosenau-Hyman equation [84], for $\epsilon = 0, \alpha = 1, \beta = 3, \kappa = 0$, and

(iii) Fuchssteiner-Fokas-Camassa-Holm equation [86], for $\epsilon = 1, \alpha = -1, \beta = 2$.

Many authors have investigated this equation using different techniques. Wang and Li [19] investigated the class of third-order nonlinear equation using factorization method, in which they include equation (5.1.1) as a special case. Some exact explicit solutions were proposed by Aslan [41] using division theorem. Tang et al. [17] have presented the periodic solutions and solitary patterns for the generalized Gilson-Pickering equation. Ebadia et al. [32] have employed the invariance and multiplier approach and furnished the soliton solutions and conservation laws for the equation (5.1.1). The bifurcations of traveling wave solutions for equation (5.1.1) were provided by Chen et al. in [7]. Irshad and Din [9] have introduced some exponential and hyperbolic function solutions of equation (5.1.1), using the Tanh-Coth method. New complex soliton solutions of (5.1.1) were introduced by Baskonus [35] using Bernoulli sub-equation function method. Symmetry reductions were proposed by Clarkson et al. [75] by utilizing the classical Lie method and nonclassical method. Fan et al. [100] have investigated some other solitary wave and traveling wave solutions using simplified G'/G-expansion method.
In view of equation (5.1.1), we generalize it as follows:

$$u_t - a(t)u_{xxt} + b(t)u_x - uu_{xxx} - c(t)uu_x - d(t)u_xu_{xx} = 0, \tag{5.1.2}$$

where $a(t), b(t), c(t), d(t)$ are arbitrary functions of t. The chapter is structured as follows: In section 5.2, we established the symmetries of the equation (5.1.2). In section 5.3, we have presented the similarity reductions for each form of the variable coefficients and proposed some exact solutions. Also, we furnished the ordinary differential equations which admit the trivial symmetries. Thus in section 5.4, we present the power series solutions of the reduced ordinary differential equations alongwith their convergence. The concluding remarks are provided in section 5.5.

5.2 Symmetry Analysis

In the present section, we introduce the symmetry analysis of the equation (5.1.2) by employing the classical Lie symmetry method. Consider the one-parameter Lie group of point transformations under which equation (5.1.2) remains invariant of the form [44]

$$\begin{aligned} x^* &= x + \epsilon\xi(t,x,u) + O(\epsilon^2), \\ t^* &= t + \epsilon\eta(t,x,u) + O(\epsilon^2), \\ u^* &= u + \epsilon\zeta(t,x,u) + O(\epsilon^2), \end{aligned} \tag{5.2.1}$$

where ϵ is a group parameter and ξ, η, ζ are the group infinitesimals of the Lie point transformations acting on the space of independent and dependent variables. On using the group of point transformations (5.2.1) in equation (5.1.2), and equating the coefficients of various partial derivatives of u with respect to the independent variables x and t, we obtain the following list of determining equations

(i) $\xi_u = 0, \eta_u = 0, \eta_x = 0, \zeta_{uu} = 0,$

(ii) $-2d(t)\zeta_{xu} + d(t)\xi_{xx} = 0,$

(iii) $-2a(t)\zeta_{xu} + a(t)\xi_{xx} = 0,$

(iv) $-u\eta_t - a(t)u\zeta_{xxu} + a(t)\xi_t + 3u\xi_x - \zeta = 0,$

(v) $\zeta_t - a(t)\zeta_{xxt} + b(t)\zeta_x - u\zeta_{xxx} - c(t)u\zeta_x = 0,$

(vi) $a(t)u\eta_t + 2a(t)u\xi_x - a^{'}(t)u\eta - a^2(t)\xi_t + a(t)\zeta - 3a(t)u\xi_x = 0,$

(vii) $-a(t)\zeta_{ut} + 2a(t)\xi_{xt} - 3u\zeta_{xu} + 3u\xi_{xx} - d(t)\zeta_x = 0,$

(viii) $-u\eta d^{'}(t) - ud(t)\zeta_u - a(t)d(t)\xi_t + d(t)\zeta = 0,$

(ix) $-u\xi_t - 2ua(t)\zeta_{xtu} + a(t)u\xi_{xxt} + 2b(t)u\xi_x - 3u^2\zeta_{xxu} + u^2\xi_{xxx} - c^{'}(t)u^2\eta - 2c(t)u^2\xi_x - d(t)u\zeta_{xx} - b(t)\zeta + b(t)a(t)\xi_t - c(t)a(t)u\xi_t + b^{'}(t)u\eta = 0.$

On solving these linear partial differential equations involving variable coefficients, we come across the following two cases:

Case 1: When $d^{'}(t) \neq 0$, the symmetries of equation (5.1.2) are as follows

$$\begin{aligned} \xi &= \int \frac{c_1}{a(t)} dt + c_2, \\ \zeta &= c_1, \\ \eta &= 0, \end{aligned} \tag{5.2.2}$$

alongwith the relation $c(t) = -\frac{1}{a(t)}$, $a(t) \neq 0$.

Case 2: When $d(t)$ is constant, the symmetries of the equation (5.1.2) are given by

$$\begin{aligned} \xi &= \left(\int \frac{c_3}{a(t)} dt + c_4 \right) x + c_6 - c_3 \int \left(\frac{1}{a(t)} \int b(t) dt \right) dt + \int \frac{c_5}{a(t)} dt, \\ \eta &= 2a(t) \int \left(\frac{1}{a(t)} \int \frac{c_3}{a(t)} dt + c_4 \right) dt - a(t) \int \frac{c_2}{a(t)} dt + c_1 a(t), \\ \zeta &= \left(c_2 - c_4 - \int \frac{c_3}{a(t)} dt \right) u + c_3 x - c_3 \int b(t) dt + c_5, \end{aligned} \tag{5.2.3}$$

where $c_1, c_2, c_3, c_4, c_5, c_6$ are arbitrary constants and arbitrary coefficients $a(t), b(t)$ and $c(t)$ are governed by the following equations

$$2c_3 \int \frac{1}{a(t)} dt + 2c_4 - c_1 a^{'}(t) - 2a^{'}(t) \int \frac{1}{a(t)} \left(\int \frac{c_3}{a(t)} dt + c_4 \right) dt + a^{'}(t) \int \frac{c_2}{a(t)} dt = 0, \tag{5.2.4}$$

$$\begin{aligned} &3b(t) \int \frac{c_3}{a(t)} dt + 3c_4 b(t) - c_2 b(t) + c_1 b^{'}(t) a(t) - a(t) b^{'}(t) \int \frac{c_2}{a(t)} dt \\ &+ 2a(t) b^{'}(t) \int \frac{1}{a(t)} \left(\int \frac{c_3}{a(t)} dt + c_4 \right) dt = 0, \end{aligned} \tag{5.2.5}$$

$$a(t) = -\frac{1}{c(t)}. \tag{5.2.6}$$

Since the explicit forms of the variable coefficients can not be worked out, therefore, the further process is illustrated by taking some special forms of these coefficients.

Subcase 1: Let $a(t) = k_1 t^m$, and on solving the equations (5.2.4)-(5.2.6), we get

$$\begin{aligned} m &= 2/3, \\ c_1 &= 0, \\ c_2 &= c_4. \end{aligned}$$

and the forms of variable coefficients are

$$
\begin{aligned}
a(t) &= k_1 t^{2/3},\\
c(t) &= -\frac{t^{-2/3}}{k_1},\\
d(t) &= 3, \qquad\qquad (5.2.7)\\
b(t) &= At^{-2/3}(3c_3t^{1/3} + k_1c_2)^{-1}.
\end{aligned}
$$

Therefore, the equations (5.2.3) reduce to

$$
\begin{aligned}
\xi &= \frac{3c_3}{k_1}t^{1/3}x + +c_2x + c_6 + \frac{3c_5}{k_1}t^{1/3} - \frac{A}{k_1}\Bigg(3\log(3c_3t^{1/3} + c_2k_1) - 3t^{1/3}\\
&\quad + \frac{c_2k_1}{c_3}\log(3c_3t^{1/3} + c_2k_1) - \frac{c_2k_1}{c_3}\Bigg)\\
\eta &= \frac{9c_3}{k_1}t^{4/3} + 3c_2t \qquad\qquad (5.2.8)\\
\zeta &= -\frac{3c_3}{k_1}t^{1/3}u + c_3x - A\log(3c_3t^{1/3} + c_2k_1) + c_5, \quad c_3 \neq 0.
\end{aligned}
$$

For the forms of infinitesimals, we take the particular case as follows:

Let $c_2 = 0$, then we have, $b(t) = At^{-1}$, and the infinitesimals are

$$
\begin{aligned}
\xi &= \frac{3c_3}{k_1}t^{1/3}x + c_6 + \frac{3c_5}{k_1}t^{1/3} - \frac{c_3A}{k_1}\Bigg(3t^{1/3}\log t - 9t^{1/3}\Bigg),\\
\eta &= \frac{9c_3}{k_1}t^{4/3}, \qquad\qquad (5.2.9)\\
\zeta &= -\frac{3c_3}{k_1}t^{1/3}u + c_3x - c_3A\log t + c_5, \quad c_3 \neq 0.
\end{aligned}
$$

Subcase 2: Let $a(t) = k_1e^{mt}$, and on solving the equations (5.2.4)-(5.2.6), we get the forms of variable coefficients as follows:

$$
\begin{aligned}
b(t) &= e^{\frac{m}{2}t},\\
c(t) &= -\frac{1}{k_1}e^{-mt}, \qquad\qquad (5.2.10)\\
d(t) &= 3,
\end{aligned}
$$

and the associated infinitesimals are

$$
\begin{aligned}
\xi &= \frac{c_2}{4}x + c_6 - \frac{c_5}{mk_1}e^{-mt},\\
\eta &= \frac{c_2}{2m}, \qquad\qquad (5.2.11)\\
\zeta &= \frac{3c_2}{4}u + c_5.
\end{aligned}
$$

Subcase 3: Let $a(t) = \frac{1}{k_1 t + k_2}$, and on solving the equations (5.2.4)-(5.2.6), we get the forms of variable coefficients as follows:

$$\begin{aligned} b(t) &= A(k_1 t + k_2)^{-\frac{3}{2}}, \\ c(t) &= -(k_1 t + k_2),\ d(t) = 3, \end{aligned} \tag{5.2.12}$$

therefore, the infinitesimals are

$$\begin{aligned} \xi &= -\frac{c_1 k_1}{2k_2^2} x + c_6 + c_5 \left(\frac{k_1 t^2}{2} + k_2 t \right), \\ \eta &= \frac{c_1 k_1 (k_1 t^2 + 2k_2 t) + c_1 k_2^2}{k_2^2 (k_1 t + k_2)}, \\ \zeta &= -\frac{5 c_1 k_1}{2k_2^2} u + c_5. \end{aligned} \tag{5.2.13}$$

5.3 Similarity reductions and Exact solutions

In this section, we establish the similarity reductions and exact solutions of the equation (5.1.2) for each considered case in the previous section. The similarity variables are obtained by solving the characteristic equations

$$\frac{dx}{\xi} = \frac{dt}{\eta} = \frac{du}{\zeta}, \tag{5.3.1}$$

where ξ, η, ζ are given by equations (5.2.2), (5.2.9), (5.2.11), and (5.2.13).
On solving the characteristic equations (5.3.1), we get the similarity variables corresponding to infinitesimals (5.2.2) as

$$\begin{aligned} \beta &= t, \\ u &= \frac{c_1 x}{\int \frac{c_1}{a(t)} dt + c_2} + F(\beta) \end{aligned} \tag{5.3.2}$$

and equation (5.1.2) reduces to an ordinary differential equation

$$F'(\beta) + \frac{c_1}{a(\beta) \int \frac{c_1}{a(\beta)} d\beta + c_2} F(\beta) + \frac{c_1 b(\beta)}{\int \frac{c_1}{a(\beta)} d\beta + c_2} = 0. \tag{5.3.3}$$

On solving this equation, we have

$$F(\beta) = -\frac{c_1 \int b(\beta) d\beta}{\int \frac{c_1}{a(\beta)} d\beta + c_2} + \frac{c_3}{\int \frac{c_1}{a(\beta)} d\beta + c_2}, \tag{5.3.4}$$

where c_1, c_2, c_3 are arbitrary constants.

Reverting back to original variables x, t, the exact solution of equation (5.1.2) is

$$u(x,t) = \frac{c_1 x}{\int \frac{c_1}{a(t)}dt + c_2} - \frac{c_1 \int b(t)dt}{\int \frac{c_1}{a(t)}dt + c_2} + \frac{c_3}{\int \frac{c_1}{a(t)}dt + c_2}. \tag{5.3.5}$$

Case 1: Herein, we are taking the following particular subcases for the infinitesimals (5.2.9).

Subcase 1: Similarity variable corresponding to parameter c_3 is

$$\begin{aligned} \theta &= xt^{-1/3} - At^{-1/3}\log t, \\ u &= \frac{k_1 A}{3}\big(xt^{-1/3} - t^{-1/3}\log t\big) + t^{-1/3}F(\theta). \end{aligned} \tag{5.3.6}$$

Using equations (5.3.6) in equation (5.1.2), we get the reduced ODE as follows:

$$FF^{'} - 3k_1 F^{'}F^{''} + k_1^2 F^{'''} - k_1 FF^{'''} = 0. \tag{5.3.7}$$

In order to solve this ODE, we consider the following transformation

$$F^{'} = U, \tag{5.3.8}$$

thus equation(5.3.7), reduces to second-order ODE as follow:

$$k_1(k_1 - \theta)(UU^{''} + (U^{'})^2) - 3k_1 UU^{'} + \theta = 0. \tag{5.3.9}$$

On solving this ODE with the help of software MAPLE, we obtain

$$\frac{dF}{d\theta} = \pm \frac{\sqrt{3}\sqrt{k_1(12c_7k_1^3 - 24c_7k_1^2\theta + 12c_7k_1\theta^2 - 4k_1\theta^3 + 3\theta^4 - 12c_8k_1)}}{6k_1(k_1 - \theta)}. \tag{5.3.10}$$

From this equation, we can workout $F(\theta)$ by integration.

Subcase 2: Similarity variables corresponding to parameters c_3, c_5, c_6 are given by

$$\begin{aligned} \theta &= xt^{-1/3} + \frac{c_6 k_1}{6c_3}t^{-2/3} + \frac{c_5}{c_3}t^{-1/3} - At^{-1/3}\log t, \\ u &= \frac{k_1 x}{3}t^{-1/3} + \frac{c_6 k_1^2}{9c_3}t^{-2/3} + \frac{k_1 c_5}{3c_3}t^{-1/3} - \frac{Ak_1}{3}t^{-1/3}\log t + yt^{-1/3}F(\theta) \end{aligned} \tag{5.3.11}$$

Using this equation in equation (5.1.2), we get the reduced ordinary differential equation as follows:

$$-FF^{'''} + Ak_1F^{'''} - 3F^{'}F^{''} + \frac{1}{k_1}FF^{'} - \frac{c_6k_1^2}{27c_3} = 0. \tag{5.3.12}$$

Software MAPLE, reduces this equation to the second order ODE

$$Ak_1F^{''} - FF^{''} - (F^{'})^2 + \frac{1}{2k_1}F^2 - \frac{c_6k_1^2}{27c_3}\theta + B = 0, \tag{5.3.13}$$

where B is an arbitrary constant.

The ODE (5.3.13) is further analyzed for its invariance analysis using the Group method, and we found that it admits the trivial symmetries only. Therefore, we adopt the power series method to construct the further solutions of equation (5.3.13) in the next section.

Case 2: In this case, we give the similarity variables corresponding to symmetries (5.2.11).

Subcase 1: Similarity variables corresponding to parameter c_5 are

$$\begin{aligned} \theta &= t, \\ u &= -mk_1xe^{mt} + F(\theta), \end{aligned} \tag{5.3.14}$$

and equation (5.1.2) reduces to

$$F^{'}(\theta) - mF(\theta) - mk_1e^{\frac{3m}{2}\theta} = 0, \tag{5.3.15}$$

which yields

$$F(\theta) = \alpha e^{m\theta} + 2k_1e^{\frac{3m}{2}\theta}, \tag{5.3.16}$$

where α is an arbitrary constant.

The exact solution of equation (5.1.2) is given by

$$u = -mk_1xe^{mt} + \alpha e^{mt} + 2k_1e^{\frac{3m}{2}t}. \tag{5.3.17}$$

Subcase 2: Similarity variables corresponding to parameter c_2 are

$$\begin{aligned} \theta &= xe^{-\frac{m}{2}t}, \\ u &= e^{\frac{3m}{2}t}F(\theta), \end{aligned} \tag{5.3.18}$$

and equation (5.1.2) reduces to an ODE

$$\frac{3m}{2}F - \frac{m}{2}\theta F^{'} - \frac{mk_1}{2}F^{''} + \frac{k_1m}{2}\theta F^{'''} + F^{'} - FF^{'''} + \frac{1}{k_1}FF^{'} - 3F^{'}F^{''} = 0. \tag{5.3.19}$$

Since this ODE also admits only the trivial symmetries, therefore, for the sake of exact solutions, we apply the power series method on this ODE.

Case 3: In this case, we present the similarity variables corresponding to symmetries (5.2.13).

Subcase 1: Similarity variables corresponding to parameter c_5 are

$$\begin{aligned} \theta &= t, \\ u &= \frac{2x}{k_1t^2 + 2k_2t} + F(\theta), \end{aligned} \tag{5.3.20}$$

and equation (5.1.2) reduces to

$$F^{'}(\theta) - \frac{2(k_1\theta + k_2)}{(k_1\theta^2 + 2k_2\theta)}F(\theta) + \frac{2(k_1\theta + k_2)^{-3/2}}{(k_1\theta^2 + 2k_2\theta)} = 0,$$

which yields

$$F(\theta) = \frac{4(k_1\theta + k_2)^{-1/2}}{k_1\theta(k_1\theta + 2k_2)} + \frac{\beta}{\theta(k_1\theta + 2k_2)}, \tag{5.3.21}$$

where β is an arbitrary constant.

The exact solution of equation (5.1.2) is given by

$$u = \frac{2x}{k_1t^2 + 2k_2t} + \frac{4(k_1t + k_2)^{-1/2}}{k_1t(k_1t + 2k_2)} + \frac{\beta}{t(k_1t + 2k_2)}. \tag{5.3.22}$$

Subcase 2: Similarity variables corresponding to parameter c_1 are

$$\begin{aligned} \theta &= x(k_1t + k_2)^{1/2}, \\ u &= (k_1t + k_2)^{-5/2}F(\theta), \end{aligned} \tag{5.3.23}$$

and equation (5.1.2) reduces to an ODE

$$-\frac{5k_1}{2}F + \frac{k_1}{2}\theta F^{'} + \frac{3k_1}{2}F^{''} - \frac{k_1}{2}\theta F^{'''} + AF^{'} - FF^{'''} + FF^{'} - 3F^{'}F^{''} = 0. \tag{5.3.24}$$

Since the non-trivial symmetries could not be found out, we apply the power series method on this ODE and present the power series solution in the next section.

5.4 Power series solutions

In this section, we will present the exact series solutions of the reduced ODEs (5.3.13), (5.3.19), and (5.3.24) by utilizing the power series method.

Case 1: Consider the power series solution of equation (5.3.13) in the form

$$F=\sum_{n=0}^{\infty}a_n\theta^n, \tag{5.4.1}$$

substituting the equation (5.4.1) into (5.3.13), we have

$$\begin{aligned} &Ak_1\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}\theta^n-\sum_{n=0}^{\infty}a_n\theta^n\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}\theta^n-\frac{c_6k_1^2}{27c_3}\theta\\ &-\left(\sum_{n=0}^{\infty}(n+1)a_{n+1}\theta^n\right)^2+\frac{1}{2k_1}\left(\sum_{n=0}^{\infty}a_n\theta^n\right)^2+B=0, \end{aligned} \tag{5.4.2}$$

which is further equivalent to

$$\begin{aligned} &Ak_1\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}\theta^n-\sum_{n=0}^{\infty}\sum_{i=0}^{n}(n-i+1)(n-i+2)a_ia_{n-i+2}\theta^n-\frac{c_6k_1^2}{27c_3}\theta\\ &-\sum_{n=0}^{\infty}\sum_{i=0}^{n}(i+1)(n-i+1)a_{i+1}a_{n-i+1}\theta^n+\frac{1}{2k_1}\sum_{n=0}^{\infty}\sum_{i=0}^{n}a_ia_{n-i}\theta^n+B=0, \end{aligned} \tag{5.4.3}$$

Next, on comparing the coefficients of $\theta^n, n=0,1$, we have

$$a_2=\frac{1}{2Ak_1-2a_0}\left(a_1^2-\frac{1}{2k_1}a_0^2-B\right)=0, \tag{5.4.4}$$

$$a_3=\frac{1}{6Ak_1-6a_0}\left(6a_1a_2-\frac{a_0a_1}{k_1}+\frac{c_6k_1^2}{27c_3}\right)=0, \tag{5.4.5}$$

where a_0, a_1 are arbitrary constants.

For $n\geq 2$, we have the general recurrence relation as follows:

$$\begin{aligned} a_{n+2}=&\frac{1}{(n+1)(n+2)(Ak_1-a_0)}\left[\sum_{i=0}^{n}(i+1)(n-i+1)a_{i+1}a_{n-i+1}-\frac{1}{2k_1}\sum_{i=0}^{n}a_ia_{n-i}\right.\\ &\left.+\sum_{i=1}^{n}(n-i+2)(n-i+1)a_ia_{n-i+2}\right]=0. \end{aligned} \tag{5.4.6}$$

Therefore, it implies that there exists a power series solution of equation (5.3.13). Next, we have to prove the convergence of the series. Let

$$|a_{n+2}|\leq M\left[\sum_{i=0}^{n}|a_i||a_{n-i}|+\sum_{i=0}^{n}|a_{i+1}||a_{n-i+1}|+\sum_{i=1}^{n}|a_i||a_{n-i+2}|\right], \tag{5.4.7}$$

where M=max$\left(\frac{1}{|2k_1(Ak_1-a_0)|},\frac{1}{|Ak_1-a_0|}\right)$.

Let us define the series

$$\mu=P(\theta)=\sum_{n=0}^{\infty}p_n\theta^n, \tag{5.4.8}$$

$p_0 = |a_0| = |\delta|, p_1 = |a_1| = |\lambda|, p_2 = |a_2|$ and so on. Therefore,

$$p_{n+2} = M\left[\sum_{i=0}^{n} p_i p_{n-i} + \sum_{i=0}^{n} p_{i+1} p_{n-i+1} + \sum_{i=1}^{n} p_i p_{n-i+2}\right], n = 2, 3, \ldots \tag{5.4.9}$$

Hence, this can be easily seen that $|a_n| \leq p_n, n = 0, 1, 2, \ldots$ or in other words, we can say that the series (5.4.8) is a dominant series of series (5.4.1). Next, we have to prove that the series (5.4.8) has positive radius of convergence.

$$\begin{aligned} p(\theta) &= p_0 + p_1\theta + p_2\theta^2 + p_3\theta^3 + \sum_{n=2}^{\infty} p_{n+2}\theta^{n+2}, \\ &= p_0 + p_1\theta + p_2\theta^2 + p_3\theta^3 + M\left[\sum_{n=2}^{\infty}\sum_{i=0}^{n} p_i p_{n-i}\theta^{n+2} + \sum_{n=2}^{\infty}\sum_{i=0}^{n} p_{i+1}p_{n-i+1}\theta^{n+2}\right. \\ &\quad \left. + \sum_{n=2}^{\infty}\sum_{i=1}^{n} p_i p_{n-i+2}\theta^{n+2}\right], \end{aligned} \tag{5.4.10}$$

Further, by some mathematical calculations, we can deduce that

$$\begin{aligned} p(\theta) &= p_0 + p_1\theta + p_2\theta^2 + p_3\theta^3 + M[2P^2 + \theta^2 P^2 - (p_1\theta + 4p_0)P - 2p_0^2 \\ &\quad - p_0 p_1\theta - p_1^2\theta^2 - (3p_1p_2 + 2p_0p_1)\theta^3]. \end{aligned} \tag{5.4.11}$$

Next, consider the implicit functional equation [99] as follows:

$$\begin{aligned} G(\theta, \mu) &= \mu - p_0 - p_1\theta - p_2\theta^2 - p_3\theta^3 - M[2\mu^2 + \theta^2\mu^2 - (p_1\theta + 4p_0)\mu - 2p_0^2 \\ &\quad - p_0p_1\theta - p_1^2\theta^2 - (3p_1p_2 + 2p_0p_1)\theta^3]. \end{aligned} \tag{5.4.12}$$

Since G is analytic in the (θ, μ)-plane and $G(\theta, p_0) = 0$ and

$$G^{'}(0, p_0) = 1 \neq 0. \tag{5.4.13}$$

Thus, we see that $\mu = P(\theta)$ is analytic in the neighborhood of the point $(0, p_0)$ of the plane and has a positive radius of convergence. Therefore, we conclude that the series (5.4.1) converges in the neighborhood of the point $(0, p_0)$ of the plane. Hence, the proof is completed.

The power series solution of (5.3.13) can be written as

$$\begin{aligned}
F(\theta) &= a_0 + a_1\theta + a_2\theta^2 + a_3\theta^3 + \sum_{n=2}^{\infty} a_{n+2}\theta^{n+2},\\
&= a_0 + a_1\theta + \frac{1}{2(Ak_1-a_0)}\Big(a_1^2 - \frac{1}{2k_1}a_0^2 - B\Big)\theta^2 + \frac{1}{6(Ak_1-a_0)}\Big(6a_1a_2 - \frac{a_0a_1}{k_1} + \frac{c_6k_1^2}{27c_3}\Big)\theta^3\\
&\quad + \sum_{n=2}^{\infty}\frac{1}{(n+1)(n+2)(Ak_1-a_0)}\Bigg[\sum_{i=0}^{n}(i+1)(n-i+1)a_{i+1}a_{n-i+1} - \frac{1}{2k_1}\sum_{i=0}^{n}a_ia_{n-i}\\
&\quad + \sum_{i=1}^{n}(n-i+2)(n-i+1)a_ia_{n-i+2}\Bigg]\theta^{n+2}, \qquad (5.4.14)
\end{aligned}$$

where a_2 and a_3 are given in equations (5.4.4) and (5.4.5) respectively. The exact explicit solution can be obtained by substituting the equations (5.4.14) in equations (5.3.11).

Case 2: Substituting equation (5.4.1) in equation (5.3.19), we have

$$\begin{aligned}
&\frac{3m}{2}\sum_{n=0}^{\infty}a_n\theta^n - \frac{m\theta}{2}\sum_{n=1}^{\infty}na_n\theta^{n-1} - \frac{mk_1}{2}\sum_{n=2}^{\infty}n(n-1)a_n\theta^{n-2} + \frac{k_1m\theta}{2}\sum_{n=3}^{\infty}n(n-1)(n-2)a_n\theta^{n-3}\\
&+\sum_{n=1}^{\infty}na_n\theta^{n-1} - \sum_{n=0}^{\infty}a_n\theta^n\sum_{n=3}^{\infty}n(n-1)(n-2)a_n\theta^{n-3} + \frac{1}{k_1}\sum_{n=0}^{\infty}a_n\theta^n\sum_{n=1}^{\infty}na_n\theta^{n-1}\\
&-3\sum_{n=1}^{\infty}na_n\theta^{n-1}\sum_{n=2}^{\infty}n(n-1)a_n\theta^{n-2} = 0, \qquad (5.4.15)
\end{aligned}$$

which is further equivalent to

$$\begin{aligned}
&\frac{3m}{2}a_0 + \frac{3m}{2}\sum_{n=1}^{\infty}a_n\theta^n - mk_1a_2 - \frac{m}{2}\sum_{n=1}^{\infty}na_n\theta^n - \frac{mk_1}{2}\sum_{n=1}^{\infty}(n+1)(n+2)a_{n+2}\theta^n\\
&+a_1 + \frac{mk_1}{2}\sum_{n=1}^{\infty}n(n+1)(n+2)a_{n+2}\theta^n + \sum_{n=1}^{\infty}(n+1)a_{n+1}\theta^n - 6a_0a_3 + \frac{a_0a_1}{k_1} - 6a_1a_2\\
&-\sum_{n=1}^{\infty}\sum_{i=0}^{n}(n-i+1)(n-i+2)(n-i+3)a_ia_{n-i+3}\theta^n + \frac{1}{k_1}\sum_{n=1}^{\infty}\sum_{i=0}^{n}(n-i+1)a_ia_{n-i+1}\theta^n\\
&-3\sum_{n=1}^{\infty}\sum_{i=0}^{n}(n-i+1)(n-i+2)(i+1)a_{i+1}a_{n-i+2}\theta^n = 0. \qquad (5.4.16)
\end{aligned}$$

Next, on comparing the coefficients of θ^n, $n=0$, we get

$$a_3 = \frac{1}{6a_0}\Big(\frac{3m}{2}a_0 - mk_1a_2 + a_1 + \frac{a_0a_1}{k_1} - 6a_1a_2\Big). \qquad (5.4.17)$$

For n$\geq$ 1, the general recurrence relation is as follows:

$$
\begin{aligned}
a_{n+3} = & \frac{1}{(n+1)(n+2)(n+3)a_0}\Big(\frac{3m}{2}a_n - \frac{m}{2}na_n + (n+1)a_{n+1} \\
& - \frac{mk_1}{2}(n+1)(n+2)a_{n+2} + \frac{mk_1}{2}n(n+1)(n+2)a_{n+2} \\
& - \sum_{i=1}^{n}(n-i+1)(n-i+2)(n-i+3)a_i a_{n-i+3} + \frac{1}{k_1}\sum_{i=0}^{n}(n-i+1)a_i a_{n-i+1} \\
& - 3\sum_{i=0}^{n}(n-i+1)(n-i+2)(i+1)a_{i+1}a_{n-i+2}\Big). \qquad (5.4.18)
\end{aligned}
$$

Therefore, it implies that there exists a power series solution of the equation (5.3.19). Thus, the power series solution of equation (5.3.19) can be expressed as:

$$
\begin{aligned}
F(\theta) = & a_0 + a_1\theta + a_2\theta^2 + a_3\theta^3 + \sum_{n=1}^{\infty} a_{n+3}\theta^{n+3} \\
= & a_0 + a_1\theta + a_2\theta^2 + \frac{1}{6a_0}\Big(\frac{3m}{2}a_0 - mk_1a_2 + a_1 + \frac{a_0a_1}{k_1} - 6a_1a_2\Big)\theta^3 \\
& + \sum_{n=1}^{\infty}\Bigg(\frac{1}{(n+1)(n+2)(n+3)a_0}\Big(\frac{3m}{2}a_n - \frac{m}{2}a_n + (n+1)a_{n+1} \\
& - \frac{mk_1}{2}(n+1)(n+2)a_{n+2} - 6a_1a_2 + \frac{mk_1}{2}n(n+1)(n+2)a_{n+2} \\
& - \sum_{i=1}^{n}(n-i+1)(n-i+2)(n-i+3)a_i a_{n-i+3} + \frac{1}{k_1}\sum_{i=0}^{n}(n-i+1)a_i a_{n-i+1} \\
& - \sum_{i=0}^{n}(n-i+1)(n-i+2)(i+1)a_{i+1}a_{n-i+2}\Big)\Bigg)\theta^4. \qquad (5.4.19)
\end{aligned}
$$

Case 3: Using equation (5.4.1) in equation (5.3.24), we have

$$
\begin{aligned}
& -\frac{5k_1}{2}\sum_{n=0}^{\infty}a_n\theta^n + \frac{k_1\theta}{2}\sum_{n=1}^{\infty}na_n\theta^{n-1} + \frac{3k_1}{2}\sum_{n=2}^{\infty}n(n-1)a_n\theta^{n-2} - \frac{k_1\theta}{2}\sum_{n=3}^{\infty}n(n-1)(n-2)a_n\theta^{n-3} \\
& + A\sum_{n=1}^{\infty}na_n\theta^{n-1} - \sum_{n=0}^{\infty}a_n\theta^n\sum_{n=3}^{\infty}n(n-1)(n-2)a_n\theta^{n-3} + \sum_{n=0}^{\infty}a_n\theta^n\sum_{n=1}^{\infty}na_n\theta^{n-1} \\
& - 3\sum_{n=1}^{\infty}na_n\theta^{n-1}\sum_{n=2}^{\infty}n(n-1)a_n\theta^{n-2} = 0, \qquad (5.4.20)
\end{aligned}
$$

which is further equivalent to

$$
\begin{aligned}
&-\frac{5k_1}{2}a_0-\frac{5k_1}{2}\sum_{n=1}^{\infty}a_n\theta^n+Aa_1+3k_1a_2+\frac{k_1}{2}\sum_{n=1}^{\infty}na_n\theta^n+\frac{3k_1}{2}\sum_{n=1}^{\infty}(n+1)(n+2)a_{n+2}\theta^n\\
&+a_0a_1-\frac{k_1}{2}\sum_{n=1}^{\infty}n(n+1)(n+2)a_{n+2}\theta^n+A\sum_{n=1}^{\infty}(n+1)a_{n+1}\theta^n-6a_0a_3-6a_1a_2\\
&-\sum_{n=1}^{\infty}\sum_{i=0}^{n}(n-i+1)(n-i+2)(n-i+3)a_ia_{n-i+3}\theta^n+\sum_{n=1}^{\infty}\sum_{i=0}^{n}(n-i+1)a_ia_{n-i+1}\theta^n\\
&-3\sum_{n=1}^{\infty}\sum_{i=0}^{n}(n-i+1)(n-i+2)(i+1)a_{i+1}a_{n-i+2}\theta^n=0. \qquad (5.4.21)
\end{aligned}
$$

Next, on comparing the coefficients of θ^n, $n=0$, we get

$$
a_3=\frac{1}{6a_0}\Big(-\frac{5k_1a_0}{2}-3k_1a_2+Aa_1-a_0a_2-6a_1a_2\Big). \qquad (5.4.22)
$$

For n$\geq$ 1, we obtained the general recurrence relation as follows:

$$
\begin{aligned}
a_{n+3}=\frac{1}{(n+1)(n+2)(n+3)a_0}\Bigg(&-\frac{5k_1}{2}a_n-\frac{k_1}{2}na_n+A(n+1)a_{n+1}\\
&+\frac{3k_1}{2}(n+1)(n+2)a_{n+2}-\frac{k_1}{2}n(n+1)(n+2)a_{n+2}\\
&-\sum_{i=1}^{n}(n-i+1)(n-i+2)(n-i+3)a_ia_{n-i+3}+\sum_{i=0}^{n}(n-i+1)a_ia_{n-i+1}\\
&-3\sum_{i=0}^{n}(n-i+1)(n-i+2)(i+1)a_{i+1}a_{n-i+2}\Bigg). \qquad (5.4.23)
\end{aligned}
$$

Therefore, it implies that there exists a power series solution of the equation (5.3.24). Hence, the power series solution of equation (5.3.24) is as follows

$$
\begin{aligned}
F(\theta)&=a_0+a_1\theta+a_2\theta^2+a_3\theta^3+\sum_{n=1}^{\infty}a_{n+3}\theta^{n+3}\\
&=a_0+a_1\theta+a_2\theta^2+\frac{1}{6a_0}\Big(-\frac{5k_1a_0}{2}-3k_1a_2+Aa_1-a_0a_2-6a_1a_2\Big)\theta^3\\
&+\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)(n+3)a_0}\Bigg[-\frac{5k_1}{2}a_n-\frac{k_1}{2}na_n+A(n+1)a_{n+1}\\
&+\frac{3k_1}{2}(n+1)(n+2)a_{n+2}-\frac{k_1}{2}n(n+1)(n+2)a_{n+2}\\
&-\sum_{i=1}^{n}(n-i+1)(n-i+2)(n-i+3)a_ia_{n-i+3}+\sum_{i=0}^{n}(n-i+1)a_ia_{n-i+1}\\
&-3\sum_{i=0}^{n}(n-i+1)(n-i+2)(i+1)a_{i+1}a_{n-i+2}\Bigg]\theta^4. \qquad (5.4.24)
\end{aligned}
$$

The convergence of the series solutions in *case 2* and *case 3* can be proved in the same way as shown in *case 1*.

5.5 Discussion

In this chapter, the Gilson-Pickering equation with time-dependent variable coefficients has been analyzed from a group theoretic point of view. This method is feasible and efficient for the analysis of nonlinear partial differential equations. Taking special forms of variable coefficients, we constructed the infinitesimals of the group of transformations which leaves the time-dependent variable coefficients Gilson-Pickering equation invariant and obtained the particular solutions. Corresponding to the Lie group infinitesimals, it is shown that the equation (5.1.2) reduces to nonlinear ordinary differential equations in each case, which is further studied with the aim of deriving certain closed-form solutions. The reduced ordinary differential equations admitting trivial symmetries have been further analyzed by using the power series method and provided the exact power series solutions. We also established the convergence of the series solutions reported in the work.

Chapter 6

Exact traveling wave solutions of BLP equation

6.1 Introduction

This chapter is devoted to the study of the (2+1) dimensional Boiti-Leon-Pempinelli equation [61]

$$\begin{aligned} u_{ty} &= (u^2 - u_x)_{xy} + 2v_{xxx}, \\ v_t &= v_{xx} + 2uv_x, \end{aligned} \tag{6.1.1}$$

which describes the evolution of the horizontal velocity component of the water waves propagating in xy-plane in an infinite narrow channel of constant depth. The (2+1)-dimensional Boiti-Leon-Pempinelli equation is a generalization of the dispersive long wave equation introduced by Boiti et al. [61]. Additionally, various interesting properties of equation (6.1.1) have been studied by many authors with the help of different methods such as [10], [14], [20], [48], [106], [112] etc.

The organization of the chapter is as follows: The exact traveling wave solutions of the equations (6.1.1) using the (G'/G^2)-expansion method are presented in section 6.2. In section 6.3, traveling wave solutions are furnished by utilizing the first integral method. The concluding remarks are presented in the last section .

6.2 Exact traveling wave solutions

Herein, we will apply the $\left(\frac{G'}{G^2}\right)$-expansion method to obtain the traveling wave solutions of system (6.1.1).
Consider the following transformations

$$v(x,y,t) = V(\zeta), u(x,y,t) = U(\zeta), \zeta = lx + my + nt, \tag{6.2.1}$$

where l, m, n are arbitrary constants. Using equations (6.2.1) into equations (6.1.1), we get

$$\begin{aligned} mnU^{''}(\zeta) &= lm(U^2(\zeta) - lU^{'}(\zeta))^{''} + 2l^3V^{'''}(\zeta), \\ nV^{'}(\zeta) &= l^2V^{''}(\zeta) + 2lV^{'}(\zeta). \end{aligned} \tag{6.2.2}$$

Integrating the first equation of (6.2.2) twice w.r.t ζ, we have

$$mnU(\zeta) = lm(U^2(\zeta) - lU^{'}(\zeta)) + 2l^3V^{'}(\zeta), \tag{6.2.3}$$

and taking all the arbitrary integration constants equal to zero, we get

$$V^{'}(\zeta) = \frac{mn}{2l^3}U(\zeta) - \frac{m}{2l^2}(U^2(\zeta) - lU^{'}(\zeta)). \tag{6.2.4}$$

Substituting equation (6.2.4) into the second equation of (6.2.2), we get

$$l^4U^{''}(\zeta) - 2l^2U^3(\zeta) + 3nlU^2(\zeta) - n^2U(\zeta) = 0. \tag{6.2.5}$$

Next, on balancing the nonlinear term and the highest order derivative term present in equation (6.2.5), we obtain

$$deg[U^{''}(\zeta)] = M + 2 = deg[U^3(\zeta)] = 3M, \tag{6.2.6}$$

which implies that $M = 1$. Thus, the exact solution of equation (6.2.5) can be written as

$$U(\zeta) = a_0 + a_1\left(\frac{G'}{G^2}\right) + b_1\left(\frac{G'}{G^2}\right)^{-1}, \tag{6.2.7}$$

where $a_0,\ a_1,\ b_1$ are unknown parameters.
Now, substituting the equation (6.2.7) into equation (6.2.5) alongwith equation (1.7.6),

and collecting the coefficients with same power of $\left(\frac{G'}{G^2}\right)^i, (i = 0, \pm1, \pm2, \pm3)$, and equating them to zero, we have a system of algebraic equations as follows:

$$
\begin{aligned}
2b_1\mu\lambda l^4 - 6a_0^2b_1l^2 - 6a_1b_1^2l^2 + 6a_0b_1nl - n^2b_1 &= 0,\\
-2l^2a_0^3 - 12a_0a_1b_1l^2 + 3nla_0^2 + 6a_1b_1nl - n^2a_0 &= 0,\\
2a_1\mu\lambda l^4 - 6a_0^2a_1l^2 - 6a_1^2b_1l^2 + 6a_0a_1nl - n^2a_1 &= 0,\\
2a_1\lambda^2l^4 - 2l^2a_1^3 &= 0,\\
-6a_0b_1^2l^2 + 3nlb_1^2 &= 0,\\
-6a_0a_1^2l^2 + 3nla_1^2 &= 0,\\
2b_1\mu^2l^4 - 2l^2b_1^3 &= 0.
\end{aligned} \tag{6.2.8}
$$

On solving the above system of nonlinear algebraic equations, we come across the following cases:

Case 1:

$$
\begin{aligned}
a_0 &= \frac{n}{2l},\\
a_1 &= \pm\lambda l,\\
b_1 &= \pm\Big(\frac{n^2}{12\lambda l^3} + \frac{\mu l}{3}\Big),\\
n &= \pm\sqrt{-4\lambda\mu l^4 \pm 12\lambda\mu l^4},
\end{aligned} \tag{6.2.9}
$$

Case 2:

$$
\begin{aligned}
a_1 &= 0,\ \ b_1 \neq 0,\\
a_0 &= \frac{n}{2l},\ \ l \neq 0,\\
b_1 &= \pm\mu l,\\
n &= \pm\sqrt{-4\mu\lambda l^4},
\end{aligned} \tag{6.2.10}
$$

Case 3: when $a_1 \neq 0, b_1 \neq 0$

$$\begin{aligned}
a_0 &= \frac{n}{2l}, \\
b_1 &= \pm\mu l, \\
a_1 &= \pm\Big(\frac{n^2}{12\mu l^3} + \frac{\lambda l}{3}\Big), \\
n &= \pm\sqrt{-4\lambda\mu l^4 \pm 12\lambda\mu l^4},
\end{aligned} \tag{6.2.11}$$

Case 4:

$$\begin{aligned}
b_1 &= 0, \ \ a_1 \neq 0, \\
a_0 &= \frac{n}{2l}, \ \ l \neq 0, \\
a_1 &= \pm\lambda l, \\
n &= \pm\sqrt{-4\mu\lambda l^4},
\end{aligned} \tag{6.2.12}$$

where $l, \ \mu, \ \lambda$ are arbitrary constants.

Now, substituting the equations (6.2.9)-(6.2.12) alongwith the functions $\left(\frac{G'}{G^2}\right)$ given in equations (1.7.7)-(1.7.10) into equation (6.2.7), we obtain the exact solutions of equation (6.1.1) as follows:

Set 1: Substituting the values of the parameters presented in *Case 1* alongwith the transformation $\zeta = lx + my \pm (\sqrt{-4\lambda\mu l^4 \pm 12\lambda\mu l^4})t$ in equation (6.2.7), we obtain the exact traveling wave solutions as under:

(i) For $\mu\lambda > 0$, the traveling wave solution of equation (6.1.1) is given in terms of trigonometric function by

$$\begin{aligned}
u &= \frac{n}{2l} \pm \lambda l\left[\sqrt{\frac{\mu}{\lambda}}\Big(\frac{Acos(\sqrt{\mu\lambda})\zeta + Bsin(\sqrt{\mu\lambda})\zeta}{(Bcos(\sqrt{\mu\lambda})\zeta - Asin\sqrt{\mu\lambda})\zeta}\Big)\right] \\
&\quad \pm \Big(\frac{n^2}{12\lambda l^3} + \frac{\mu l}{3}\Big)\left[\sqrt{\frac{\mu}{\lambda}}\Big(\frac{Acos(\sqrt{\mu\lambda})\zeta + Bsin(\sqrt{\mu\lambda})\zeta}{Bcos(\sqrt{\mu\lambda})\zeta - Asin(\sqrt{\mu\lambda})\zeta}\Big)\right]^{-1}, \\
v &= \int \Big(\frac{mn}{2l^3}u(\zeta) - \frac{m}{2l^2}(u^2(\zeta) - lu'(\zeta))\Big)d\zeta.
\end{aligned} \tag{6.2.13}$$

(ii) For $\mu\lambda < 0$, the traveling wave solution of equation (6.1.1) is given by

$$u = \frac{n}{2l} \pm \frac{l}{2}\left(2\sqrt{|\mu\lambda|} - \frac{4A\sqrt{|\mu\lambda|}e^{2\zeta\sqrt{|\mu\lambda|}}}{Ae^{2\zeta\sqrt{|\mu\lambda|}} - B}\right)$$
$$\pm\left(\frac{n^2}{12\lambda l^3} + \frac{\mu l}{3}\right)\left[\frac{1}{2\lambda}\left(2\sqrt{|\mu\lambda|} - \frac{4A\sqrt{|\mu\lambda|}e^{2\zeta\sqrt{|\mu\lambda|}}}{Ae^{2\zeta\sqrt{|\mu\lambda|}} - B}\right)\right]^{-1},$$
$$v = \int \left(\frac{mn}{2l^3}u(\zeta) - \frac{m}{2l^2}(u^2(\zeta) - lu'(\zeta))\right)d\zeta. \tag{6.2.14}$$

(iii) For $\lambda \neq 0, \mu = 0$, the traveling wave solution is

$$u = \frac{n}{2l} \mp \left(\frac{Al}{(A\zeta + B)}\right) \pm \left(\frac{n^2}{12\lambda l^3} + \frac{\mu l}{3}\right)\left(\frac{-Al}{(A\zeta + B)}\right)^{-1},$$
$$v = \int \left(\frac{mn}{2l^3}u(\zeta) - \frac{m}{2l^2}(u^2(\zeta) - lu'(\zeta))\right)d\zeta. \tag{6.2.15}$$

Set 2: Substituting the values of the parameters provided in *Case 2* alongwith the transformation $\zeta = lx + my \pm (\sqrt{-4\mu\lambda l^4})t$ in equation (6.2.7), we get the traveling wave solutions as follows:

(i) For $\mu\lambda > 0$, the traveling wave solution of equation (6.1.1) is given in terms of trigonometric function by

$$u = \frac{n}{2l} \pm \mu l\left[\sqrt{\frac{\mu}{\lambda}}\left(\frac{Acos(\sqrt{\mu\lambda})\zeta + Bsin(\sqrt{\mu\lambda})\zeta}{Bcos(\sqrt{\mu\lambda})\zeta - Asin\sqrt{\mu\lambda})\zeta}\right)\right]^{-1}. \tag{6.2.16}$$

(ii) For $\mu\lambda < 0$, the traveling wave solution of equation (6.1.1) is given as

$$u = \frac{n}{2l} \pm \mu l\left[\frac{1}{2\lambda}\left(2\sqrt{|\mu\lambda|} - \frac{4A\sqrt{|\mu\lambda|}e^{2\zeta\sqrt{|\mu\lambda|}}}{Ae^{2\zeta\sqrt{|\mu\lambda|}} - B}\right)\right]^{-1}. \tag{6.2.17}$$

(iii) For $\lambda \neq 0, \mu = 0$, the traveling wave solution is given by

$$u = \frac{n}{2l} \pm \mu l\left(\frac{-A}{\lambda(A\zeta + B)}\right)^{-1}, \tag{6.2.18}$$

where v can be found by putting the values of u respectively, in equation (6.2.4).

Set 3: Inserting the values of the parameters given in *Case 3* alongwith $\zeta = lx + my \pm (\sqrt{-4\lambda\mu l^4 \pm 12\lambda\mu l^4})t$ in equation (6.2.7), we get the traveling wave solutions as follows:

(i) For $\mu\lambda > 0$, the exact solution of equation (6.1.1) is given in terms of trigonometric function by

$$u = \frac{n}{2l} \pm \Big(\frac{n^2}{12\mu l^3} + \frac{\lambda l}{3}\Big)\left[\sqrt{\frac{\mu}{\lambda}}\Big(\frac{Acos(\sqrt{\mu\lambda})\zeta + Bsin(\sqrt{\mu\lambda})\zeta}{Bcos(\sqrt{\mu\lambda})\zeta - Asin\sqrt{\mu\lambda})\zeta}\Big)\right]$$
$$\pm \mu l\left[\sqrt{\frac{\mu}{\lambda}}\Big(\frac{Acos(\sqrt{\mu\lambda})\zeta + Bsin(\sqrt{\mu\lambda})\zeta}{Bcos(\sqrt{\mu\lambda})\zeta - Asin\sqrt{\mu\lambda})\zeta}\Big)\right]^{-1}. \qquad (6.2.19)$$

(ii) For $\mu\lambda < 0$, the exact solution of equation (6.1.1) is given by

$$u = \frac{n}{2l} \pm \frac{1}{2\lambda}\Big(\frac{n^2}{12\mu l^3} + \frac{\lambda l}{3}\Big)\left(2\sqrt{|\mu\lambda|} - \frac{4A\sqrt{|\mu\lambda|}e^{2\zeta\sqrt{|\mu\lambda|}}}{Ae^{2\zeta\sqrt{|\mu\lambda|}} - B}\right)$$
$$\pm \mu l\left[\frac{1}{2\lambda}\left(2\sqrt{|\mu\lambda|} - \frac{4A\sqrt{|\mu\lambda|}e^{2\zeta\sqrt{|\mu\lambda|}}}{Ae^{2\zeta\sqrt{|\mu\lambda|}} - B}\right)\right]^{-1}. \qquad (6.2.20)$$

(iii) For $\lambda \neq 0, \mu = 0$, the exact solution is

$$u = \frac{n}{2l} \mp \Big(\frac{n^2}{12\mu l^3} + \frac{\lambda l}{3}\Big)\left(\frac{A}{\lambda(A\zeta + B)}\right) \pm \mu l\left(\frac{-A}{\lambda(A\zeta + B)}\right)^{-1}, \qquad (6.2.21)$$

where v can be calculated by putting the values of u respectively, in equation (6.2.4).

Set 4 : Substituting the values of the parameters given in *Case 4* and $\zeta = lx + my \pm (\sqrt{-4\mu\lambda l^4})t$ in equation (6.2.7), we have

(i) For $\mu\lambda > 0$, the exact solution of equation (6.1.1) is given in terms of trigonometric function by

$$u = \frac{n}{2l} \pm \lambda l\left[\sqrt{\frac{\mu}{\lambda}}\Big(\frac{Acos(\sqrt{\mu\lambda})\zeta + Bsin(\sqrt{\mu\lambda})\zeta}{Bcos(\sqrt{\mu\lambda})\zeta - Asin\sqrt{\mu\lambda})\zeta}\Big)\right]. \qquad (6.2.22)$$

(ii) For $\mu\lambda < 0$ the exact traveling wave solution of equation (6.1.1) is given by

$$u = \frac{n}{2l} \pm \lambda l\left[\frac{1}{2\lambda}\left(2\sqrt{|\mu\lambda|} - \frac{4A\sqrt{|\mu\lambda|}e^{2\zeta\sqrt{|\mu\lambda|}}}{Ae^{2\zeta\sqrt{|\mu\lambda|}} - B}\right)\right]. \qquad (6.2.23)$$

(iii) For $\lambda \neq 0, \mu = 0$, the traveling wave solution is given as

$$u = \frac{n}{2l} \mp \frac{lA}{(A\zeta + B)}, \qquad (6.2.24)$$

where v can be determined by inserting the values of u respectively, in equation (6.2.4).

6.3 Traveling wave solutions by first integral method

In this section, the exact solutions of equations (6.1.1) have been obtained by using the first integral method.

Applying the traveling wave transformations (6.2.1) on equation (6.1.1), we get equation (6.2.5). Next, by using the methodology of the first integral method, we get from equation (1.6.5),

$$\dot{X}(\zeta) = Y(\zeta),$$
$$\dot{Y}(\zeta) = \frac{2}{l^2}U^3(\zeta) - \frac{3n}{l^3}U^2(\zeta) + \frac{n^2}{l^4}U(\zeta). \tag{6.3.1}$$

As per the notion of the first integral method, we assume that, $X(\zeta)$ and $Y(\zeta)$ are the nontrivial solutions of equations (6.3.1), $q(X,Y) = \sum_{i=0}^{m} a_i(X)Y^i$ is an irreducible polynomial in the complex plane $\mathbb{C}[X,Y]$ such that

$$q(X(\zeta), Y(\zeta)) = \sum_{i=0}^{m} a_i(X)Y^i = 0, \tag{6.3.2}$$

where $a_i(X)$ $(i = 0, 1, 2, \ldots, m)$ are the polynomials of X and $a_m(X) \neq 0$. Hence, the equation (6.3.2) is known as the first integral to equations (6.3.1).

Case 1: Taking $m = 1$ in equation (6.3.2), and using the Division theorem, there exists a polynomial $g(X) + h(X)Y$ in the complex plane $\mathbb{C}[X,Y]$ such that

$$\frac{dq}{d\zeta} = \frac{\partial q}{\partial X}\frac{\partial X}{\partial \zeta} + \frac{\partial q}{\partial Y}\frac{\partial Y}{\partial \zeta} = (g(X) + h(X)Y)\sum_{i=0}^{1} a_i(X)Y^i. \tag{6.3.3}$$

Next, by comparing the coefficients of Y^i $(i = 0, 1)$, on both sides of the equation (6.3.3), we have

$$\dot{a}_1(X) = h(X)a_1(X), \tag{6.3.4}$$

$$\dot{a}_0(X) = g(X)a_1(X) + h(X)a_0(X), \tag{6.3.5}$$

$$a_1(X)\left[\frac{2}{l^2}X^3 - \frac{3n}{l^3}X^2 + \frac{n^2}{l^4}X\right] = g(X)a_0(X). \tag{6.3.6}$$

Since $a_i(X)$ $(i = 0, 1)$ are the polynomials, therefore, from equation (6.3.4) we can conclude that $h(X) = 0$, which implies $a_1(X)$ must be a constant. Further, for convenience let us consider $a_1(X) = 1$, then the equations (6.3.5) and (6.3.6) become

$$\dot{a}_0(X) = g(X), \tag{6.3.7}$$

$$\frac{2}{l^2}X^3 - \frac{3n}{l^3}X^2 + \frac{n^2}{l^4}X = g(X)a_0(X). \tag{6.3.8}$$

On balancing the degree of $a_0(X)$ and $g(X)$, we conclude that, the degree of $g(X)$ is 1 only. Thus, we take $g(X) = AX + B,\ A \neq 0$, and the equation (6.3.7) gives

$$a_0(X) = \frac{AX^2}{2} + BX + B_0, \tag{6.3.9}$$

where B_0 is an arbitrary integration constant.

Next, substituting the value of $g(X)$ and $a_0(x)$ into equation (6.3.8), equating all the coefficients of powers of X to zero, we obtain a set of nonlinear algebraic equations as

$$\begin{aligned} \frac{2}{l^2} - \frac{A^2}{2} &= 0,\\ -\frac{3n}{l^3} - \frac{3AB}{2} &= 0,\\ \frac{n^2}{l^4} - AB_0 - B^2 &= 0,\\ BB_0 &= 0. \end{aligned} \tag{6.3.10}$$

On solving it, we can easily get

$$A = \frac{2}{l}, B = \frac{-n}{l^2}, B_0 = 0, l \neq 0, \tag{6.3.11}$$

$$A = -\frac{2}{l}, B = \frac{n}{l^2}, B_0 = 0, l \neq 0. \tag{6.3.12}$$

Now, using equations (6.3.11) and (6.3.12) in equation (6.3.2), we have

$$Y(\zeta) = \frac{-X^2}{l} + \frac{n}{l^2}X,\ Y(\zeta) = \frac{X^2}{l} - \frac{n}{l^2}X. \tag{6.3.13}$$

Combining equation (6.3.13) with equation (6.3.1), we obtain

$$U(\zeta) = \frac{\frac{n}{l}\exp\left(\frac{n\zeta}{l^2} + \frac{n\zeta_o}{l}\right)}{\exp\left(\frac{n\zeta}{l^2} + \frac{n\zeta_o}{l}\right) - 1}, \tag{6.3.14}$$

$$U(\zeta) = \frac{\frac{n}{l}\exp\left(-\frac{n\zeta}{l^2} - \frac{n\zeta_o}{l}\right)}{\exp\left(-\frac{n\zeta}{l^2} - \frac{n\zeta_o}{l}\right) - 1}, \tag{6.3.15}$$

where ζ_o is an arbitrary integration constant.

Therefore, the exact solutions of equations (6.1.1) become

$$u(x,y,t) = \frac{\frac{n}{l}\exp\left(\frac{n(lx+my+nt)}{l^2} + \frac{n\zeta_o}{l}\right)}{\exp\left(\frac{n(lx+my+nt)}{l^2} + \frac{n\zeta_o}{l}\right) - 1}, \tag{6.3.16}$$

$$u(x,y,t) = \frac{\frac{n}{l}\exp\left(-\frac{n(lx+my+nt)}{l^2} - \frac{n\zeta_o}{l}\right)}{\exp\left(-\frac{n(lx+my+nt)}{l^2} - \frac{n\zeta_o}{l}\right) - 1}. \tag{6.3.17}$$

Corresponding to solutions (6.3.16) and (6.3.17), we can obtain the forms of $v(x,y,t)$ through equation (6.2.4) as follows:

$$v(x,y,t) = \frac{mn}{2l^2}\left(\frac{\exp\left(\frac{n(lx+my+nt)}{l^2} + \frac{n\zeta_o}{l}\right) + 1}{\exp\left(\frac{n(lx+my+nt)}{l^2} + \frac{n\zeta_o}{l}\right) - 1}\right), \tag{6.3.18}$$

$$v(x,y,t) = \frac{mn}{2l^2}. \tag{6.3.19}$$

Case 2: Taking $m = 2$ in equation (6.3.2). On comparing the coefficients of Y^i $(i = 0, 1, 2, 3)$ on both sides of (6.3.3), we get

$$\dot{a}_2(X) = h(X)a_2(X), \tag{6.3.20}$$

$$\dot{a}_1(X) = g(X)a_2(X) + h(X)a_1(X), \tag{6.3.21}$$

$$\dot{a}_o(X) = g(X)a_1(X) + h(X)a_o(X) - 2a_2(X)\left[\frac{2}{l^2}X^3 - \frac{3n}{l^3}X^2 + \frac{n^2}{l^4}X\right], \tag{6.3.22}$$

$$a_1(X)\left[\frac{2}{l^2}X^3 - \frac{3n}{l^3}X^2 + \frac{n^2}{l^4}X\right] = g(X)a_o(X). \tag{6.3.23}$$

Since $a_i(X)$, $(i = 0, 1, 2)$ are polynomials, from equation (6.3.20), we can analyze that $h(X) = 0$, so that $a_2(X)$ must be a constant. For convenience, suppose that $a_2(X) = 1$. Then equation (6.3.21) and (6.3.22) become

$$\dot{a}_1(X) = g(X), \tag{6.3.24}$$

$$\dot{a}_o(X) = g(X)a_1(X) - \left[\frac{4}{l^2}X^3 - \frac{6n}{l^3}X^2 + \frac{2n^2}{l^4}X\right], \tag{6.3.25}$$

$$a_1(X)\left[\frac{2}{l^2}X^3 - \frac{3n}{l^3}X^2 + \frac{n^2}{l^4}X\right] = g(X)a_o(X). \tag{6.3.26}$$

Next, on balancing the degrees of $g(X)$, $a_o(X)$ and $a_1(X)$, we conclude that the degree of $g(X) = 0$ or 1 only.

Subcase 1: For $deg[g(X)] = 1$, assume that $g(X) = AX + B$, $A \neq 0$, then equations (6.3.24) and (6.3.25) gives

$$a_1(X) = \frac{AX^2}{2} + BX + B_o, \tag{6.3.27}$$

$$a_o(X) = \left(\frac{A^2}{8} - \frac{1}{l^2}\right) X^4 + \left(\frac{AB}{2} + \frac{2n}{l^3}\right) X^3 + \left(\frac{AB_o}{2} + \frac{B^2}{2} - \frac{n^2}{l^4}\right) X^2 + BB_oX + B_1, \tag{6.3.28}$$

where B_o and B_1 are arbitrary integration constants.

Further, putting the equations (6.3.27), (6.3.28) into equation (6.3.26) and equating each coefficient of powers of X to zero, we have a nonlinear algebraic system as follows:

$$\begin{aligned} \frac{A}{l^2} - \frac{A^3}{8} + \frac{A}{l^2} &= 0, \\ -\frac{7nA}{2l^3} + \frac{3B}{l^2} - \frac{5A^2B}{8} &= 0, \\ \frac{3n^2A}{2l^4} - AB^2 - \frac{5nB}{l^3} + \frac{2B_0}{l^2} - \frac{A^2B_0}{2} &= 0, \\ \frac{2n^2B}{l^4} - \frac{3nB_0}{l^3} - \frac{B^3}{2} - \frac{3ABB_0}{2} &= 0, \\ BB_1 &= 0. \end{aligned} \tag{6.3.29}$$

After solving this system, we get

$$A = \frac{4}{l}, \; B = \frac{-2n}{l^2}, \; B_o = 0, \; B_1 = 0, \; l \neq 0, \tag{6.3.30}$$

$$A = \frac{-4}{l}, \; B = \frac{2n}{l^2}, \; B_o = 0, \; B_1 = 0, \; l \neq 0. \tag{6.3.31}$$

Using equations (6.3.30) and (6.3.31) in equation (6.3.2), we have

$$Y(\zeta) = \frac{-X^2}{l} + \frac{n}{l^2}X, \; Y(\zeta) = \frac{X^2}{l} - \frac{n}{l^2}X. \tag{6.3.32}$$

Combining equation (6.3.32) with equation (6.3.1), we acquire the exact solutions of (6.2.5) as

$$U(\zeta) = \frac{n}{l}\left(\frac{\exp\left(\frac{n\zeta}{l^2} + \frac{n\zeta_o}{l}\right)}{\exp\left(\frac{n\zeta}{l^2} + \frac{n\zeta_o}{l}\right) - 1}\right), \tag{6.3.33}$$

$$U(\zeta) = \frac{n}{l}\left(\frac{\exp\left(-\frac{n\zeta}{l^2} - \frac{n\zeta_o}{l}\right)}{\exp^{\left(-\frac{n\zeta}{l^2} - \frac{n\zeta_o}{l}\right)} - 1}\right), \tag{6.3.34}$$

where ζ_o is an arbitrary integration constant.

The exact solutions of (6.1.1) are given by

$$u(x,y,t) = \frac{n}{l}\left(\frac{\exp\left(\frac{n(lx+my+nt)}{l^2} + \frac{n\zeta_o}{l}\right)}{\exp\left(\frac{n(lx+my+nt)}{l^2} + \frac{n\zeta_o}{l}\right) - 1}\right), \tag{6.3.35}$$

$$u(x,y,t) = \frac{n}{l}\left(\frac{\exp\left(-\frac{n(lx+my+nt)}{l^2} - \frac{n\zeta_o}{l}\right)}{\exp\left(-\frac{n(lx+my+nt)}{l^2} - \frac{n\zeta_o}{l}\right) - 1}\right), \tag{6.3.36}$$

Now, using equations (6.3.35) and (6.3.36) into equation (6.2.4) respectively, $v(x,y,t)$ becomes

$$v(x,y,t) = \frac{mn}{2l^2}\left(\frac{\exp\left(\frac{n(lx+my+nt)}{l^2} + \frac{n\zeta_o}{l}\right) + 1}{\exp\left(\frac{n(lx+my+nt)}{l^2} + \frac{n\zeta_o}{l}\right) - 1}\right), \tag{6.3.37}$$

$$v(x,y,t) = \frac{mn}{2l^2}. \tag{6.3.38}$$

Remark: In *case 1* and *subcase 1*, we are able to recover the solutions reported by Wazwaz et al. in [10].

Subcase 2: For $deg[g(X)] = 0$, suppose that $g(X) = A$, then equations (6.3.24) and (6.3.25) give

$$a_1(X) = AX + B_0, \tag{6.3.39}$$

$$a_0(X) = -\frac{1}{l^2}X^4 + \frac{2n}{l^3}X^3 + \left(\frac{A^2}{2} - \frac{n^2}{l^4}\right)X^2 + AB_0X + B_1. \tag{6.3.40}$$

where B_0 and B_1 are arbitrary integration constants. Using equations (6.3.39) and (6.3.40) into equation (6.3.26) and equating each coefficient of powers of X to zero, we obtain a nonlinear algebraic system

$$\begin{aligned} \frac{3A}{l^2} &= 0, \\ -\frac{5nA}{l^3} + \frac{2B_0}{l^2} &= 0, \\ \frac{2n^2A}{l^4} - \frac{3nB_0}{l^3} - \frac{A^3}{2} &= 0, \\ \frac{n^2B_0}{l^4} - A^2B_0 &= 0, \\ AB_1 &= 0, \end{aligned} \tag{6.3.41}$$

which on solving yield

$$A = 0, B_0 = 0, B_1 = B_1. \tag{6.3.42}$$

On using equation (6.3.42) in equation (6.3.2), we get

$$Y^2(\zeta) - \frac{1}{l^2}X^4 + \frac{2n}{l^3}X^3 - \frac{n^2}{l^4}X^2 + B_1 = 0. \tag{6.3.43}$$

Combining equation (1.6.4) with equation (6.3.43), we obtain

$$\left(\frac{dU}{d\zeta}\right)^2 = \frac{1}{l^2}U^4(\zeta) - \frac{2n}{l^3}U^3(\zeta) + \frac{n^2}{l^4}U^2(\zeta) - B_1. \tag{6.3.44}$$

With the aid of software MAPLE, we attain the one implicit and two explicit solutions of equation (6.3.44) as follows:

$$U(\zeta) = \frac{n + \sqrt{n^2 \pm 4l^3\sqrt{B_1}}}{2l}, \tag{6.3.45}$$

$$U(\zeta) = -\frac{-n + \sqrt{n^2 \pm 4l^3\sqrt{B_1}}}{2l}, \tag{6.3.46}$$

$$\zeta - \left[\int^{U(\zeta)} \pm \frac{l^2}{\sqrt{a^4l^4 - B_1l^4 - 2a^3ln + a^2n^2}} da\right] - C_1 = 0. \tag{6.3.47}$$

Thus, on making use of equations (6.3.45) and(6.3.46) in equations (6.2.1) and (6.2.4), we get the exact solutions of equations (6.1.1) as follows:

$$u(x,y,t) = \frac{n + \sqrt{n^2 \pm 4l^3\sqrt{B_1}}}{2l}, \tag{6.3.48}$$

$$\begin{aligned} v(x,y,t) = {} & \tfrac{m}{2l}\left(\tfrac{n+\sqrt{n^2 \pm 4l^3\sqrt{B_1}}}{2l}\right) + \tfrac{mn}{2l^3}\left(\tfrac{n+\sqrt{n^2 \pm 4l^3\sqrt{B_1}}}{2l}\right)\zeta \\ & -\tfrac{m}{2l^2}\left(\tfrac{n+\sqrt{n^2 \pm 4l^3\sqrt{B_1}}}{2l}\right)^2\zeta, \end{aligned} \tag{6.3.49}$$

$$u(x,y,t) = -\frac{-n + \sqrt{n^2 \pm 4l^3\sqrt{B_1}}}{2l}, \tag{6.3.50}$$

and

$$\begin{aligned} v(x,y,t) = {} & -\tfrac{m}{2l}\left(\tfrac{-n+\sqrt{n^2 \pm 4l^3\sqrt{B_1}}}{2l}\right) - \tfrac{mn}{2l^3}\left(\tfrac{-n+\sqrt{n^2 \pm 4l^3\sqrt{B_1}}}{2l}\right)\zeta \\ & -\tfrac{m}{2l^2}\left(\tfrac{-n+\sqrt{n^2 \pm 4l^3\sqrt{B_1}}}{2l}\right)^2\zeta. \end{aligned} \tag{6.3.51}$$

Similarly, using equation (6.3.47) into equation (6.2.1) and (6.2.4), $u(x,y,t)$ and $v(x,y,t)$ can be obtained implicitly.

6.4 Discussion

In this chapter, the (2+1)-dimensional Boiti-Leon-Pempinelli system has been successfully investigated by using the $(\frac{G'}{G^2})$-expansion method and the first integral method and proposed the exact traveling wave solutions. We have found that the obtained exact solutions are expressed in terms of rational functions, exponential functions, and trigonometric functions. Some solutions can be recovered by giving the specific values to the parameters, and some new solutions are obtained. This study demonstrates that these techniques are practically appropriate, direct, concise, and reliable to furnish the exact traveling wave solutions of nonlinear partial differential equations. Also, we are able to recover the solutions reported by Wazwaz et al., wherein authors used the exp-function method. With the help of the first integral method, we have been able to contribute some new explicit/implicit solutions of equations (6.1.1). These techniques can be extended to solve the nonlinear problems which arise in the soliton theory and other areas.

Chapter 7

Conclusions

The importance of nonlinear differential equations due to their occurrence in the study of many physical phenomena and also various limitations posed by linear differential equations have been the primary reasons for study put up in the thesis entitled "Study of Some Nonlinear Partial Differential Equations for Lie Symmetries and Exact Solutions". The investigation of Lie symmetries and exact solutions of nonlinear partial differential equations has great theoretical and practical importance. Exact solutions of differential equations play an important role in the proper understanding of qualitative features of many phenomena and processes in various areas of natural science. More specifically, the thesis deals with nonlinear partial differential equations representing some interesting physical systems, which are-the (2+1)-dimensional dispersive long wave system, the (3+1)-dimensional Kadomtsev-Petviashvilli (KP) equation with variable coefficients and an arbitrary nonlinear term, Schr*ö*dinger equation with variable coefficients, Gilson-Pickering equation with variable coefficients and the (2+1)-dimensional Boiti-Leon-Pempinelli system, from the viewpoint of their underlying Lie point symmetries and exact solutions. To determine the admissible symmetries, we adopted the classical Lie symmetry approach on the system of NLPDEs. After obtaining the Lie point symmetries admitted by the nonlinear systems under investigation, the attempt has been to reduce the number of independent variables of the nonlinear systems which result in PDEs or ODEs. The resulting PDEs are further investigated by utilizing the Lie method and obtained the exact solutions. And the resulting ODEs have been examined subsequently for exact solutions by using the power series method. Also, the (2+1) dimensional BLP system has been investigated by using the two techniques, i.e. (G'/G^2)-expansion method and first integral method, and obtained the various exact traveling wave solutions.

Lie point symmetries and exact solutions of the nonlinear systems under examination are not only interesting from a mathematical point of view but also important for applications. In fact, an exhaustive and systematic study of these systems has been made and a variety of new exact solutions are presented. It may be noted that solutions obtained for various nonlinear systems in this thesis include, explicit/implicit solutions, power series solutions, and traveling wave solutions.

In chapter 2, the DLW system has been investigated by using the group method and a variety of new exact solutions have been presented. Also, we have been able to recover some solutions which are already present in the literature. The study in Chapters 3, 4, 5, is devoted to the time-dependent variable coefficients nonlinear partial differential equations. Various systems of governing equations have been furnished for admissible forms of the variable coefficients for which the nonlinear systems possess the Lie point symmetries. Most of the solutions obtained involve an arbitrary coefficient function, and it may enable us to control and discuss the behavior of the solution given by the choice of these arbitrary variable coefficients. In fact, various other power series solutions have also been presented. Also, the solutions proposed in the last chapter include exponential functions, rational functions, and trigonometric functions.

Finally, it is worth mentioning here that in spite of the focus on exact solutions, the authors found it really difficult task to handle the resulting systems of ODEs for the extraction of the exact solutions. In some cases, the obtained exact solutions are very specific in nature, and a systematic search for further reduction of the order of ODEs using the Lie method led only to the trivial symmetries. Keeping in view these limitations, undoubtedly, it turned out to be a useful exercise of obtaining the power series solutions of these ODEs. The study of reduced ODEs and their exact solutions bringforth tremendous scope for future work.

Future Scope

There are a number of potential directions for future research based on the material within the current thesis, of which we discuss a handful of possibilities here.

First, one may focus on a nonclassical symmetry analysis of the systems of nonlinear partial differential equations studied in chapters 2, 3, 4, and 5. This includes making reductions to the PDEs for admitted symmetries and solving the reduced systems of equations to obtain invariant solutions. One may also seek additional symmetries in each model, such as the approximate symmetries, residual symmetries, nonlocal symmetries, and conservation laws.

The numerical investigation of the studied equations can also be performed. For a better understanding of the solutions, one can plot the results in three-dimensional surfaces, so that the physical significance of each solution may be discussed to establish the authenticity of the solutions. The soliton dynamics in a single-mode optical fiber of the nonlinear Schr*ö*dinger equation (3.1.2) may also be studied. Solitons have the most important applications in high-rate telecommunications with optical nonlinear fibers, where they are used as the carriers for the transmission of information. One may check the integrability of the reduced ODEs (see for example equations (3.3.14), (3.3.17), (4.3.18), and (4.3.29)) using the Painlev*é* approach. We can also analyze the classification of group invariant solutions of differential equations by means of the optimal systems. As the objective of the present work was confined to the applications of Lie group theory with the view of deducing the symmetries and then attempting some tractable forms of solutions, such an endeavor remained beyond the scope of the thesis.

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