SURVEYING

(VOLUME II)

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TO MY MOTHER

PREFACE TO THE TWELFTH EDITION

In the Twelfth Edition of the book, the subject matter has been thoroughly revised and updated. Many new articles and solved examples have been added. The entire book has been typeset using laser printer. The authors are thankful to Shri Mool singh Gahlot for the fine laser typesetting done by him.

JODHPUR
15th Aug. 1994

B.C. PUNMIA
A.K. JAIN
A.K. JAIN.

PREFACE TO THE FIFTEENTH EDITION

In the Fifteenth Edition, the subject matter has been thoroughly revised, updated and rearranged. In each Chapter, many new articles have been added. *Four new* Chapters have been added at the end of the book : Chapter 13 on '*Field Astronomy*', Chapter 14 on '*Photogrammetric Surveying*', Chapter 15 on '*Electromagnetic Distance Measurement (EDM)*' and Chapter 16 on '*Remote Sensing*'. All the diagrams have been redrawn using computer graphics and the book has been computer type-set in a bigger format keeping in pace with the Modern trend. Account has been taken throughout of the suggestions offered by many users of the book and grateful acknowledgement is made to them. The authors are thankful to Shri M.S. Gahlot for the fine Laser type setting done by him. The Authors are also thankful Shri R.K. Gupta, Managing Director Laxmi Publications, for taking keen interest in publication of the book and bringing it out nicely and quickly.

JODHPUR
Independence Day
15th August, 2005

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Curve Surveying I : Simple Circular Curves

1.1. GENERAL

Curves are generally used on highways and railways where it is necessary to change the direction of motion. A curve may be circular, parabolic or spiral and is always tangential to the two straight directions.

Circular curves are further divided into three classes : (*i*) simple, (*ii*) compound, and (*iii*) reverse.

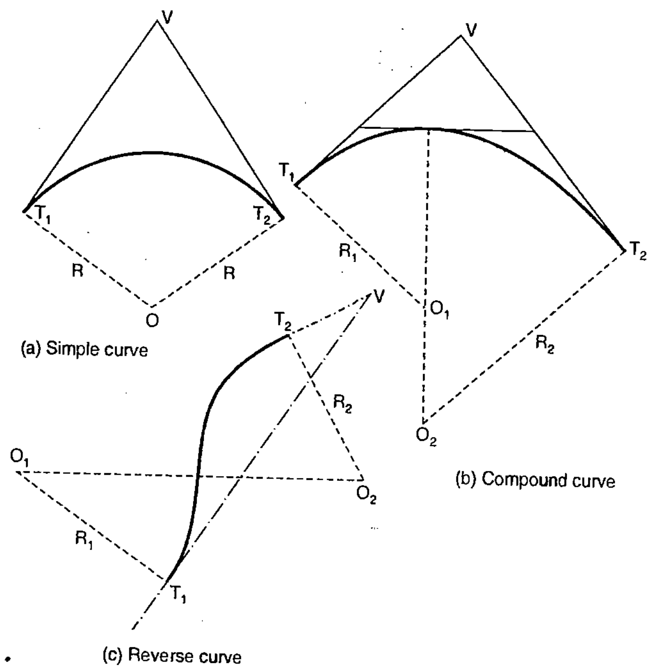


FIG. 1.1. CURVES.

A simple curve [Fig. 1.1 (*a*)] is the one which consists of a single arc of a circle. It is tangential to both the straight lines.

A compound curve [Fig. 1.1 (*b*)] consists of two or more simple arcs that turn in the same direction and join at common tangent points.

A reverse curve [Fig. 1.1 (*c*)] is the one which consists of two circular arcs of same or different radii, having their centres to the different sides of the common tangent. Both the arcs thus bend in different directions with a common tangent at their junction.

SIMPLE CURVES

1.2. DEFINITIONS AND NOTATIONS (Fig. 1.2)

1. *Back tangent.* The tangent (AT_1) previous to the curve is called the back tangent or first tangent.

2. *Forward tangent.* The tangent (T_2B) following the curve is called the forward tangent or second tangent.

3. *Point of intersection.* If the two tangents AT_1 and BT_2 are produced, they will meet in a point, called the point of intersection (P.I.) or vertex (V).

4. *Point of curve (P.C.).* It is the beginning of the curve where the alignment changes from a tangent to a curve.

5. *Point of tangency (P.T.).* It is end of the curve where the alignment changes from a curve to tangent.

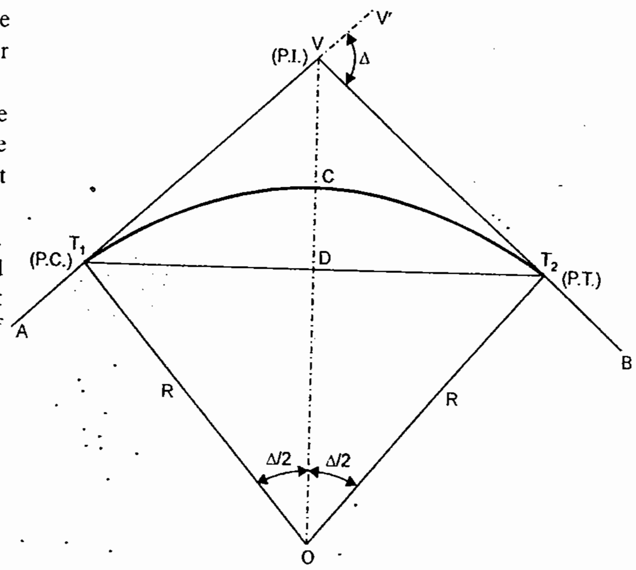


FIG. 1.2. PARTS OF A CIRCULAR CURVE.

6. *Intersection angle.* The angle $V'VB$ between the tangent AV produced and VB is called the intersection angle (Δ) or the *external deflection angle* between the two tangents.

7. *Deflection angle to any point.* The deflection angle to any point on the curve is the angle at P.C. between the back tangent and the chord from P.C. to point on the curve.

8. *Tangent distance (T).* It is the distance between P.C. to P.I. (also the distance from P.I. to P.T.).

9. *External distance (E).* It is distance from the mid-point of the curve to P.I.

10 *Length of curve (L)* It is the total length of the curve from P.C. to P.T.

11. *Long chord.* It is chord joining P.C. to P.T.

12. *Mid ordinate (M).* It is the ordinate from the mid-point of the long chord to the mid-point of the curve.

13. *Normal chord (C).* A chord between two successive regular stations on a curve.

14. *Sub-chord (c).* Sub-chord is any chord shorter than the normal chord.

15. *Right-hand curve.* If the curve deflects to the right of the direction of the progress of survey, it is called the right-hand curve.

16. *Left-hand curve.* If the curve deflects to the left of the direction of the progress of survey, if is called the left-hand curve.

1.3. DESIGNATION OF CURVE

The *sharpness* of the curve is designated either by its *radius* or by its *degree of curvature*. The former system is adopted in Great Britain while the later system is used in America, Canada, India and some other countries.

The degree of curvature has several slightly different definitions. According to the *arc definition* generally used in highway practice, the degree of the curve is defined as the central angle of the curve that is subtended by an arc of 100 ft length. According to the *chord definition* generally used in railway practice, the degree of the curve is defined as the central angle of the curve that is subtended by its chord of 100 ft length.

The relation between the radius (R) and degree of the curve (D) can easily by derived with reference to Fig. 1.3.

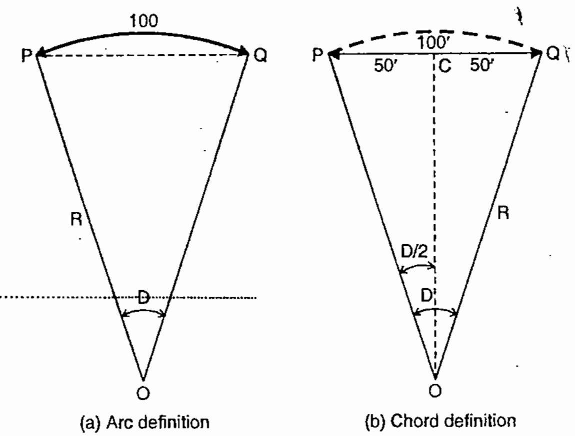


FIG. 1.3. DEGREE OF CURVE (FEET UNITS).

Arc definition. From familiar proportion [Fig. 1.3 (*a*)], we have

$$100 : 2\pi R = D° : 360°$$

or

$$R = \frac{360°}{D} \times \frac{100}{2\pi} = \frac{5729.578}{D} \text{ ft.} \quad ...(1.1)$$

Thus, radius of 1° curve is 5729.578 ft.

To the first approximation, we have

$$R = \frac{5730}{D} \quad ...[1.1\ (a)]$$

Chord definition. From triangle *POC* [Fig. 1.3 (*b*)],

$$\sin \tfrac{1}{2} D = \frac{50}{R} \quad [1.1\ (b)]$$

or $$R = \frac{50}{\sin \frac{1}{2} D} \quad \ldots \text{(exact)} \qquad \ldots(1.2)$$

When D is small, $\sin \frac{1}{2} D$ may be taken approximately equal to $\frac{1}{2} D$ radians.

$$\therefore \quad R = \frac{50}{\frac{D}{2} \times \frac{\pi}{180}}, \text{ where } D \text{ is in degrees}$$

$$= \frac{50 \times 360}{D \times \pi} = \frac{5729.578}{D} = \frac{5730}{D} \text{ (approx.)} \qquad \ldots[1.2\ (a)]$$

It will be seen that for smaller values of D, both equations 1.1 *a* and 1.2 *a* are the same. Both the expressions are not applicable to the curves of comparatively small radius. For more accurate work exact expressions should be used.

In actual practice, every curve is chosen so that either its radius or its degree of curvature is expressed in round numbers. If the radius is even, it is known as *even radius curve*. If the degree is even, it is known as *even degree curve*. The even degree curves are more easily staked out while the even radius curves simplify the computation of complex alignment arrangements. Few railway curves are less than 1° or greater than 6°. Some highway curves are as great as 20°. To satisfy the requirements of safety with greater speeds on highways and railways, the curvature must be reduced to the minimum allowed by the topography.

Metric Degree of Curve

In metric system, two definitions for the "degree of curve" are in use :

1. Angle at the centre subtended by an arc (or chord) of 20 metres.
2. Angle at the centre subtended by an arc (or chord) of 10 metres.

If 20 metres arc (or chord) length is the basis for the degree of the curve, we get

$$D° : 360° = 20 : 2\pi R$$

From which, $$R = \frac{1145.92}{D} \approx \frac{1146}{D} \text{ metres (approx)} \qquad \ldots(1.3)$$

If the definition is based on 10 m arc length, we have

$$D° : 360° = 10° : 2\pi R$$

From which, $$R = \frac{572.958}{D} \approx \frac{573}{D} \text{ metres} \qquad \ldots[1.3\ (a)]$$

For all numerical work in this book, the definition given by equation 1.3, based on 20 m basis has been used.

1.4. ELEMENTS OF SIMPLE CURVE (Fig. 1.2)

(1) Length of the curve (*l*) :

Length, $$l = T_1CT_2 = R\Delta \text{ where } \Delta \text{ is in radians} = \frac{\pi R}{180°} \Delta \qquad \ldots(1.4)$$

where Δ is in degrees.

If the curve is designated by its degree of curvature, the length of the curve will depend upon the criteria used for the definition of the degree of the curve.

(a) *Arc definition* :

Length of arc = 100 ft.

Since any two central angles of the same circle are proportional to the corresponding intercepted arcs (or chords), we have

$$\frac{\Delta}{D} = \frac{l}{100}$$

or $$l = \frac{100\,\Delta}{D} \text{ ft.} \qquad \ldots[1.4\ (a)]$$

(b) *Arc definition* :

Length of arc = 20 m

$$\therefore \quad l = \frac{20\,\Delta}{D} \text{ metres} \qquad \ldots[1.4\ (b)]$$

For the chord definition, l is the total length as if measured along the 100 ft chords of an inscribed polygon. Equations 1.4 (*a*) and (*b*) can be approximately used for chord definition when the chord begins and ends with a sub-chord.

(2) Tangent Length (*T*)

Tangent length, $$\therefore \quad T = T_1V = VT_2 = OT_1 \tan \frac{\Delta}{2} = R \tan \frac{\Delta}{2} \qquad \ldots(1.5)$$

(3) Length of the long chord (*L*)

$$L = T_1T_2 = 2\,OT_1 \sin \frac{\Delta}{2} = 2\,R \sin \tfrac{1}{2} \Delta \qquad \ldots(1.6)$$

(4) Apex distance or external distance (*E*)

$$E = CV = VO - CO = R \sec \frac{\Delta}{2} - R$$

$$= R\left(\sec \frac{\Delta}{2} - 1\right) = R \text{ exsec } \frac{\Delta}{2} \qquad \ldots(1.7)$$

(5) Mid-ordinate (*M*)

$$M = CD = CO - DO = R - R \cos \frac{\Delta}{2}$$

$$= R\left(1 - \cos \frac{\Delta}{2}\right) = R \text{ versin } \frac{\Delta}{2} \qquad \ldots(1.8)$$

The mid-ordinate of the curve is also known as the *versed sine of the curve.*

1.5. SETTING OUT SIMPLE CURVES

The methods of setting out curves can be mainly divided into two heads depending upon instruments used :

(1) *Linear methods*: In the linear methods, only a chain or tape is used. Linear methods are used when (*a*) a high degree of accuracy is not required, (*b*) the curve is short.

(2) *Angular methods*. In angular method, an instrument such as a theodolite is used with or without a chain (or tape).

Before a curve is set out, it is essential to locate the tangents, points of intersection (P.I.), point of the curve (P.C.) and point of tangency (P.T.).

Location of tangent. Before setting out the curve, the surveyor is always supplied with a working plan upon which the general alignment of tangent is known in relation to the traverse controlling the survey of that area. Knowing offsets to certain points on both the tangents, the tangents can be staked on the ground by the tape measurements. The tangents may then be set out by theodolite by trial and error so that they pass through the marks as nearly as possible. The total deflection angle (Δ) can then be measured by setting the theodolite on the P.I.

Location of tangent points. After having located the P.I. and measured Δ, the tangent length (*T*) can be calculated from equation 1.5, *i.e.*,

$$T = R \tan \frac{\Delta}{2}$$

The point T_1 (Fig. 1.2) can be located by measuring back a distance $VT_1 = T$ on the rear tangent.

Similarly, the point T_2 can be located by measuring a distance $VT_2 = T$ on the forward tangent.

Knowing the chainage of P.I., the chainage of point T_1 can be known by subtracting the tangent length from it. The length of the curve is then added to the chainage of T_1 to get the chainage of T_2. The tangent points must be located with greater precision.

Peg Interval. For the ease in calculations and setting out, it is essential that the pegs on the curve are at regular interval from the begining to the end. Such interval is known as *peg interval* and the chord joining two such adjacent pegs is known as the *full chord or normal chord.* The length of the normal chord is generally taken equal to 100 ft in English units or 20 metres in metric units, so that angle subtended by the normal chord at the centre is equal to the degree of the curve. The stations having the chainages in the multiples of chain lengths are known as *full stations.* Except by chance, the tangent points will not be full stations (*i.e.*, their chainages will not be multiples of full chains). The distance between the point T_1 and the first peg will be less than the length of the normal chord so that the first peg may be a full station. Thus, the first chord joining the point of curve T_1 to the first peg will be a *sub-chord.* Similarly, the last chord, joining the last peg on the curve and the tangent point T_2 will be a sub-chord. All other intermediate chords will be normal chords or units chords. Thus, if the chainage T_1 is n chains + m links, the first chord length will be the remaining portion of the chain length *i.e.*, (100 – m) links. Similarly, if the chainage of T_2 is n' chains + m' links, the last chord length will be m' links.

The length of the normal or unit chord should be so selected that there is no appreciable difference between the lengths of the chord and the arc. If the length of the chord is not greater than one-tenth of the radius, it will give sufficiently accurate results, the error being 8 mm in 20 m. For more accurate results, the length of normal chord should be limited to 1/20 of its radius so that the error is only 2 mm in 20 m.

Linear methods of Setting Out

Following are some of the linear methods for setting out simple circular curves :

(1) By ordinates or offsets from the long chord.

(2) By successive bisection of arcs.

(3) By offsets from the tangents.

(4) By offsets from chords produced (or by deflection distances).

Location of tangent points. If an angle measuring instrument is not available, the following procedure may be adopted for the location of tangent points (Fig. 1.4) :

(1) Produce two straights to meet at *V*.

(2) Select two inter-visible points *E* and *G* on the two straights, equidistant from *V*. *VE* and *VG* should be as long as possible.

(3) Join. *EG*, measure it and bisect it at *F*. Join *VF* and measure it.

From similar triangles, *VEF* and VT_1O we have

$$\frac{VT_1}{OT_1} = \frac{VF}{EF}$$

$$\therefore \quad VT_1 = T = \frac{VF}{EF} \cdot OT_1 = \frac{VF}{EF} \cdot R$$

Thus, the tangent poins T_1 and T_2 can be located by measuring VT_1 and VT_2 each equal to *T* along the straights.

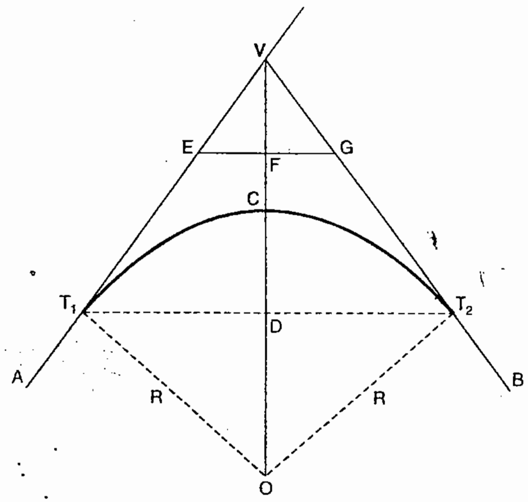


FIG. 1.4. LOCATION OF TANGENT POINTS.

1.6. BY ORDINATES FROM THE LONG CHORD : (Fig. 1.5)

Let R = Radius of the curve.

O_0 = Mid-ordinate.

O_x = Ordinate at distance x from the mid-point of the chord.

T_1 and T_2 = Tangent points.

L = Length of the long chord actually measured on the ground.

Bisect the long chord at point *D*.

From triangle OT_1D,

$$OT_1^2 = T_1D^2 + DO^2$$

or

$$R^2 = \left(\frac{L}{2}\right)^2 + (CO - CD)^2 = \left(\frac{L}{2}\right)^2 + (R - O_0)^2$$

$$\therefore \quad (R - O_0) = \sqrt{R^2 - \left(\frac{L}{2}\right)^2}$$

or $O_0 = R - \sqrt{R^2 - \left(\frac{L}{2}\right)^2}$ ...(1.9)

In order to calculate the ordinate O_x to any point E, draw the line EE_1 parallel to the long chord T_1T_2. Join EO to cut the long chord in G.

Then $O_x = EF = E_1D$

$= E_1O - DO$

$= \sqrt{(EO)^2 - (EE_1)^2} - (CO - CD)$

$= \sqrt{R^2 - x^2} - (R - O_0)$ ...(exact) ...(1.10)

To set out the curve, the long chord is divided into an even number of equal parts. Offsets calculated from equation 1.10 are then set out at each of these points.

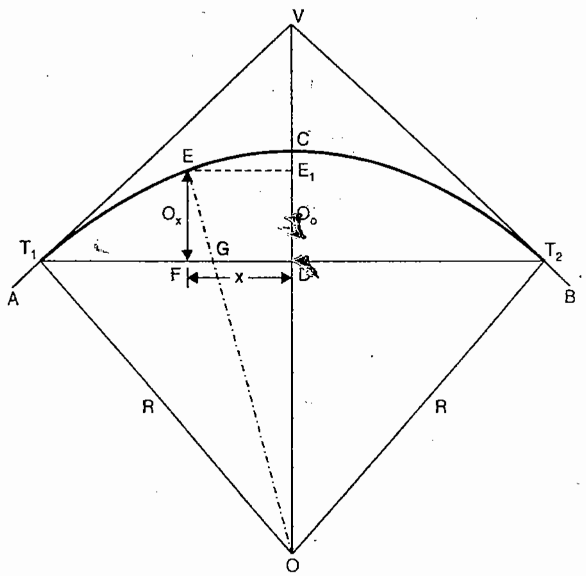


FIG. 1.5. BY ORDINATES FROM THE LONG CHORD.

Approximate Method

If the radius of the curve is large as compared to the length of the long chord, the offsets may be approximately calculated by assuming that the perpendicular ordinate EF (*i.e.* O_x) is approximately equal to the radial ordinate EG. Then taking $T_1F = x$ measured from T_1 we have

$$EG \times 2R = T_1F \times FT_2$$

or $$O_x \times 2R = x\,(L - x)$$

$\therefore$ $$O_x = \frac{x\,(L-x)}{2R} \quad \text{...(approx.)} \qquad \text{...(1.11)}$$

It should be clearly noted that the distance x in this method is measured from the tangent point T_1, while it is measured from the mid-point of the chord in the previous case (equation 1.10).

1.7. BY SUCCESSIVE BISECTION OF ARCS OR CHORDS

Procedure (Fig. 1.6)

1. Join the tangent points T_1, T_2 and bisect the long chord at D. Erect the perpendicular DC and make it equal to the versed sine of the curve. Thus,

$$CD = R\left(1 - \cos\frac{\Delta}{2}\right) = R - \sqrt{R^2 - \left(\frac{L}{2}\right)^2}$$

2. Join T_1C and T_2C and bisect them at D_1 and D_2 respectively. At D_1 and D_2, set out perpendicular offsets $C_1D_1 = C_2D_2 = R\left(1 - \cos\frac{\Delta}{4}\right)$ to get points C_1 and C_2 on the curve.

3. By the successive bisection of these chords, more points may be obtained.

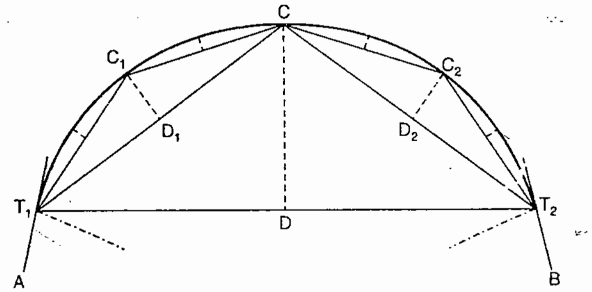


FIG. 1.6. SUCCESSIVE BISECTION OF ARCS.

1.8. BY OFFSETS FROM THE TANGENTS

If the deflection angle and the radius of curvature are both small, the curves can be set out by offsets from the tangent. The offsets from the tangents can be of two types:

(*i*) Radial offsets (*ii*) Perpendicular offsets.

(*i*) Radial Offsets (Fig. 1.7)

Let O_x = Radial offset DE at any distance x along the tangent

$T_1D = x$

From triangle T_1DO,

$$DO^2 = T_1O^2 + T_1D^2$$

or $$(DE + EO)^2 = T_1O^2 + T_1D^2$$

or $(O_x + R)^2 = R^2 + x^2$

$\therefore$ $$O_x = \sqrt{R^2 + x^2} - R$$

....(exact) ..(1.12)

In order to get an approximate expression for O_x, expand $\sqrt{R^2 + x^2}$. Thus,

$$O_x = R\left(1 + \frac{x^2}{2R^2} - \frac{x^4}{8R^4} + \ldots\right) - R$$

Neglecting the other terms except the first two, we get

$$O_x = R + \frac{x^2}{2R} - R$$

$$O_x = \frac{x^2}{2R} \quad \text{...(approx.)}$$

...[1.12 (*a*)]

When the radius is large, the above approximate expression can also be obtained as under :

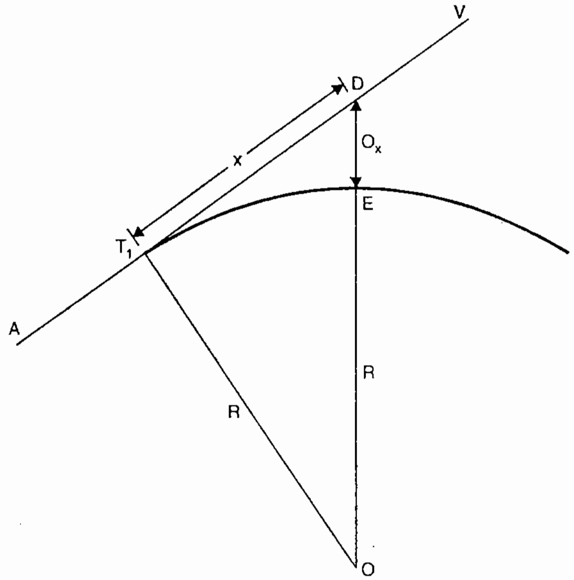


FIG. 1.7. SETTING OUT BY RADIAL OFFSETS.

$$T_1D^2 = DE(2R + DE)$$

or $$x^2 = O_x\,(2R + O_x)$$

Neglecting O_x in comparison to $2R$, we get

$$O_x = \frac{x^2}{2R} \quad \text{...(approximate)}$$

(ii) Perpendicular Offsets (Fig. 1.8)

Let $DE = O_x$ = Offset perpendicular to the tangent

$T_1D = x$, measured along the tangent

Draw EE_1 parallel to the tangent.

From triangle EE_1O, we have

$$E_1O^2 = EO^2 - E_1E^2$$

or $$(T_1O - T_1E_1)^2 = EO^2 - E_1E^2$$

or $$(R - O_x)^2 = R^2 - x^2$$

From which, $O_x = R - \sqrt{R^2 - x^2}$ ...(exact) ...(1.13)

The corresponding approximate expression for O_x may be obtained by expanding the term $\sqrt{R^2 - x^2}$. Thus,

$$O_x = R - R\left(1 - \frac{x^2}{2R^2} - \frac{x^4}{8R^4} \ldots\ldots\right)$$

Neglecting the other terms except the first two of the expansion

$$O_x = R - R + \frac{x^2}{2R}$$

or $$O_x = \frac{x^2}{2R} \quad \text{...[1.13 (a)]}$$

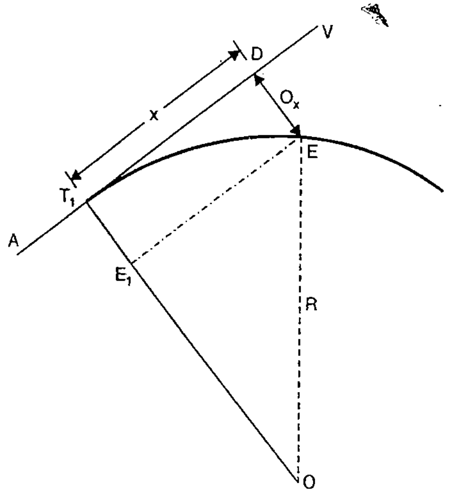


FIG. 1.8. SETTING OUT BY PERPENDICULAR OFFSETS

It should be noted that if the curve is set out by the approximate expression given above, the points on the curve will lie on a parabola and not on the arc of a circle. However, if the versed sine of the curve is less than one-eighth of its chord, the curve approximates very closely to a circle.

∴ To set out the curve, distances x_1, x_2, x_3....etc., are measured from the first tangent point along the tangent and the perpendicular offsets calculated above are erected with the help of an optical square at the corresponding points. When the distace x increases, the offsets become too large to set out accurately. In that case, the central position of the curve may be set out from a third tangent drawn through the apex of the curve.

The method is useful for small curves only.

1.9. BY DEFLECTION DISTANCES (OR OFFSETS FROM THE CHORDS PRODUCED)

The method is very much useful for long curves and is generally used on highway curves when a theodolite is not available.

Let $T_1A_1 = T_1A$ = initial sub-chord $= C_1$

A, B, D etc.= points on the curve

$AB = C_2$

$BD = C_3$ etc.

T_1V = Rear Tangent

$\angle A_1T_1A = \delta$ = deflection angle of the first chord

$A_1A = O_1$ = first offset

$B_2B = O_2$ = second offset

$D_2D = O_3$ = third offset, etc.

Now, arc $A_1A = O_1 = T_1A\,.\,\delta$ ...(i)

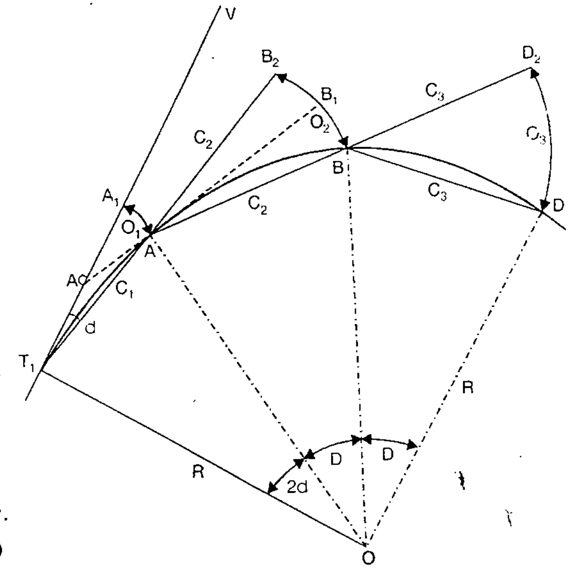


FIG. 1.9. SETTING OUT THE CURVE BY DEFLECTION DISTANCES.

Since T_1V is the tangent to the circle at T_1

$$\angle T_1OA = 2\angle A_1T_1A = 2\delta$$

∴ $$T_1A = R\,.\,2\delta$$

$$\delta = \frac{T_1A}{2R} \quad \text{...(ii)}$$

Substituting the value of δ in (i), we get

$$\text{Arc } A_1A = O_1 = T_1A\,.\,\frac{T_1A}{2R} = \frac{T_1A^2}{2R}$$

Taking arc T_1A = chord T_1A (very nearly), we get

$$O_1 = \frac{C_1^2}{2R} \quad \text{...[1.14 (a)]}$$

In order to obtain the value of the second offset O_2 for getting the point B on the curve, draw a tangent AB_1 to the curve at A to cut the rear tangent in A'. Join T_1A and prolong it to a point B_2 such that $AB_2 = AB = C_2$ = length of the second chord. Then $O_2 = B_2B$.

As from equation 1.14 (a), the offset B_1B from the tangent AB_1 is given by

$$B_1B = \frac{C_2^2}{2R} \quad \text{...(iii)}$$

Again, $\angle B_2AB_1 = \angle A'AT_1$ being opposite angles.

Since T_1A' and $A'A$ are both tangents, they are equal in length.

$\therefore \quad \angle A'T_1A = \delta = \angle A'AT_1$

$\therefore \quad \angle B_2AB_1 = \angle A'AT_1 = \delta$

$\therefore \quad$ arc $B_2B_1 = AB_2 \cdot \delta = C_2 \cdot \delta$

Substituting the value of δ from (*ii*), we get

$$B_2B_1 = C_2 \cdot \frac{T_1A}{2R} = \frac{C_2 \cdot C_1}{2R} \quad \text{...(iv)}$$

$\therefore \quad$ arc $B_2B = B_2B_1 + B_1B$

or
$$O_2 = \frac{C_2C_1}{2R} + \frac{C_2^2}{2R} = \frac{C_2}{2R}(C_1 + C_2) \quad \text{...[1.14 (b)]}$$

Similarly, the third offset $O_3 = D_2D$ is given by

$$O_3 = \frac{C_3}{2R}(C_2 + C_3)$$

The last or *n*th offset is given by

$$O_n = \frac{C_n}{2R}(C_{n-1} + C_n) \quad \text{...[1.14 (c)]}$$

Generally, the first chord is a sub-chord, say of length c, and the intermediate chords are normal chords, say of length C. In that case, the above formulae reduce to

$$O_1 = \frac{c^2}{R}$$

and
$$O_2 = \frac{C}{2R}(c + C) \quad \text{...[1.14 (d)]}$$

$$O_3 = O_4 = \ldots O_{n-1} = \frac{C}{2R}(2C) = \frac{C^2}{R} \quad \text{...[1.14 (e)]}$$

and
$$O_n = \frac{c'}{2R}(C + c') \quad \text{...[1.14 (f)]}$$

where c' is the last sub-chord.

Procedure for Setting Out the Curve

(1) Locate the tangent points T_1 and T_2 and find out their chainages as explained earlier. Calculate the lenght (c) of the first sub-chord so that the first peg is the full station.

(2) With zero mark at T_1, spread the chain (or tape) along the first tangent to point A_1 on it such that $T_1A_1 = c =$ length of the first sub-chord.

(3) With T_1 as centre and T_1A_1 as radius, swing the chain such that the arc $A_1A =$ calculated offset O_1. Fix the point A on the curve.

(4) Spread the chain along T_1A and pull it straight in this direction to point B_2 such that the zero of the chain is at A and the distance $AB_2 = C \doteq$ length of the normal chord.

(5) With zero of the chain centred at A and AB_2 as radius, swing the chain to a point B such that $B_2B = O_2$ = length of the second offset. Fix the point B on the curve.

(6) Spread the chain along AB and repeat the steps (4) and (5) till the point of tangency (T_2) is reached. All intermediate offsets will be equal to $\frac{C^2}{R}$, while the last offset will be equal to $\frac{c'}{2R}(C + c')$.

The last point so fixed must coincide with the point of tangency (T_2) fixed originally by measurements from the vertex. If the discrepancy (sometimes called as the closing error) is more, the curve should be re-set. If the error is less, it should be distributed to all the points by moving them sideways by an amount proportional to the square of their distance from the point T_1.

The method is mostly used in road surveys and is very satisfactory, specially when a theodolite is not available. However, it has a great defect in that the error in fixing point is carried forward.

INSTRUMENTAL METHODS

The following are instrumental methods commonly used for setting out a circular curve :

(1) Rankine's method of tangential (or deflection) angle.

(2) Two theodolite method.

(3) Tacheometric method.

1.10. RANKINE'S METHOD OF TANGENTIAL (OR DEFLECTION) ANGLES

A deflection angle to any point on the curve is the angle at P.C. between the back tangent and the chord from P.C. to that point.

Rankine's method is based on the principle that the deflection angle to any point on a circular curve is measured by one-half the angle subtended by the arc from P.C. to that point. It is assumed that the length of the arc is approximately equal to its chord.

Let us first derive expression for the tangential angles.

Let $\quad T_1V$ = Rear tangent

T_1 = Point to curve (P.C.)

$\delta_1, \delta_2, \delta_3$ = The tangential angles or the angles which each of the successive chords T_1A, AB, BC etc. makes with the respective tangents to the curve at T_1, A, B etc.

$\Delta_1, \Delta_2, \Delta_3 \ldots$ = Total tangential angles or the deflection angles to the points A, B, C etc.

C_1, C_2, C_3 = Lengths of the chords T_1A, AB, BC...

A_1A = Tangent to the curve at A.

From the property of a circle,

$$\angle VT_1A = \tfrac{1}{2}\angle T_1OA$$

or
$$\angle T_1OA = 2\angle VT_1A = 2\delta_1$$

Now $\frac{\angle T_1OA}{C_1} = \frac{180°}{\pi R}$

or $\angle T_1OA = 2\delta_1 = \frac{180°\ C_1}{\pi R}$

From which $\delta_1 = \frac{90 C_1}{\pi R}$ degrees

$= \frac{90 \times 60 C_1}{\pi R} = 1718.9 \frac{C_1}{R}$ minutes.

Similarly,

$\delta_2 = 1718.9 \frac{C_2}{R}$; $\delta_3 = 1718.9 \frac{C_3}{R}$,

or, in general,

$\delta = 1718.9 \frac{C}{R}$ minutes ...(1.1)

where C is the length of the chord.

For the first chord T_1A, the deflection angle = its tangential angle

or $\Delta_1 = \delta_1$ ...(1)

For the second point B, let the deflection angle by = Δ_2.

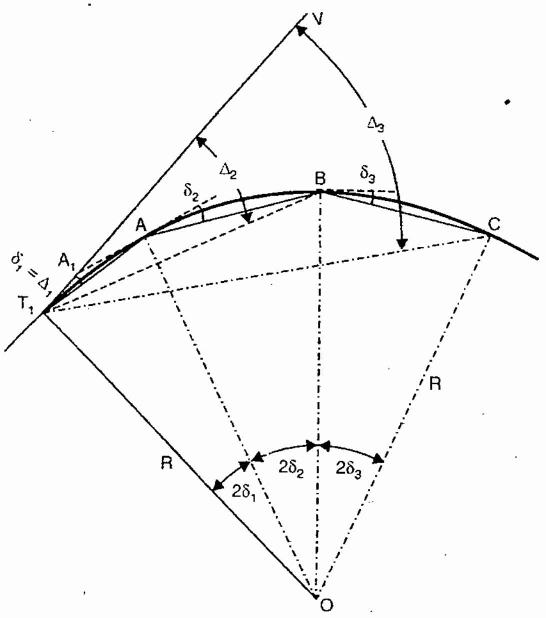


FIG. 1.10. RANKINE'S METHOD OF TANGENTIAL ANGLES.

Since δ_2 = tangential angle for the chord AB,

$\angle AOB = 2\delta_2$

$\therefore$ $\angle AT_1B$ = Half the angle subtended by AB at the centre = δ_2

Now $\Delta_2 = \angle VT_1B = \angle A_1T_1A + \angle AT_1B$

or $\Delta_2 = \delta_1 + \delta_2 = \Delta_1 + \delta_2$ ...(2)

Similarly, $\Delta_3 = \delta_1 + \delta_2 + \delta_3 = \Delta_2 + \delta_3$ ...(3)

and $\Delta_n = \delta_1 + \delta_2 + \ldots \delta_n = \Delta_{n-1} + \delta_n$ ...(1.16)

Hence, the deflection angle for any chord is equal to the deflection angle for the previous chord plus the tangential angle for that chord.

Check : Deflection angle of the long chord, *i.e.*,

$\angle VT_1T_2 = \Delta_n = \frac{\Delta}{2}$, where Δ is the intersection angle or the external deflection angle for the curve.

If the degree of the curve is equal to D for a 20 m chord,

$\delta_2 = \delta_3 \ldots = \delta_{n-1} = \frac{1}{2} D$

Similarly, if c and c' are the first and the last sub-chords

$\delta_1 = \frac{c}{20} \cdot \frac{D}{2} = \frac{cD}{40}$, where c is metres ; $\delta_n = \frac{c'}{20} \cdot \frac{D}{2} = \frac{c'D}{40}$

$\therefore$ $\Delta_1 = \delta_1 = \frac{cD}{40}$

$\Delta_2 = \Delta_1 + \delta_2 = \frac{cD}{40} + \frac{1}{2} D$

$\Delta_3 = \Delta_2 + \delta_3 = \frac{cD}{40} + \frac{1}{2} D + \frac{1}{2} D = \frac{cD}{40} + D$

$\Delta_n = \Delta_{n-1} + \delta_n = \frac{cD}{40} + (n-2)\frac{D}{2} + \frac{c'D}{40}$

Similarly, if the degree of the curve is equal to D for a 100 ft chord,

$\delta_1 = \frac{c \times D}{200}$; $\delta_2 = \delta_3 = \ldots \delta_{n-1} = \frac{D}{2}$

$\delta_n = \frac{c'D}{200}$.

Procedure for Setting out the Curve

(1) Set the theodolite at the point of curve (T_1). With both plates clamped to zero, direct the theodolite to bisect the point of intersection (V). The line of sight is thus in the direction of the rear tangent.

(2) Release the vernier plate and set angle Δ_1 on the vernier. The line of sight is thus directed along chord T_1A.

(3) With the zero end of the tape pointed at T_1 and an arrow held at a distance $T_1A = c$ along it, swing the tape around T_1 till the arrow is bisected by the cross-hairs. Thus, the first point A is fixed.

(4) Set the second deflection angle Δ_2 on the vernier so that the line of sight is directed along T_1B.

(5) With the zero end of the tape pinned at A, and an arrow held at distance $AB = C$ along it, swing the tape around A till the arrow is bisected by the cross-hairs, thus fixing the point B.

(6) Repeat steps (4) and (5) till the last point T_2 is reached.

Check : The last point so located must coincide with the point of tangency (T_2) fixed independently by measurements from the point of intersection. If the discrepancy is small, last few pegs may be adjusted. If it is more, the whole curve should be reset.

In the case of the left hand curve, each of the calculated values of the deflection angle (*i.e.*Δ_1, Δ_2 etc.) , should be subtracted from 360°. The angles so obtained are to be set on the vernier of theodolite for setting out the curve.

In the above method, three men are required : the surveyor to operate the theodolite, and two chainmen to measure the chord lengths with chain or tape. This method is most frequently used for setting out circular curves of large radius and of considerable length.

Field Notes

The record of deflection angles for various points is usually kept in the following form (next page) :

Point	Chainage	Chord Length (C)	Tangential angle (δ)			Deflection angle (Δ)			Actual theodolite reading			Remarks
			°	′	″	°	′	″	°	′	″	

Curve Location from the Point of Intersection

If the P.I. is accessible, this method has the advantage that the curve may be set out in one operation while the theodolite is set at P.I. for the purpose of finding the angle of intersection, thus saving much time. The method is uncommon, since the points on the curve are at equal distances apart and are not full stations.

Let us locate any point P on the curve by observations from a theodolite set at the P.I.

Let $\alpha = \angle T_1VP$, usually called the deflection angle.

(When the point P is to the other side of C, it is measured with the tangent VT_2).

Let $\theta = \angle T_1OP$

Drop PP_1 perpendicular to VT_1.

Draw PD parallel to VT_1

$$\tan\alpha = \frac{PP_1}{VP_1} = \frac{T_1D}{VT_1 - P_1T_1}$$

$$= \frac{T_1D}{T - PD}$$

$$= \frac{R\,(1-\cos\theta)}{\left(R\tan\frac{\Delta}{2} - R\sin\theta\right)}$$

$$= \frac{(1-\cos\theta)}{\left(\tan\frac{\Delta}{2} - \sin\theta\right)} \quad \ldots(1.17)$$

$$= \frac{\text{versin}\,\theta}{\tan\frac{\Delta}{2} - \sin\theta} \quad \ldots(1.17\ a)$$

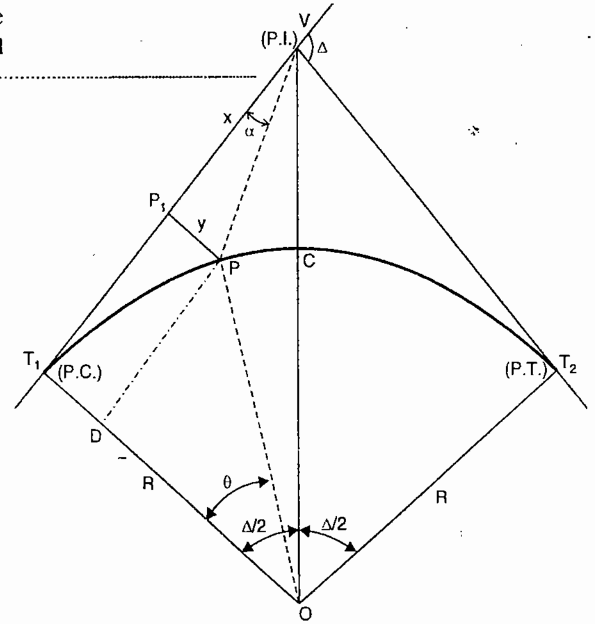


FIG. 1.11. CURVE LOCATION FROM P.I.

The above equation gives deflection angles for various points. It should be noted that these deflection angles are independent of the radius or length of the curve. If the curve is divided into ten equal parts.

$$\theta_1 = \frac{1}{10}\Delta$$

$$\theta_2 = \frac{2}{10}\Delta$$

....

$$\theta_{10} = \Delta$$

Knowing θ_1, θ_2 etc., the deflection angles α_1, α_2 etc., can be calculated from equation 1.17.

Method of setting out the Curve

(1) Set the theodolite at the point of intersection. Measure the angle of intersection Δ. Calculate the various elements of the curve and locate T_1 and T_2 as usual. Fix the values of θ_1, θ_2 etc. by dividing the curve into suitable equal parts. Calculate the arc (or chord) length by dividing the length of the curve by the same number of equal parts of the curve.

(2) With both the plates clamped to zero, direct the line of sight to the point of tangency (T_1). Set the angle α_1 on the vernier.

(3) With the zero of the tape pinned at T_1 and the arrow kept at the arc (or chord) distance along it, swing the tape till the arrow is bisected by line of sight, thus fixing the first point.

(4) Set the angle α_2 on the vernier, thus directing the line of sight towards second point on the curve.

(5) With the zero of the tape pinned at the first point fixed in step (3), and the arrow kept at the arc (or chord) distance along it, swing the tape till the arrow is bisected by the line of sight, thus fixing the second point.

(6) Repeat steps (4) and (5) to set out other points.

The middle point of the curve (*i.e.* point 5) is located independently by deflection angle $\alpha_5 \left(= 90^\circ - \frac{\Delta}{2} \right)$ and the measurement of the external distance E from P.I.

Curve Location from Point of Tangency (T_2)

If the entire curve is visible from the P.I., the curve can be located by one sigle set-up there. Also for long curves, it is better to set out the second half of the curve by starting the measurements and deflections from P.T. so that any small error can be adjusted at the middle of the curve where a slight deviation in the alignment is of less consequence than at or near points of tangency to the curve.

In this method, the theodolite is set up at P.T. and is properly oriented by sighting on the P.C. when the vernier is reading 0° 00′ and telescope is normal. If this is done, the curve notes are the same whether deflections are from P.C. or P.T. (See the two theodolite method). Beginning with the first station on the curve from P.C., stations are established in order along the curve towards P.T.

1.11. TWO THEODOLITE METHOD

In this method, two theodolites are used one at P.C. and the other at P.T. The method is used when the ground is unsuitable for chaining and is *based on the principle that the angle between the tangent and the chord is equal to the angle which that chord subtends in the opposite segment.*

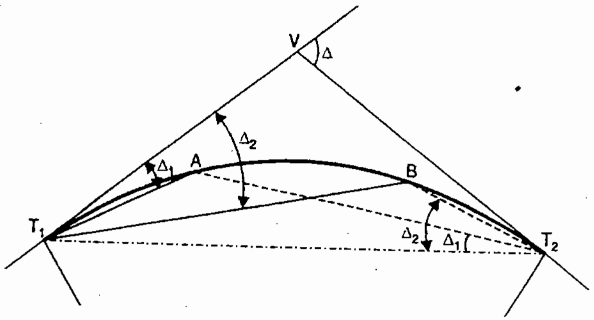


FIG. 1.12. TWO THEODOLITE METHOD.

Thus, in Fig. 1.12

$\angle VT_1A = \Delta_1$ = Deflection angle for A

But $\angle AT_2T_1$ is the angle subtended by the chord T_1A in the opposite segment.

$\therefore \quad \angle AT_2T_1 = \angle VT_1A = \Delta_1$

Similarly, $\quad \angle VT_1B = \Delta_2 = \angle T_1T_2B$

Hence the angle between the long chord and the line joining any point to T_2 is equal to the deflection angle to the point measured with respect to the rear tangent.

Method of Setting Out the Curve

(1) Set up one transit at P.C. (T_1) and the other at P.T.(T_2).

(2) Clamp both the plates of each transit to zero reading.

(3) With the zero reading, direct the line of sight of the transit at T_1 towards V. Similarly, direct the line of sight of the other transit at T_2 towards T_1 when the reading is zero. *Both the transits are thus correctly oriented.*

(4) Set the reading of each of the transits to the deflection angle for the first point A. The line of sight of both the theodolites are thus directed towards A along T_1A and T_2A respectively.

(5) Move a ranging rod or an arrow in such a way that it is bisected simultaneously by cross-hairs of both the instruments. Thus, point A is fixed.

(6) To fix the second point B, set reading Δ_2 on both the instruments and bisect the ranging rod.

(7) Repeat steps (4) and (5) for location of all the points.

The method is expensive since two instruments and two surveyors are required. However, the method is most accurate since each point is fixed independently of the others. An error in setting out one point is not carried right through the curve as in the method of tangential angles.

1.12. TACHEOMETRIC METHOD

By the use of a tacheometer, chaining may be completely dispensed with, though the method is much less accurate than Rankine's. In this method, a point on the curve is fixed by the deflection angle from the rear tangent and measuring, tacheometrically, the

distance of that point from P.C. (T_1) and *not* from the precceding point as in Rankine's method. Thus, in this method also, each point is fixed independently of the others and the error in setting out one point is not carried right through the curve as in the Rankine's method.

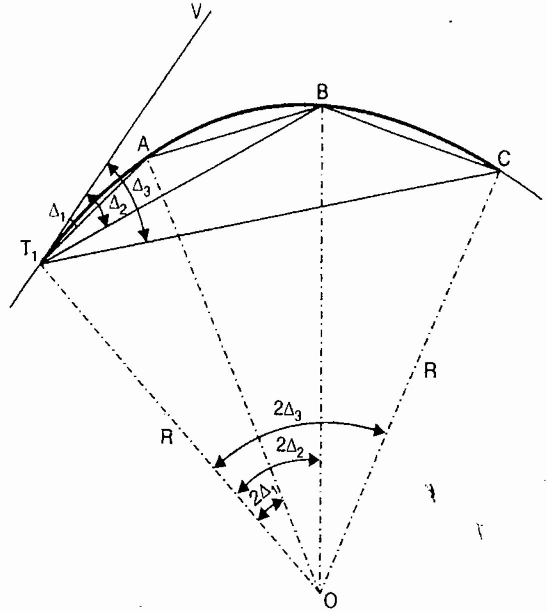


FIG. 1.13. SETTING OUT BY TACHEOMETRIC METHOD.

In Fig. 1.13, T_1A, T_1B, T_1C etc. are the *whole chords* joining point A, B, C etc. to the point of curvature (T_1).

Evidently $T_1A = L_1 = 2R \sin \Delta_1$

$T_1B = L_2 = 2R \sin \Delta_2$

$T_1C = L_3 = 2R \sin \Delta_3$ etc. etc.

and $\quad T_1T_2 = L_n = 2R \sin \Delta_n$

$= 2R \sin \frac{\Delta}{2} = L$

= length of the long chord.

Having known these lengths, the repective staff intercepts $s_1, s_2, s_3, \ldots, s_n$ can be calculated from the tacheometric formulae :

$L = \frac{f}{i} s + (f + d)$, when the line of sight is horizontal

or $\quad L = \frac{f}{i} s \cos^2 \theta + (f + d) \cos \theta$, when the line of sight is inclined.

Procedure for Setting Out the Curve

(1) Set the tacheometer at T_1 and sight the point of intersection (V) when the reading is zero. The line of sight is thus oriented along the rear tangent.

(2) Set the angle Δ_1 on the vernier, thus directing the line of sight along T_1A.

(3) Direct a staffman to move in the direction T_1A till the calculated staff intercept s_1 is obtained. The staff is generally held vertical. Thus, the first point A is fixed.

(4) Set the angle Δ_2 now, thus directing the line of sight along T_1B. Move the staff backward or forward along T_1B until the staff intercept s_2 is obtained, thus fixing the point B.

(5) Fix other points similarly.

Since the staff intercept increases with its distance from the tacheometer, accurate staff reading is not possible when the distances along the whole chords become too large. In that case, the curve is to be located by shifting tacheometer to the last point located on the curve.

Example 1.1. *Calculate the ordinates at 10 metres distances for a circular curve having a long chord of 80 metres and a versed sine of 4 metres.*

Solution. (Fig. 1.5)

From equation 1.9, the versed sine is given by

$$O_0 = R - \sqrt{R^2 - \left(\frac{L}{2}\right)^2}$$

or $$4 = R - \sqrt{R^2 - (40)^2}$$

or $$R^2 - (40)^2 = (R-4)^2 = R^2 + 16 - 8R$$

$\therefore$ $$R = \frac{1616}{8} = 202 \text{ metres}$$

$$(R - O_0) = 202 - 4 = 198 \text{ m}$$

From equation 1.10, we have

$$O_x = \sqrt{R^2 - x^2} - (R - O_0)$$

$\therefore$ $$O_{10} = \sqrt{(202)^2 - (10)^2} - 198 = 201.75 - 198 = 3.75 \text{ m}$$

$$O_{20} = \sqrt{(202)^2 - (20)^2} - 198 = 201.01 - 198 = 3.01 \text{ m}$$

$$O_{30} = \sqrt{(202)^2 - (30)^2} - 198 = 199.76 - 198 = 1.76 \text{ m}$$

Check : $$O_{40} = \sqrt{(202)^2 - (40)^2} - 198 = 198 - 198 = 0.$$

Example 1.2. *Determine the offsets to be set out at $\frac{1}{2}$ chain interval along the tangents to locate a 16-chain curve, the length of each chain being 20 m.*

Solution. (*a*) ***Radial offsets*** (Fig. 1.7)

From equation 1.12, we have

$$O_x = \sqrt{R^2 + x^2} - R \qquad \text{Here } R = 16 \text{ chains}$$

$O_{0.5} = \sqrt{(16)^2 + (0.5)^2} - 16 = 0.0078$	chains	= 0.16 m
$O_1 = \sqrt{(16)^2 + (1)^2} - 16 = 0.031$	chains	= 0.62 m
$O_{1.5} = \sqrt{(16)^2 + (1.5)^2} - 16 = 0.0702$	chains	= 1.40 m
$O_2 = \sqrt{(16)^2 + (2)^2} - 16 = 0.1245$	chains	= 2.49 m
$O_{2.5} = \sqrt{(16)^2 + (2.5)^2} - 16 = 0.1941$	chains	= 3.88 m
$O_3 = \sqrt{(16)^2 + (3)^2} - 16 = 0.2788$	chains	= 5.58 m

etc. etc.

(*b*) ***Perpendicular offsets*** (Fig. 1.8)

From equation 1.13,

$$O_x = R - \sqrt{R^2 - x^2}$$

$O_{0.5} = 16 - \sqrt{(16)^2 - (0.5)^2} = 0.0078$	chains	= 0.16 m
$O_1 = 16 - \sqrt{(16)^2 - (1)^2} = 0.0311$	chains	= 0.62 m
$O_{1.5} = 16 - \sqrt{(16) - (1.5)^2} = 0.0704$	chains	= 1.41 m
$O_2 = 16 - \sqrt{(16)^2 - (2)^2} = 0.1255$	chains	= 2.51 m
$O_{2.5} = 16 - \sqrt{(16)^2 - (2.5)^2} = 0.1965$	chains	= 3.93 m
$O_3 = 16 - \sqrt{(16)^2 - (3)^2} = 0.284$	chains	= 5.68 m

etc. etc.

(*c*) ***By approximate method*** :

$$O_x = \frac{x^2}{2R} \qquad \ldots(1.13\ a)$$

$O_{0.5} = \frac{(0.5)^2}{32} = 0.0078$	chains	= 0.15 m
$O_1 = \frac{1^2}{32} = 0.0312$	chains	= 0.62 m
$O_{1.5} = \frac{(1.5)^2}{32} = 0.0704$	chains	= 1.41 m
$O_2 = \frac{(2)^2}{32} = 0.125$	chains	= 2.50 m
$O_{2.5} = \frac{(2.5)^2}{32} = 0.1953$	chains	= 3.91 m
$O_3 = \frac{3^2}{32} = 0.281$	chains	= 5.62 m

etc. etc.

Example 1.3. *Two tangents intersect at chainage 59 + 60, the deflection angle being 50° 30′. Calculate the necessary data for setting out a curve of 15 chains radius to connect the two tangents if it is intended to set out the curve by offets from chords. Take peg interval equal to 100 links, length of the chain being equal to 20 metres (100 links).*

Solution.

Tangent length $(T) = R \tan \frac{\Delta}{2} = 15 \tan 25° 15'$

$= 7.074$ chains $= 141.48$ m

Length of the curve $(l) = \frac{\pi R \Delta}{180°} = \frac{\pi \times 15 \times 50° 30'}{180°}$

$= 13.221$ chains $= 264.42$ m

	(Chains)	(Links)	(Metres)
Chainage of P.I	= 59	+ 60.0	1192.00
Deduct tangent length (T)	= 7	+ 07.4	141.48
Chainage of P.C.	= 52	+ 52.6	1050.52
Add length of curve (l)	= 13	+ 22.1	= 264.42
Chainage of P.T.	= 65	+ 74.7	= 1314.94

The chainage of each peg will be multiple of 20 metres.

Length of first sub-chord $(c) = 1060 - 1050.52 = 9.48$ m.

or more conveniently, $c = (53 + 00) - (52 + 52.6) = 47.4$ links $= 9.48$ m

Length of last sub-chord $(c') = 1314.94 - 1300 = 14.94$ m

or more conveniently, $c' = (65 + 74.7) - (65 + 00) = 74.7$ links $= 14.94$ m

Number of full chords $= \dfrac{1300 - 1060}{20} = \dfrac{240}{20} = 12$, each of 20 m length

Total number of chords $= 1 + 12 + 1 = 14$

Length of first offset $O_1 = \dfrac{c^2}{2R} = \dfrac{(9.48)^2}{2 \times 300} = 0.15$ m

where $R = 15$ chains $= 15 \times 20$ m $= 300$ m

Length of second offset $O_2 = \dfrac{C}{2R}(c + C) = \dfrac{20}{2 \times 300}(9.48 + 20) = 0.98$ m

$$O_3, O_4 = \ldots\ldots O_{12} = \frac{C^2}{2R} = \frac{(20)^2}{300} = 1.33 \text{ m}$$

Last offset $O_n = \dfrac{c'}{2R}(C + c') = \dfrac{14.94}{2 \times 300}(20 + 14.94) = 0.87$ m

Example 1.4. *Calculate the necessary data for setting out the curve of example 1.3 if it is intended to set out the curve by Rankine's method of tangential angles. If the theodolite has a least count of 20", tabulate the actual readings of deflection angles to be set out.*

Solution.

As calculated earlier

$$c = 9.48 \text{ m}$$
$$c' = 14.94 \text{ m}$$
$$C = 20 \text{ m}$$

The tangential angle $\delta = 1718.9 \dfrac{C}{R}$ min.

where $R = 15 \times 20 = 300$ m

δ_1 for the first chord $= 1718.9 \dfrac{9.48}{300} = 54.32$ min $= 54' 19''$

$$\delta_2 = \delta_3 \ldots \delta_{13} = \delta = 1718.9 \frac{20}{300} = 114.593 \text{ min} = 1° 54' 35.6''$$

δ_{14} for last chord $= 1718.9 \dfrac{14.94}{300} = 85.592$ min $= 1° 25' 35''$.

The deflection angles for various chords are as follows :

	(Deflection angle) °	′	″	(Theodolite reading) °	′	″
$\Delta_1 = \delta_1 =$	00	54	19.0	00	54	20
$\Delta_2 = \Delta_1 + \delta =$	02	48	54.6	02	49	00
$\Delta_3 = \Delta_2 + \delta =$	04	43	30.2	04	43	40
$\Delta_4 = \Delta_3 + \delta =$	06	38	5.8	06	38	00
$\Delta_5 = \Delta_4 + \delta =$	08	32	41.4	08	32	40
$\Delta_6 = \Delta_5 + \delta =$	10	27	17.0	10	27	20
$\Delta_7 = \Delta_6 + \delta =$	12	21	52.6	12	22	00
$\Delta_8 = \Delta_7 + \delta =$	14	16	28.2	14	16	20
$\Delta_9 = \Delta_8 + \delta =$	16	11	3.8	16	11	00
$\Delta_{10} = \Delta_9 + \delta =$	18	05	39.4	18	05	40
$\Delta_{11} = \Delta_{10} + \delta =$	20	00	15.0	20	00	20
$\Delta_{12} = \Delta_{11} + \delta =$	21	54	50.6	21	55	00
$\Delta_{13} = \Delta_{12} + \delta =$	23	49	26.2	23	49	20
$\Delta_{14} = \Delta_{13} + \delta_{14} =$	25	15	1.2	25	15	00

Check : $\Delta_{14} = \frac{1}{2}\Delta = \frac{1}{2}(50° 30') = 25° 15'$.

1.13. OBSTACLES TO THE LOCATION OF CURVES

CASE 1. WHEN THE P.I. IS INACCESSIBLE

When the point of intersection is inaccessible, a line is run (or traverse if necessary) to connect the two tangents. The deflection angles it makes with the tangents are measured with transit set-ups at its end. Its length is also measured very accurately. The deflection angle Δ of the curve is then equal to the sum of the two deflection angles. The distances from the ends of the line to the P.I. are calculated by the solution of the triangle. The *procedure* is as follows : (Fig. 1.14).

1. Select two intervisible points A and B and the two tangents so that line AB is moderately on the level ground. If the ground towards the P.I. is unsuitable for

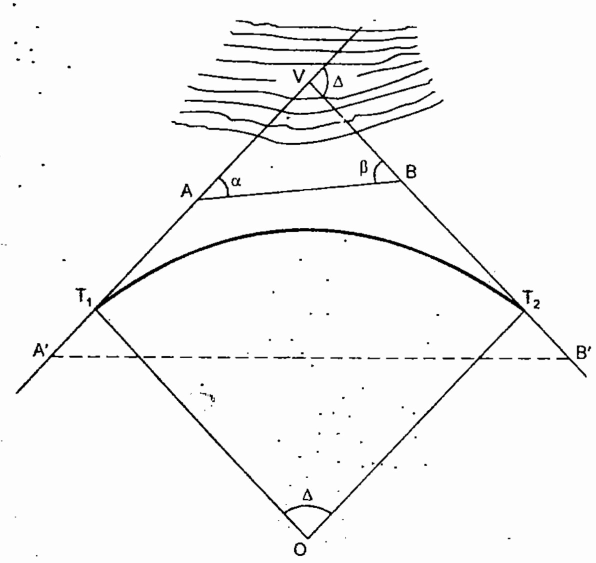


FIG. 1.14. CURVE LOCATION WHEN P.I. IS INACCESSIBLE.

chaining, the points (A' and B') may be chosen towards the centre of the curve

2. Set the transit at A and orient the line of sight in the direction AT_1 when the reading on the circle is zero. Transit the telescope so that the line of sight is now in the direction AV. Measure the deflection angle α accurately.

3. Similarly, set the transit at B and measure the deflection angle β. The total deflection angle of the curve will then be

$$\Delta = \alpha + \beta.$$

4. Measure the distance AB accurately with the help of a tape.

5. To get the lengths AV and BV, solve the triangle AVB.

Thus,
$$AV = \frac{AB}{\sin\,[180° - (\alpha + \beta)]} \cdot \sin\beta = \frac{AB}{\sin\Delta} \cdot \sin\beta$$

and
$$BV = \frac{AB}{\sin\,[180° - (\alpha + \beta)]} \cdot \sin\alpha = \frac{AB}{\sin\Delta} \cdot \sin\alpha.$$

6. Calculate the tangent lengths VT_1 and VT_2 from the formula

$$VT_1 = VT_2 = T = R\tan\frac{\Delta}{2}$$

7. Calculate the length AT_1 and BT_2. Thus,

$$AT_1 = VT_1 - AV$$

and
$$BT_2 = VT_2 - BV.$$

From A and B, measure the distances AT_1 and BT_2 along the tangents to get the tangent points T_1 and T_2 respectively.

8. The curve can now be located from the point of curve (T_1).

If it is not possible to obtain a clear line AB, run a traverse between A and B. By traverse calculations, find the length and bearing of the line AB. Knowing the bearings of the tangents, angles α and β can be calculated.

CASE 2. WHEN THE P.C. IS INACCESSIBLE

If the point of curve (P.C.) is inaccessible, the following steps are neccessary to determine its chainage. Unless this is done, the length of the first sub-chord and the position of the pegs on the curves cannot be determined.

1. Calculate the tangent length T and measure it back along the rear tangent. Noting that P.C. falls in the obstacle, select a point A very near the obstacle and measure the distance AV.

Then $\quad T_1A = T - AV \quad \ldots(i)$

2. Select another point B on the tangent and to the other side of the obstacle and find chainage.

3. By any method of chaining past the obstacle (*e.g.*, the solution

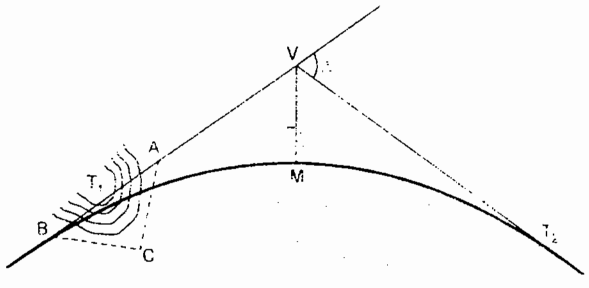


FIG. 1.15 P.C. INACCESSIBLE

of the triangle ABC), compute the length AB.

4. The chainage of T_1 = chainage of $B + AB - T_1A$.

Thus, the chainage of P.C. is known.

5. Compute length of the curve and find the chainage of T_2.

6. Set out the curve in the reverse direction from T_2.

CASE 3. WHEN THE P.T. IS INACCESSIBLE

In order to continue the work past the forward tangent, it is neccessary to know the chainage of the point of tangency (T_2). If it is inaccessible following steps are necessary to determine its chainage (Fig. 1.16).

1. Determine the chainage of the point of intersection (V).

2. On the forward tangent, select a point A very near the obstacle. Measure the distance VA.

Thus, distance $AT_2 = T - VA$.

3. Chainage of T_2 = chainage of T_1 + length of the curve.

4. Select any point B on the tangent, but to the other side of the obstacle. By any method of chaining past the obstacle (*e.g.*, solution of triangle ABC), calculate the distance AB.

5. The chainage of B = chainage of $T_2 + AB - AT_2$.

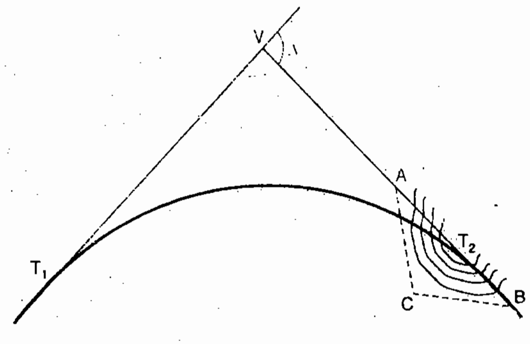


FIG. 1.16 P.T. INACCESSIBLE

6. From the point B, locate the first accessible full chain peg on the forward tangent.

CASE 4. WHEN BOTH P.C. AND P.T. ARE INACCESSIBLE

This may happen simultaneously, particularly in course of setting out in cities. The problem is two-fold :

(*i*) Determination of continuous chainage along the curve, and

(*ii*) Setting out the curve.

Procedure. (Fig. 1.15)

1. Determine the chainage of T_1 as explained in case of (2) above when the P.C. is inaccessible. To this, add the length (l) of the curve to get the chainage of point T_2.

2. Set up the theodolite at point of intersection (V) and bisect the angle T_1VT_2. Calculate the apex distance VM and measure it along the bisector to get apex M of the curve.

The chainage of M = chainage to $T_1 + \frac{l}{2}$.

3. Set the theodolite at M and orient it by back-sighting to V. Set out the curve in both directions from M.

CASE 5. WHEN BOTH P.C. AND P.I. ARE INACCESSIBLE (Fig. 1.17)

1. Select any point A on the rear tangent. Run any convenient line AB and measure its length. Measure angles VAB and VBA by the theodolite set-ups at A and B. Calculate AT_1 and the angle Δ as discussed in case (1) of § 1.13. Knowing the chainage of A and distance AT_1, calculate the chainage of T_1 and also of T_2.

2. Imagine the curve produced backward to C on the perpendicular offset AC. Then

$$\sin COT_1 = \frac{AT_1}{R}$$

or $$\angle COT_1 = \alpha = \sin^{-1}\left(\frac{AT_1}{R}\right)$$

and $$AC = R(1 - \cos\alpha) = R \text{ versin } \alpha$$

Thus, make the perpendicular offset AC at A, as calculated above.

3. From C, draw a chord CD parallel to AT_1, making $CD = 2\,AT_1$. *Then, point D will be on the curve.*

4. Set a theodolite at D and deflect from DC an angle equal to COT_1 (*i.e.*, $= \alpha$) for a tangent to the curve at D. Prepare a table of deflection angles with respect to D and set out the whole curve from it.

Alternative to step (4), adopt the procedure for setting out the curve from D as described in case 6 below.

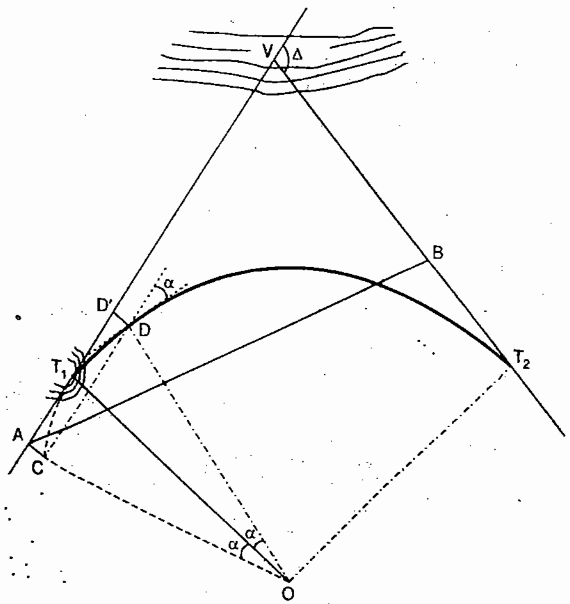


FIG. 1.17. BOTH P.C. AND P.I. ARE INACCESSIBLE.

CASE 6. WHEN THE COMPLETE CURVE CANNOT BE SET OUT FROM P.C.

In case of very long curves or obstructions intervening the line of sight, it may not be possible to set out the whole curve from one single set up of the instrument at P.C. In such a case, it is necessary to set up the instrument at one or more points along the curve. We will consider two cases of the set ups of the theodolite at intermediate points on the curve :

(*a*) When the P.C. is visible from the intermediate point.

(*b*) When the P.C. is not visible from the intermediate point.

CASE (*a*) When the P.C. is visible from the intermediate point.

First method :

Let C be the last point set out from the P.C. (T_1) and let its deflection angle be Δ_c. Assuming that T_1 is visible from C, the procedure for setting out the rest of the curve is as follows :

1. Shift the theodolite at C and set it there.

2. Set the vernier to read 0° and backsight on T_1 with the telescope inverted.

3. Transit the telescope. The line of sight is now directed along T_1C produced.

4. Unclamp the upper plate, and set the vernier to read deflection angle Δ_d to the forward point D as if it were located from T_1 and locate the point D as usual.

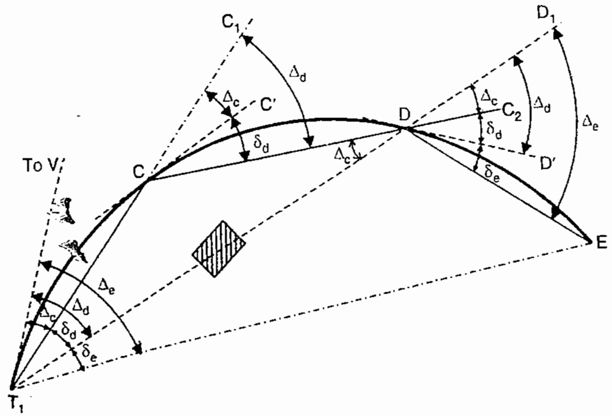


FIG. 1.18. SETTING OUT FROM INTERMEDIATE POINT

A careful study of Fig. 1.18 will reveal that when the angle Δ_d is on the circle, the line of sight is towards D. However, the proof is given below :

Let CC' be the tangent to the curve at C

$$\angle C_1CC' = \Delta_c$$

$$\delta_d = \angle C'CD = \text{tangential angle for chord } CD.$$

Since $\angle CT_1D$ is the angle that the chord CD substends in the opposite segment, we have

$$\angle CT_1D = C'CD = \delta_d$$

Hence $$\angle VT_1D = \text{deflection angle for } D = \angle VT_1C + \angle CT_1D = \Delta_c + \delta_d = \Delta_d$$

But $$\angle C_1CD = \angle C_1CC' + \angle C'CD = \Delta_c + \delta_d$$

Hence $$\angle VT_1D = \angle C_1CD.$$

Thus when the instrument is transferred to any point on the curve and oriented as explained above, no new calculations are required for continuing the curve, but the previously calculated deflection angles can be used.

Second method to case (a) :

In the above method, it is assumed that the instrument is in good adjustment. If it is not, the curve can be set out as below :

(1) While the last point C is sighted from T_1, fix a point C_1 in the direction T_1C produced.

(2) Set the theodolite at C. Clamp both the plates with zero reading and bisect C_1 accurately. The instrument is thus correctly oriented.

(3) Release the vernier plate and set the vernier to deflection angle Δ_d to set out the point D.

Third method to case (a).

(1) Let C be the last point set out from T_1. Set the theodolite at C. With vernier fixed to read 180°, and telescope normal, take a backsight to T_1. The theodolite is now correctly oriented. Swing the telescope clockwise in azimuth through 180°. The line of sight will now be along CC_1.

(2) Set the vernier to Δ_d and set out the point D.

CASE (*b*) When the P.C. is not visible from intermediate point (Fig. 1.18)

Let it be required to set out the curve from a point D from which T_1 is not visible. Let C be any previously located point on the curve.

(1) Set the transit at D.

(2) Set the vernier to reading Δ_c (equal to the deflection angle to the point C) and take a backsight to C.

(3) Plunge the telescope. The telescope is thus correcty oriented. Unclamp the upper plate and set the vernier to read the deflection angle to the next station E, as if it were located from T_1.

From Fig. 1.18, it is clear that

$$\angle VT_1E = \Delta_e = \angle VT_1C + \angle CT_1D + \angle DT_1E = \Delta_c + \delta_d + \delta_e$$

But $$\Delta_c = CDT_1 = D_1DC_2\ ;\ \delta_d = C_2DD'\ ;\ \delta_e = \angle D'DE$$

$$\therefore \quad \angle VT_1E = \angle D_1DC_2 + \angle C_2DD' + \angle D'DE = \angle D_1DE$$

Thus, $\angle D_1DE = \Delta_e$ = angle set out by the theodolite. Other points can similarly be established.

CASE 7. WHEN THE OBSTACLE TO CHAINING OCCURS

In Fig. 1.19, let C be the point upto which there is no obstruction in chaining. Let there be an obstruction between C and the next point D to be located.

(1) Having set off the point C, find, by setting off the successive deflection angles, a clear line of sight to a point on the curve. Let E be the point clear off the obstruction and is visible from T_1. Set out angle Δ_e being the deflection angle to point E.

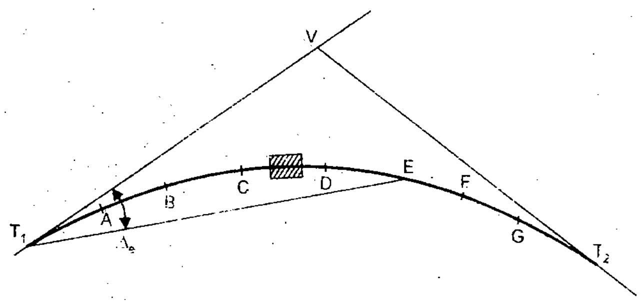


FIG. 1.19. OBSTACLE TO CHAINING

(2) Calculate length of the whole chord T_1E from the expression

$$T_1E = 2R \sin \Delta_e$$

Along the line of sight T_1E, measure the distance calculated above, thus getting the point E.

If chaining is not possible or convenient, set out distance T_1E tacheometrically.

(3) Set out other points F, G, H etc., from T_1 as usual.

The points covered by the obstruction are set out after removing the obstruction. *Alternatively*, if the obstruction is long, the part of the curve to the other side of the obstruction can be set out from T_2.

1.14. SPECIAL PROBLEMS IN SIMPLE CURVES

In the final location survey, some minor changes in the preliminary location alignment are usually introduced. This may result in changes in the radii of some of the curves, and changes in the positions of some tangents thus requiring changes in the adjacent curves. The more difficult problems that are likely to occur in *realigment* are discussed below.

(1) PASSING A CURVE THROUGH A FIXED POINT

Given the angle Δ and two tangents of undetermined length, it is required to find the radius R and tangent distance T of a curve that will pass through a fixed point, say P (Fig. 1.11).

Let P be the point located by the angle α and the distance $VP (= z)$ from the P.I. or located by the co-ordinates x and y with reference to the rear tangent T_1V.

Let the distance $\quad VP = z$

and $\quad \angle T_1VP = \alpha$

From triangle VOP, $\angle PVO = \frac{1}{2}(180° - \Delta) - \alpha = 90° - \left(\alpha + \frac{\Delta}{2}\right)$

$$\angle POV = \frac{\Delta}{2} - \theta$$

$$\therefore \quad \angle VPO = 180° - \left[\left\{90° - \left(\alpha + \frac{\Delta}{2}\right)\right\} + \left\{\frac{\Delta}{2} - \theta\right\}\right] = 90° + (\alpha + \theta)$$

By the sine rule,

$$\frac{\sin VPO}{\sin PVO} = \frac{OV}{OP}$$

or $$\frac{\sin \{90° + (\alpha + \theta)\}}{\sin \left\{90° - \left(\alpha + \frac{\Delta}{2}\right)\right\}} = \frac{R \sec \frac{\Delta}{2}}{R}$$

or $$\frac{\cos (\alpha + \theta)}{\cos \left(\alpha + \frac{\Delta}{2}\right)} = \sec \frac{\Delta}{2}$$

or $$\cos (\alpha + \theta) = \frac{\cos \left(\alpha + \frac{\Delta}{2}\right)}{\cos \frac{\Delta}{2}} \qquad ...(1.18)$$

From the above equation, θ can be calculated.

Again, $\quad T_1D = P_1P = VP \sin \alpha = z \sin \alpha$

Treating T_1D as the mid-ordinate of a curve whose radius is R and central angle 2θ, we have

$$T_1D = R - R\cos\theta = R(1-\cos\theta) = = R \text{ versin } \theta$$

or
$$R = \frac{T_1D}{\text{versin } \theta} = \frac{z \sin\alpha}{\text{versin } \theta} = \frac{z\sin\alpha}{(1-\cos\theta)} \quad ...(1.19)$$

From which R can be determined.

If the co-ordinates x and y are given, first calculate the angle α from the relation $\tan\alpha = \frac{y}{x}$ and then determine θ from equation 1.18. The radius R is then given by

$$R = \frac{T_1D}{\text{versin } \theta} = \frac{y}{\text{versin } \theta} = \frac{y}{1-\cos\theta} \quad ...[1.19\ (a)]$$

The tangent $T = R\tan\frac{\Delta}{2}$.

(2) PASSING A CURVE TANGENTIAL TO THREE LINES (Fig. 1.20)

Given three lines T_1A, AB, BT_2, their deviation angle α and β and the length of AB ($= d$), it is required to find the radius R of a curve that will be tangential to the three lines.

Let T_1, C and T_2 be the tangential points.

$$AC = AT_1 = R\tan\tfrac{1}{2}\alpha \quad ...(i)$$

$$BC = BT_2 = R\tan\tfrac{1}{2}\beta \quad ...(ii)$$

Adding (*i*) and (*ii*), we get

$$d = AB = AC + BC$$

$$= R(\tan\tfrac{1}{2}\alpha + \tan\tfrac{1}{2}\beta)$$

From which, $$R = \frac{d}{\tan\frac{1}{2}\alpha + \tan\frac{1}{2}\beta} \quad ...(1.20)$$

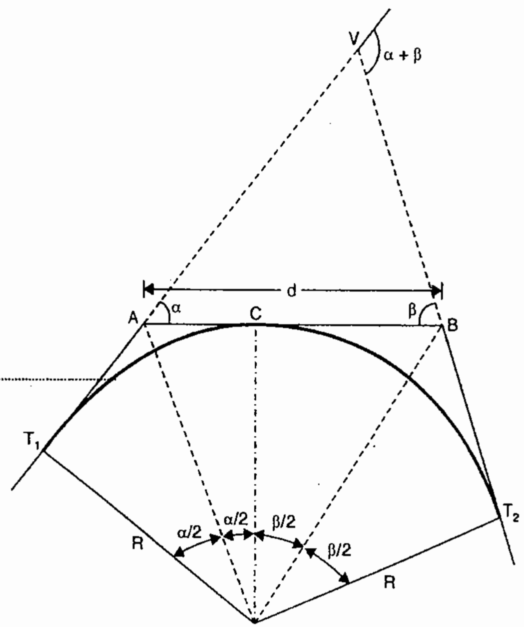


FIG. 1.20. CURVE TANGENTIAL TO THREE LINES

(3) SHIFTING FORWARD TANGENT PARALLEL OUTWARD : RADIUS UNCHANGED

Given the distance p by which the forward tangent is shifted outward, it is required to locate the new position of P.C. if the radius is unchanged.

Let T_1T_2 be the original curve and $T_1'T_2'$ be the new curve when the tangent VT_2 is shifted to a new position $V'T_2'$ parallel to itself by a distance p. Let O' be the new centre. In Fig. 1.21, the firm lines show the elements of the original curve, while the dotted lines and the letters with dash correspond to the condition when the forward tangent is shifted.

Thus, T_2A = perpendicular distance = p

$$T_1T_1' = OO' = T_2T_2' = VV' = \frac{p}{\sin\Delta} \quad ...(i)$$

$\therefore$ Chainage of T_1' = chainage of $T_1 + \frac{p}{\sin\Delta}$

Thus, the new P.C. (T_1') can be located.

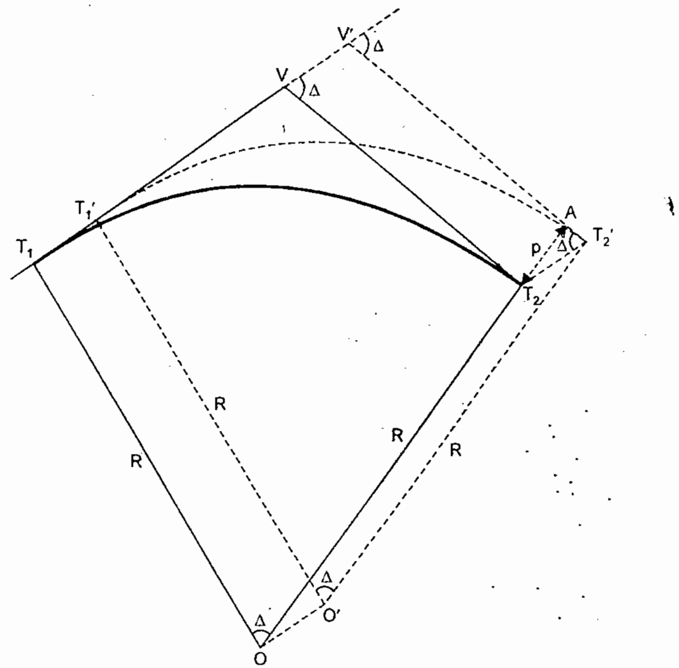


FIG. 1.21. SHIFTING FORWARD TANGENT OUTWARD (SAME RADIUS)

If, however, the tangent is shifted inward, equation (*i*) still holds good, but to find the chainage of T_1' the distance $\frac{p}{\sin\Delta}$ will have to be subtracted from chainage of T_1.

(4) SHIFTING FORWARD TANGENT PARALLEL OUT-WARD : RADIUS CHANGED

Given the distance p by which the forward tangent is shifted outward it is required to find the new radius R' without changing the position of P.C. (Fig. 1.22)

In Fig. 1.22, firm lines show various elements before the change, while dotted lines correspond to the case when the forward tangent is shifted outward. Since the P.C. (T_1) is not to be changed, the centre O' of the new curve lies on the line T_1O. Similarly, the new P.T. (T_2') lies on the chord T_1T_2 produced. Let $AT_2 = p$ = perpendicular distance through which the forward tangent has been shifted. All letters with dash correspond to the new or changed curve.

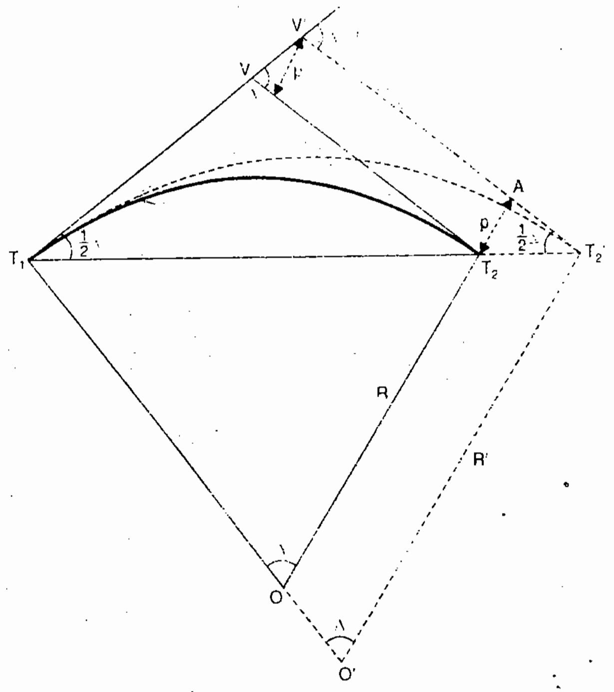


FIG. 1.22. SHIFTING FORWARD TANGENT OUTWARD (NEW RADIUS).

Evidently

$$\angle VT_1T_2 = V'T_2'T_1 = \tfrac{1}{2}\Delta$$

and $$T_2T_2' = \frac{p}{\sin\frac{1}{2}\Delta}$$

Similarly, $$VV' = \frac{p}{\sin\Delta}$$

New tangent length

$$T' = T_1V' = T_1V + VV'$$

or $$T' = R\tan\tfrac{1}{2}\Delta + \frac{p}{\sin\Delta}$$

$$R' = \frac{T'}{\tan\frac{1}{2}\Delta}$$

Hence $$R' = \frac{R\tan\frac{1}{2}\Delta + \frac{p}{\sin\Delta}}{\tan\frac{1}{2}\Delta} = R + \frac{p}{\sin\Delta\,.\,\tan\frac{1}{2}\Delta}$$

Substituting $\tan\frac{1}{2}\Delta = \dfrac{1-\cos\Delta}{\sin\Delta}$ in the above, we get

$$R' = R + \frac{p}{1-\cos\Delta} = R + \frac{p}{\text{versin}\,\Delta} \quad \text{...(1.21)}$$

Thus, the new radius R' is known. In order to locate T_2', set a theodolite at T_1, direct the line of sight towards T_2 and measure the distance $T_2T_2' = \dfrac{p}{\sin\frac{1}{2}\Delta}$ along it.

(5) CHANGING THE DIRECTION OF FORWARD TANGENT : P.T. UNCHANGED

Given the angle θ by which the forward tangent VT_2 is rotated to a position $V'T_2$, it is required to find the new radius (R') and the new P.C. (T_1')

In Fig. 1.23, the firm lines represent the original curve, while dotted lines correspond to the case when the forward tangent is rotated about T_2. All letters with dash correspond to the new curve.

We have $$\Delta' = \Delta + \theta$$

By sine rule, $$V'T_2 = VT_2\,.\,\frac{\sin\Delta}{\sin\Delta'} \qquad \text{But } V'T_2 = T' \text{ and } VT_2 = T$$

$$\therefore \quad T' = T\frac{\sin\Delta}{\sin\Delta'}$$

$$\therefore \quad R'\tan\frac{\Delta'}{2} = R\tan\frac{\Delta}{2}\,.\,\frac{\sin\Delta}{\sin\Delta'} \quad \text{or} \quad R' = R\,.\,\frac{\tan\frac{1}{2}\Delta\,\sin\Delta}{\tan\frac{1}{2}\Delta'\,\sin\Delta'} \quad \text{...(1.22)}$$

But $$\tan\tfrac{1}{2}\Delta = \frac{1-\cos\Delta}{\sin\Delta}, \quad \text{and} \quad \tan\tfrac{1}{2}\Delta' = \frac{1-\cos\Delta'}{\sin\Delta'}$$

Substituting the above values, we get

$$R' = R\,.\,\frac{1-\cos\Delta}{1-\cos\Delta'} = R\,\frac{\text{versin}\,\Delta}{\text{versin}\,\Delta'} \quad \text{...[1.22 }(a)]$$

To get the distance T_1T_1' through which P.C. is shifted, consider triange $VV'T_2$.

By sine rule,

$$VV' = VT_2\,.\,\frac{\sin\theta}{\sin\Delta'} = T\,.\,\frac{\sin\theta}{\sin\Delta'}$$

$$\therefore \quad T_1T_1' = T_1V - T_1'V$$
$$= T - (T_1'V' - VV')$$
$$= T - \left(T' - T\,.\,\frac{\sin\theta}{\sin\Delta'}\right)$$
$$= T\left(1 + \frac{\sin\theta}{\sin\Delta'}\right) - T'$$
$$= T\left(1 + \frac{\sin\theta}{\sin\Delta'} - \frac{\sin\Delta}{\sin\Delta'}\right)$$
$$= T\left(1 - \frac{\sin\Delta - \sin\theta}{\sin\Delta'}\right)$$

since $T' = T\dfrac{\sin\Delta}{\sin\Delta'}$

Thus point T_1' can be located.

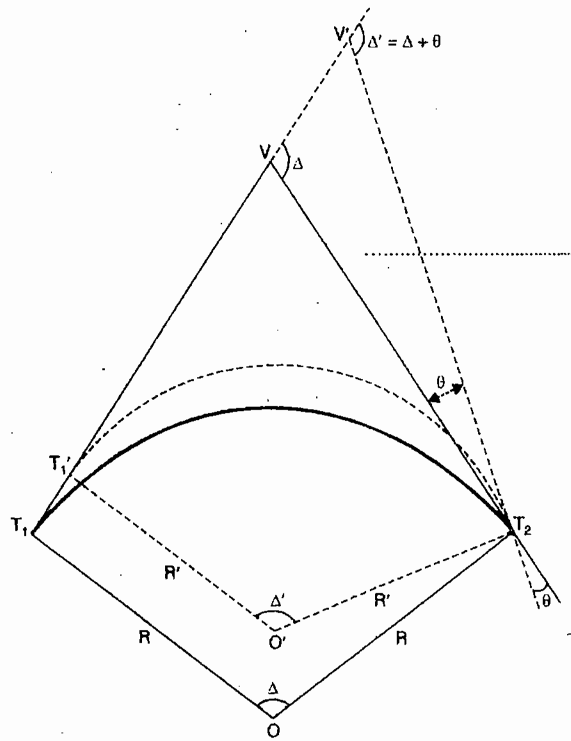


FIG. 1.23. CHANGING DIRECTION OF FORWARD TANGENT : P.T. UNCHANGED.

(6) CHANGING THE DIRECTION OF FORWARD TANGENT : P.T. MOVED AHEAD

Given the angle θ by which the forward tangent VT_2 is rotated to a new position $V'T_2'$ without changing the P.C. , it is required to find the new radius (R') and the new P.T. (T_2')

In Fig. 1.24. the firm lines show the original elements while the dotted lines correspond to the case when the forward tangent is rotated through θ. Since the position of the P.C. is unchanged, the new centre (*O*) will lie on T_1O. Let T_2' be the new P.T.

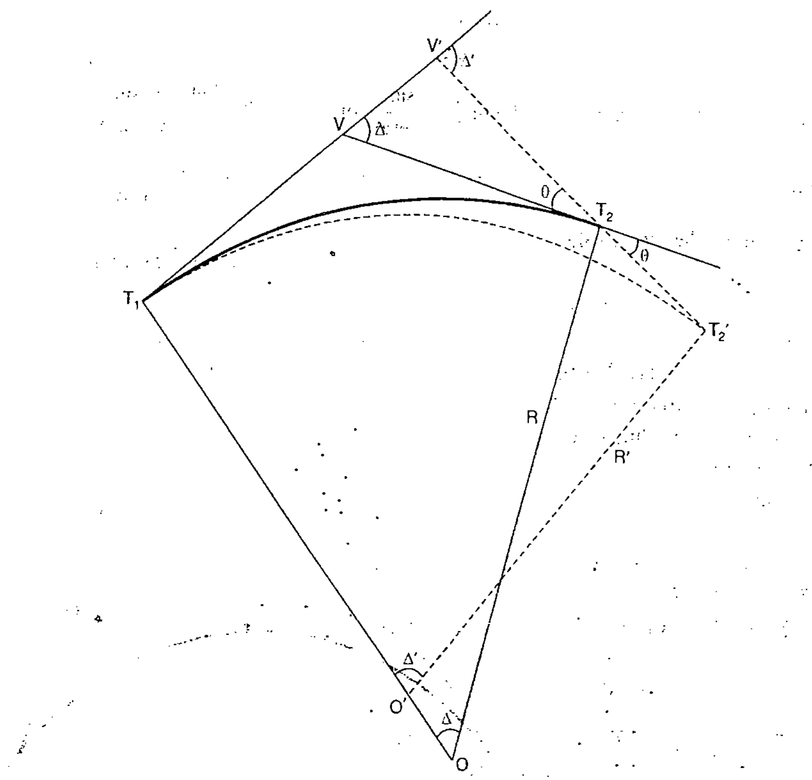


FIG. 1.24. CHANGING DIRECTION OF FORWARD TANGENT : P.C. UNCHANGED.

We have $\Delta' = \Delta + \theta$

By sine rule, $VV' = VT_2 \cdot \dfrac{\sin\theta}{\sin\Delta'} = T\dfrac{\sin\theta}{\sin\Delta'}$

Now $T' = T_1V' = T_1V + VV'$

or $$T' = T + T\frac{\sin\theta}{\sin\Delta'} = T\left(1 + \frac{\sin\theta}{\sin\Delta'}\right) \quad ...(1)$$

But $$T' = R'\tan\frac{\Delta'}{2} \text{ and } T = R\tan\frac{\Delta}{2}$$

Substituting these values, we get

$$R'\tan\frac{\Delta'}{2} = R\tan\frac{\Delta}{2}\left(1 + \frac{\sin\theta}{\sin\Delta'}\right)$$

or $$R' = R\frac{\tan\frac{\Delta}{2}}{\tan\frac{\Delta'}{2}}\left(1 + \frac{\sin\theta}{\sin\Delta'}\right) \quad ...(1.23)$$

Again, to locate the position of T_2' consider the triangle $VV'T_2$, from which

$$V'T_2 = VT_2 \cdot \frac{\sin\Delta}{\sin\Delta'} = T \cdot \frac{\sin\Delta}{\sin\Delta'}$$

Now $$T_2T_2' = V'T_2' - V'T_2 = T' - T\frac{\sin\Delta}{\sin\Delta'} = T\left(1 + \frac{\sin\theta}{\sin\Delta'}\right) - T\frac{\sin\Delta}{\sin\Delta'}$$

$$= T\left(1 - \frac{\sin\Delta - \sin\theta}{\sin\Delta'}\right)$$

Thus, T_2' can be located.

Example 1.5. *The following notes refer to setting out of a circular curve to the right, of 15 chains radius between two straights AB, BC and intersection B of which was inaccessible.*

Measurement ab = 6.21 chains from a in AB to b in BC

Theodolite at a : interior angle α = 23° 43'

Theodolite at b : interior anlge β = 25° 54'

Chainage of a = 29.059

Owing to obstructions it will be impossible to set out angles from the tangent at the first tangent point A beyond that to the peg at 31.0 chains on the curve, and the theodolite will be set up at this peg in order to continue the curve, to the second tangent point C.

Describe concisely the procedure of setting out from an intermediate peg on the curve, and show in tabular form the tangential angles to be set out at A and at 31.0 chains for peg at even chains on the curve giving also the nearest readings for a vernier reading to 20". (U.L.)

Solution.

Given : $\alpha = 23°\,43'$, $\beta = 25°\,24'$, $R = 15$ chains

∴ $\Delta = \alpha + \beta = 23°\,43' + 25°\,54' = 49°\,37'$

From triangle *Bab*, we have $Ba = ab\dfrac{\sin\beta}{\sin\Delta} = 6.21\dfrac{\sin 25°\,54'}{\sin 49°\,37'} = 3.561$ chains

$$\text{Length of curve} = \frac{\pi R\Delta}{180°} = \frac{\pi \times 15 \times 49°\,37'}{180°} = 12.989 \text{ chains}$$

$$\text{Tangent length } T = BA = R\tan\tfrac{1}{2}\Delta = 15\tan\frac{49°\,37'}{2} = 6.934 \text{ chains}$$

Chainage of $a = 29.059$ chains
Add the distance $aB = 3.561$

Chainage of $B = 32.620$
Subtract the tangent length $= 6.934$

Chainage of $A = 25.686$
Add length of curve $= 12.989$

Chainage of $C = 38.675$

Taking the length of full chord equal to one chain, the length of the first sub-chord $= 26.0 - 25.686 = 0.314$ chain and the length of the last sub-chord $= 38.675 - 38.0 = 0.675$ chain.

The tangential angle for the full chord $= \dfrac{1718.9}{R} \times C$ min

$$= \frac{1718.9 \times 1}{15} = 114' \; 35''.6 = 1° \; 54' \; 35''.6$$

Tangential angle for the first sub-chord $= \dfrac{1718.9}{15} \times 0.314 = 35' \; 59''$

Tangential angle for the last sub-chord $= \dfrac{1718.9}{15} \times 0.675 = 1° \; 17' \; 23''$.

The table below shows the tangential angles for various points calculated as though the instrument would be stationed wholly at A. The method of setting out the points beyond chainage 31.0 has already been discussed in § 1.13.

Inst. At	Point on Curve	Chain-age	Tangential angle (δ)			Total tangential angle (Δ)			Actual theodolite reading			Remarks
			°	'	"	°	"	'	°	'	"	
A	1	26.0	0	35	59.0	0	35	59.0	0	36	00	
	2	27.0	1	54	35.6	2	30	34.6	2	30	40	
	3	28.0	1	54	35.6	4	25	10.2	4	25	20	
	4	29..0	1	54	35.6	6	19	45.8	6	19	40	
	5	30.0	1	54	35.6	8	14	21.4	8	14	20	
6	6	31.0	1	54	35.6	10	08	57.0	10	09	00	Inst. Oriented at 6 by back sighting to A
	7	32.0	1	54	35.6	12	03	32.6	12	03	40	
	8	33.0	1	54	35.6	13	58	08.2	13	58	00	
	9	34.0	1	54	35.6	15	52	43.8	15	52	40	
	10	35.0	1	54	35.6	17	47	19.4	17	47	20	
	11	36.0	1	54	35.6	19	41	55.0	19	42	00	
	12	37.0	1	54	35.6	21	36	30.6	21	36	40	
	13	38.0	1	54	35.6	23	31	06.2	23	31	00	Check :
	C	38.678	1	17	23.0	24	48	29.2	48	48	20	$\frac{1}{2}\Delta = 24° \; 48' \; 30''$

Example 1.6. *Two straights T_1V and VT_2 having bearings of 40° and 100° respectively, are to be connected by a 4° curve (based on chord of 20 m). Due to inaccessible intersection point, the following traverse is run from a point P on the rear tangents to a point S on the forward tangent.*

Line	*Length (m)*	*Bearing*
PQ	*110*	*60°*
QR	*90*	*130°*
RS	*180*	*30°*

The chainage of P is 1618.8 metres. Determine the chainage P.I., P.C. and P.T.

Solution.

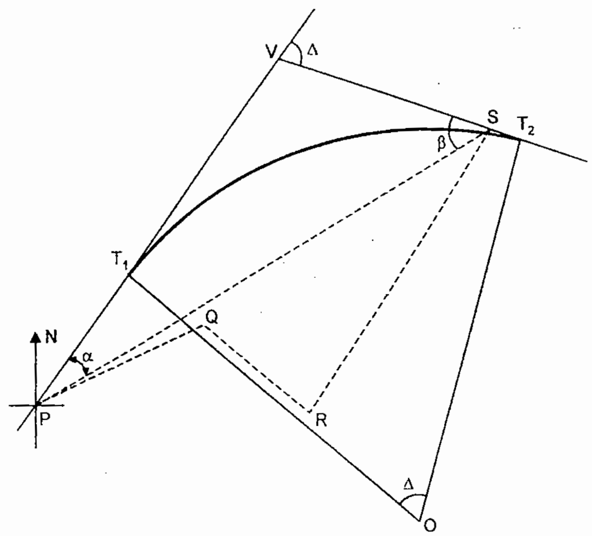


FIG. 1.25

The length and bearing of the line SP can be determined by considering the closed traverse $PQRS$, in which the lengths and bearings of PQ, QR and RS are known. The calculations for the omitted data of SP are done in the tabular from below :

Line	Length	Bearing	Latitude	Departure
PQ	110	N 60° E	+ 55.00	+ 95.26
QR	90	S 50° E	− 57.85	+ 68.24
RS	180	N 30° E	+ 155.89	+ 90.00
		Total	+ 153.04	+ 254.20
SP			− 153.04	− 254.20

The bearing θ of SP is given by

$$\theta = \tan^{-1}\frac{D}{L} = \tan^{-1}\frac{254.2}{153.04} = 58° 54'$$

Bearing of SP = S 58° 54′ W = 238° 54′

Bearing of PS = N 58° 54′ E

Length of $PS = D$ cosec 58° 54′ = 254.20 cosec 58° 54′ = 296.9 m

$\angle VPS$ = Bearing of PS – Bearing of PV = 58° 54′ – 40° = 18° 54′ = α

$\angle VSP$ = Bearing of SV – Bearing of SP

= (100° + 180°) – 238° 54′ = 41° 06′ = β

Total deflection angle Δ = 100° – 40° = 60° = $\alpha + \beta$

From triangle VPS, we have

$$PV = PS \cdot \frac{\sin\beta}{\sin\Delta} = 296.9\,\frac{\sin 41° 06'}{\sin 60°} = 225.4 \text{ m}$$

Radius of the curve is given by

$$R = \frac{1146}{D} = \frac{1146}{4} = 286.5 \text{ m}$$

Tangent length $T = T_1V = R\tan\frac{1}{2}\Delta$ = 286.5 tan 30° = 165.4 m

$$\text{Length of the curve} = \frac{\Delta}{D}\times 20 = \frac{60}{4}\times 20 = 300 \text{ m}$$

Chainge of P = 1618.8 metres
Add length PV = 225.4

Chainage of V = 1844.2
Subtract tangent length = 165.4

Chainage of T_1 = 1678.8
Add length of curve = 300.0

Chainage of T_2 = 1978.8

Example 1.7. *Two straight lines PQ and QR on the centre-line of a proposed road on a rocky headland are to be connected by a circular curve of 600 ft radius. From the traverse notes, it is found that if the bearing of PQ is assumed to be N 0° 0′ E the bearing of QR will be N 48° 20′ E while if P be taken as the origin of co-ordinates, the latiude and departure of R will be + 725 ft and + 365 ft respectively.*

Determine the distance of the tangent points of the curve from the stations P and R.

Solution.

With reference to Fig. 1.26, we have

$$\Delta = \angle R'QR = 48° 20'$$

Latitude of $R = PR'$ = 725′

Departure of $R = PR''$ = 365′

Length of $PR = \sqrt{(725)^2 + (365)^2}$

= 811.71 ft.

Bearing of $PR = \tan^{-1}\frac{D}{L}$

$= \tan^{-1}\frac{365}{725}$ = N 26° 43′ E

$\therefore$ $\angle QPR$ = 26° 43′

$\angle RQP$ = 180° – 48° 20′ = 131° 40′

$\angle QRP$ = 48° 20′ – 26° 43′ = 21° 37′

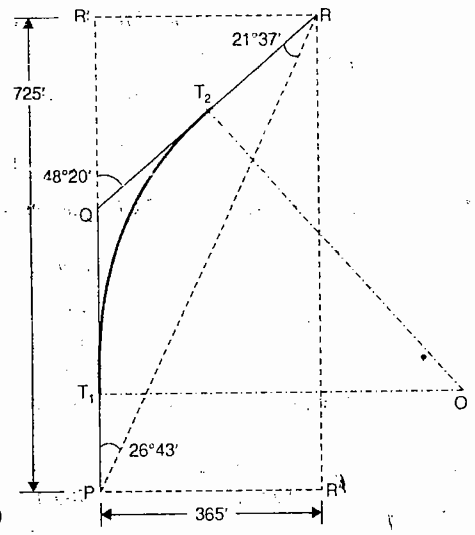


FIG. 1.26

From triangle QPR,

$$QP = \frac{PR}{\sin 48° 20'}\times\sin 21° 37'$$

$$= 811.71\,\frac{\sin 21° 37'}{\sin 48° 20'} = 400.30$$

and
$$QR = \frac{PR}{\sin 48° 20'}\times\sin 26° 43'$$

$$= 811.71\,\frac{\sin 26° 43'}{\sin 48° 42'} = 488.51$$

For the given circular curve, tangent distance is given by

$$T = QT_1 = QT_2 = R\tan\frac{\Delta}{2} = 600\tan\frac{48° 20'}{2} = 269.23$$

Distance $PT_1 = QP - QT_1$ = 400.30 – 269.23 = **131.07**

and Distance $RT_2 = QR - QT_2$ = 488.51 – 269.23 = **219.28**

Example 1.8. *Two straights T_1V and VT_2 are intersected by a third line AB. The angles VAB and VBA are measured to be 26° 24′ and 34° 36′, and the distance AB = 358 metres. Calculate the radius of the simple circular curve which will be tangential to the three lines T_1A, AB and BT_2 and the chainages of P.C. and P.T. if the chainage of V = 6857.3 metres.*

Solution. (Fig. 1.20)

Let the curve T_1CT_2 be tangential to three lines at T_1, C and T_2.

$\angle VAB = \alpha$ = 26° 24′

$\angle VBA = \beta$ = 34° 36′

For the arc T_1C, central angle $T_1OC = \alpha$ = 26° 24′

Tangent $T_1A = AC = R\tan\frac{1}{2}\alpha = R\tan 13° 12'$ ...(1)

Similarly, for the arc CT_2, the central angle

$T_2OC = \beta$ = 34° 36′

Tangent $T_2B = BC = R \tan \frac{1}{2}\beta = R \tan 17° 18'$ ...(2)

Adding (1) and (2), we get

$$AB = 358.0 = AC + BC = R \tan 13° 12' + R \tan 17° 18'$$

$$\therefore \quad R = \frac{358.0}{\tan 13° 12' + \tan 17° 18'} = \mathbf{655.7\ m}$$

For the whole curve T_1CT_2 tangent length $T_1V = R \tan \frac{1}{2}\Delta$

$$= 655.7 \tan \frac{(26° 24 + 34° 36')}{2} = 387.5 \text{ m}$$

$$\text{Length of the curve} = \frac{\pi R \Delta}{180°} = \frac{\pi \times 655.7 \times 61°}{180°} = 700 \text{ m}$$

Chainage of V = 6857.3 metres
Subtract tangent length = 387.5

Chainage of T_1 = 6469.8
Add curve length = 700.0

Chainage of T_2 = 7169.8

Example 1.9. ***Two straights T_1V and VT_2 of a road curve meet at an angle of 80°. Find the radius of curve which will pass through a point P, 30 metres from the P.I. (V), the angle T_1VP being 30°.***

Solution. (Fig. 1.11)

Let R = Radius of curve

Distance VP = 30 m

$\angle T_1VP = 30°$; $\angle T_1VT_2 = 80°$; $\Delta = 180° - 80° = 100°$.

From triangle VOP, $\angle PVO = (\frac{1}{2} \times 80°) - 30° = 10°$

Let $\angle T_1OP = \theta$

$\therefore \quad \angle POV = \frac{1}{2}\Delta - \theta = \frac{1}{2} \times 100° - \theta = 50° - \theta$

$\therefore \quad \angle VPO = 180° - (10° + 50° - \theta) = 120° + \theta$

By Sine Rule,
$$\frac{OV}{OP} = \frac{\sin (120° + \theta)}{\sin 10°}$$

or
$$\frac{R \sec 50°}{R} = \frac{\sin (120° + \theta)}{\sin 10°}$$

From which,

$$\sin(120° + \theta) = \frac{\sin 10°}{\cos 50°}$$

or
$$\cos (30° + \theta) = \frac{\cos 80°}{\cos 50°}$$

$\therefore \quad (30° + \theta) = 74° 19'$

$\therefore \quad \theta = 74° 19' - 30° = 44° 19'$

Now, from Eq. 1.19,
$$R = \frac{VP \cdot \sin 30°}{(1 - \cos \theta)} = \frac{30 \times \sin 30°}{1 - \cos 44° 19'} = 52.7 \text{ m.}$$

Example 1.10. *Two straights AV and VB, having bearings 146° 36′ and 86° 06′ respectively intersect at V and are connected by a curve of 200 metre radius. The co-ordinates of A and B are as under :*

Point	*Co-ordinate (metres)*	
	N	*E*
A	*212.6*	*60.4*
B	*100.2*	*486.8*

Give, in a tabular form the necessary calculations for setting out the curve by means of a 20″ theodolite, if the chainage of A = 4262.5 metres and the pegs are to be at interval of 20 metres.

Solution. (Fig. 1.27)

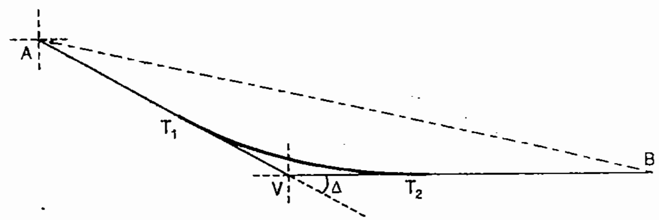


FIG. 1.27

In Fig. 1.27, AV and VB are two straights intersecting at V. The deflection angle Δ is given by

$$\Delta = \text{Bearing of } AV - \text{Bearing of } VB$$
$$= 146° 36' - 86° 06' = 60° 30' \text{ (left)}$$

Join A and B

Latitude of AB = North co-ordinate of B – North co-ordinate of A
= 100.2 – 212.6 = – 112.4

Departure of AB = East co-ordinate of B – East co-ordinate of A
= 486.8 – 60.4 = + 426.4

Bearing of AB is given by

$$\theta = \tan^{-1}\frac{D}{L} = \tan^{-1}\frac{426.4}{112.4} = S\ 75° 14' E = 104° 46'$$

$$\text{Length of } AB = \frac{L}{\cos \theta} = \frac{112.4}{\cos 75° 14'} = 441 \text{ m}$$

In the triangle AVB,

$$\angle BAV = \text{Bearing of } AV - \text{Bearing of } AB = 146° 36' - 104° 46' = 41° 50'$$

$$\angle ABV = \Delta - \angle BAV = 60° 30' - 41° 50' = 18° 40'$$

By sine rule, $AV = AB \dfrac{\sin ABV}{\sin \Delta} = 441 \dfrac{\sin 18° 40'}{\sin 60° 30'} = 162.2$ m

Let T_1 be the P.C. and T_2 be the P.T.

$$T_1V = VT_2 = R \tan \frac{\Delta}{2} = 200 \tan 30° 15' = 116.6 \text{ m.}$$

Length of curve $= \dfrac{\pi R \Delta}{180°} = \dfrac{\pi \times 200 \times 60° 30'}{180°} = 211.2$

Chainage of A = 4262.5 metres
Add length of AV = 162.2

Chainage of V = 4424.7
Subtract tangent length = 116.6

Chainage of T_1 = 4308.1
Add length of curve = 211.2

Chainage of T_2 = 4519.3

Since the chainage of the points on the curve is to be multiple of 20 m
Chainage of first point = 4320 m

Length of first sub-chord
= 4320 − 4308.1 = 11.9 m

Length of last sub-chord
= 4519.3 − 4500 = 19.3 m

Length of full chord = 20 m

Tangential angle δ_1 for first sub-chord

$$= \frac{1718.9\, c'}{R} = \frac{1718.9 \times 11.9}{200}$$

$$= 102'.27 = 1° 42' 16''.4$$

Tangential angle δ for normal chord

$$= \frac{1718.9 \times 20}{200} = 171'.89$$

$$= 2° 51' 53''.4$$

Tangential angle for last sub-chord

$$= \delta_n = \frac{1718.9 \times 19.3}{200}$$

$$= 165'.87 = 2° 45' 52''$$

No. of full chords $= \dfrac{4500 - 4320}{20} = 9$

∴ Total number of chords
= 1 + 9 + 1 = 11.

Since it is a left hand curve, theodolite readings will be (360° − Deflection angle) as tabulated below

Point	Chainage (m)	Chord Length (m)	δ			Δ			Actual theodolite Reading			Remarks
			°	′	″	°	′	″	°	′	″	
T_1	4308.1								360	0	0	
1	4320	11.9	1	42	16.4	1	42	16.4	358	17	40	
2	4340	20	2	51	53.4	4	34	09.8	355	26	00	
3	4360	20	2	51	53.4	7	26	03.2	352	34	00	
4	4380	20	2	51	53.4	10	17	56.6	349	42	00	
5	4400	20	2	51	53.4	13	09	50.0	346	50	00	
6	4420	20	2	51	53.4	16	01	43.4	343	58	20	
7	4440	20	2	51	53.4	18	53	36.8	341	06	20	
8	4460	20	2	51	53.4	21	45	30.2	338	14	20	
9	4480	20	2	51	53.4	24	37	23.6	335	22	40	
10	4500	20	2	51	53.4	27	29	17.0	332	30	40	
T_2	4519.3	19.3	2	45	52.0	30	15	9.0	329	45	00	

Example 1.11. *On the basis of preliminary survey, it was proposed to connect two straights, having deflection angle of 110°, by a circular curve of 400 metres radius. Later, however, it was decided to shift the forward tangent outward paralled to itself by a distance of 50 metres. Calculate (a) the new radius of the curve, and (b) chainages of new P.I. and P.T., if the position of original P.C. (chainage 9218.4 metres) is not to be changed.*

Solution. (Fig. 1.22).

Let V' and T_2' be the new P.I. and P.T., and R' be the new radius.

$$\angle VT_1T_2 = V'T_2'T_1 = \tfrac{1}{2}\Delta = 55°$$

$$T_2T_2' = \frac{p}{\sin \frac{1}{2}\Delta} = \frac{50}{\sin 55°} = 16 \text{ m}$$

$$VV' = \frac{p}{\sin \Delta} = \frac{50}{\sin 110°} = 53.2 \text{ m}$$

New tangent $T' = T_1V + VV'$

or $R' \tan \dfrac{\Delta}{2} = R \tan \dfrac{\Delta}{2} + 53.2$

or $R' = \dfrac{400 \tan 55° + 53.2}{\tan 55°} = \dfrac{571.3 + 53.2}{1.428} = \mathbf{437.3\ m}$

Length of the new curve $= \dfrac{\pi R' \Delta}{180°} = \dfrac{\pi \times 437.3 \times 110°}{180°} = 839.6$ m

Chainage of T_1	= 9218.4 metres
Add length T_1V	= 571.3
Chainage of old P.I.	= 9789.7
Add VV'	= 53.2
Chainage of new P.I.	= 9842.9
Chainage of new P.T.	= 9218.4 + 839.6 = **10058 m.**

Example 1.12. *Two tangents of a circular curve of radius 300 metres have a deflection angle of 90°. It is proposed to change the position of the forward tangent by rotating it through 20°, thus making the deflection angle equal to 110°. Calculate the radius of the new curve if P.C. is unchanged.*

If the chainage of original P.I. is 3240.8, calculate the chainages of new P.I. and new P.T.

Solution. (*a*) (Fig. 1.24)

$$\Delta' = \Delta + \theta = 90° + 20° = 110°$$

$$VV' = VT_2 \frac{\sin\theta}{\sin\Delta'} = T \frac{\sin\theta}{\sin\Delta'}$$

But $$T = R\tan\frac{\Delta}{2} = 300 \tan 45° = 300 \text{ m}$$

$\therefore$ $$VV' = 300 \frac{\sin 20°}{\sin 110°} = 300 \frac{\sin 20°}{\cos 20°} = 300 \tan 20° = 109.2 \text{ m}$$

Now $$T' = T_1V + VV' = T + VV'$$

or $$R' \tan\frac{\Delta'}{2} = 300 + 109.2 = 409.2$$

$\therefore$ $$R' = \frac{409.2}{\tan\frac{110°}{2}} = 286.5 \text{ m}$$

(*b*) Length of the new curve $= \dfrac{\pi R'\Delta'}{180°} = \dfrac{\pi(286.5)(110°)}{180°} = 550$ m

Chainage of old P.I.	= 3240.8
Add VV'	= 109.2
Chainage of new P.I.	= **3350.0**
Chainage of old P.I.	= 3240.8
Subtract old tangent length	= 300.0
Chainage of P.C.	= 2940.8
Add length of new curve	= 550.0
Chainage of new P.T.	= **3490.8 m**

Example 1.13. *With the same data as in example 1.12, calculate the new radius, if P.T. is unchanged and P.C. is changed. Also calculate the chainage of new P.I., new P.C. and the P.T.*

Solution. (Fig. 1.23)

$$\Delta' = \Delta + \theta = 90° + 20° = 110°$$

$$T = R\tan\frac{\Delta}{2} = 300 \tan\frac{90°}{2} = 300 \text{ m}$$

$$VV' = VT_2 \frac{\sin\theta}{\sin\Delta'} = T \cdot \frac{\sin 20°}{\sin 110°} = 300 \tan 20° = 109.2 \text{ m}$$

Also $$V'T_2 = VT_2 \cdot \frac{\sin\Delta}{\sin\Delta'}$$

or $$T' = T\frac{\sin\Delta}{\sin\Delta'} = 300 \frac{\sin 90°}{\sin 110°}$$

or $$R' \tan\frac{\Delta'}{2} = \frac{300}{\cos 20°}$$

$\therefore$ $$R' = \frac{300}{\cos 20°} \cdot \cot\frac{\Delta'}{2} = \frac{300 \cot 55°}{\cos 20°} = 223.5 \text{ m}$$

Length of new tangent $= T' = R' \tan\dfrac{\Delta'}{2} = 223.5 \tan 55° = 319.3$ m

Length of new curve $= \dfrac{\pi R'\Delta'}{180°} = \dfrac{\pi(223.5)(110°)}{180°} = 429.2$ m

Chainage of old P.I.	= 3240.8
Add VV'	= 109.2
$\therefore$ Chainage of new P.I.	= **3350.0**
Subtract new tangent length	= 319.3
Chainage of new P.C.	= **3030.7**
Add length of new curve	= 429.2
Chainage of P.T.	= **3459.9**

PROBLEMS

1. Two roads meet at an angle of 127° 30′. Calculate the necessary data for setting out a curve of 15 chains radius to connect the two straight portions of the road (*a*) if it is intended to set out the curve by chain and offsets only, (*b*) if a theodolite is available. Explain carefully how you would, in both cases, set out the curve in the field. (*U.L.*)

2. Calculate the ordinates at 5 m distances for a circular curve having a long chord of 40 metres and a versed sine of 2 m.

3. What are the common difficulties in setting out simple curves ? Describe briefly the method employed in overcoming them.

4. The chainage at the point of intersection of the tangents to a railway curve is 3876 links, and the angle between them is 124°.

Find the chainage at the beginning and end of the curve if it is 40 chains radius, and calculating the angle which are required in order to set out this cure (*a*) with a theodolite, (*b*) with a chain and tape only.

5. The tangents to a railway meet at an angle of 148°. Owing to the position of a building, a curve is to be chosen that will pass near a point 10 metres from the point of intersection of the tangents on the bisector of the angle 148°. Calculate the suitable radius of the curve.

6. The intersection point *C* of two railway straights *ABC* and *CDE* is inaccessible and so convenient points *B* and *C* in the straights are selected giving $BD = 6.10$ chains, and $\angle CBD = 9° 24'$ and $\angle CDB = 10° 36'$ and the forward chainage of $B = 90.50$ chains. The conditions of the site are such that it is decided to make *B* the first tangent point. Determine the radius of a circular curve to connect the straights, tabulate all data necesary to set out pegs at 1 chain intervals of through chainage and show that your calculations are checked.

7. Two straights of a proposed road deflect through an angle of 120°. Originally, they were to be connected by a curve of 520 metres radius. However, due to the revision of the scheme, the deflection angle is to be increased to 132°. Calculate the suitable radius of the curve such that the original starting point of the curve (P.C.) does not change.

8. On the basis of preliminary survey, it was proposed to connect two straights, having deflection angle of 112°, by a circular curve of 400 metres radius, and the direction of both the tangents were set out in the field. However, while setting out the curve, it was thought desirable to change the radius to 450 metres without changing the direction of the forward tangent. Calculate the distance by which the forward tangent must be shifted parallel to itself so that the point of curvature (P.C.) remains unaltered.

ANSWERS

1. (*a*) $O_1 = 3\frac{1}{3}$ links, O_2 ...... $O_{13} = 6\frac{2}{3}$ links, $O_{14} = 4\frac{1}{3}$ links
   (*b*) $\delta_1 = \delta_2 .... = \delta_{13} = 1° 54'.6$, $\delta_{14} = 1° 25'.3$.
2. $O_o = 2$ m ; $O_5 = 1.88$ m ; $O_{10} = 1.50$ m ; $O_{15} = 0.88$ m.
4. 17.492 chains and 56.587 chains
   (*a*) $\delta_1 = 21'.83$; δ_2 to $\delta_{39} = 42'.97$; $\delta_{40} = 25'.22$
   (*b*) $O_1 = 0.32$ links; $O_2 = 1.88$ links; O_3 to $O_{39} = 2.5$ links;
   $O_{40} = 1.16$ links.
5. $R = 248.1$ m.
6. 18.61 chains ; $\delta_1 = 46' 11''$; $\delta_2 = 2° 18' 34''$.
7. 513.2 m.
8. 68.73 m.

2

Curve Surveying II : Compound and Reverse Curves

2.1. ELEMENTS OF A COMPOUND CURVE

In Fig. 2.1, T_1DT_2 is a *two centred compound curve* having two circular arcs T_1D and DT_2 meeting at a common point *D* known as the point of compound curvature (P.C.C.). T_1 is the *point of curve* (P.C.) and T_2 is the *point of tangency* (*P.T.*). O_1 and O_2 are the centres of the two arcs.

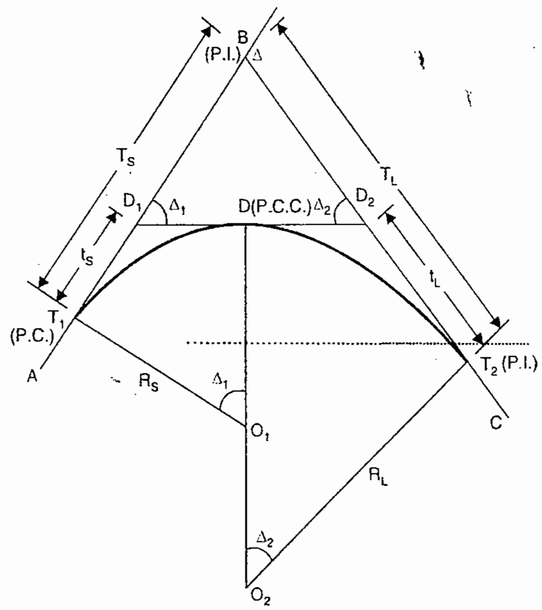


FIG. 2.1. TWO CENTRED COMPOUND CURVE.

R_S = the smaller radius (T_1O_1)
R_L = the longer radius (T_2O_2)
D_1D_2 = common tangent
Δ_1 = deflection angle between the rear and the common tangent
Δ_2 = deflection angle between the common and the forward tangent
Δ = total deflection angle
t_S = the length of the tangent to the arc (T_1D) having a smaller radius
t_L = the length of the tangent to the arc DT_2 having a longer radius
T_S = tangent distance T_1B corresponding to the shorter radius
T_L = tangent distance BT_2 corresponding to the longer radius

From Fig. 2.1, we have

$$t_S = T_1D_1 = D_1D = R_S \tan \tfrac{1}{2} \Delta_1 \quad ...[2.1\ (a)]$$

$$t_L = T_2D_2 = D_2D = R_L \tan \tfrac{1}{2} \Delta_2 \quad \text{...[2.1 (b)]}$$

$$\Delta = \Delta_1 + \Delta_2 \quad \text{...(2.2)}$$

From triangle BD_1D_2, we have

$$D_1B = D_1D_2 \cdot \frac{\sin \Delta_2}{\sin \Delta} = (t_S + t_L) \frac{\sin \Delta_2}{\sin \Delta}$$

and
$$D_2B = D_1D_2 \cdot \frac{\sin \Delta_1}{\sin \Delta} = (t_S + t_L) \frac{\sin \Delta_1}{\sin \Delta}$$

∴
$$T_S = T_1D_1 + D_1B = t_S + (t_S + t_L) \frac{\sin \Delta_2}{\sin \Delta} \quad \text{...[2.3 (a)]}$$

and
$$T_L = T_2D_2 + D_2B = t_L + (t_S + t_L) \frac{\sin \Delta_1}{\sin \Delta} \quad \text{...[2.3 (b)]}$$

2.2. RELATIONSHIP BETWEEN THE PARTS OF A COMPOUND CURVE

There are seven quantities of two centred compound curve, *i.e.* Δ, R_S, T_S, Δ_1, R_L, T_L and Δ_2. When four of these, including an angle, are given or assumed, the other three may be determined. Usually, the lengths of the two radii, R_S and R_L are established from the plan and the angle Δ is measured in the field. The fourth quantity out of Δ_1, Δ_2, T_S and T_L is usually selected from the plan. The calculation of the rest of the three quantities may be made as explained below :

Case (1) : *Given* : Δ, R_S, R_L, Δ_1 (or Δ_2)

Required : Δ_2 (or Δ_1), T_S and T_L

This is the most common case.

From equation 2.2, $\Delta_2 = \Delta - \Delta_1$

(or $\Delta_1 = \Delta - \Delta_2$)

From equation 2.3 and 2.1, we have

$$T_S = R_S \tan \tfrac{1}{2}\Delta_1 + \left(R_S \tan \tfrac{1}{2} \Delta_1 + R_L \tan \tfrac{1}{2} \Delta_2\right) \frac{\sin \Delta_2}{\sin \Delta}$$

and
$$T_L = R_L \tan \tfrac{1}{2} \Delta_2 + \left(R_S \tan \tfrac{1}{2} \Delta_1 + R_L \tan \tfrac{1}{2} \Delta_2\right) \frac{\sin \Delta_1}{\sin \Delta}$$

Case (2) : *Given* : Δ, R_S, R_L and T_S.

Required : Δ_1, Δ_2 and T_L.

In Fig. 2.2, prolong the short curve T_1D to a point D' until it has the central angle Δ. Its tangent $B'D'$ will then be parallel to the tangent BT_2.

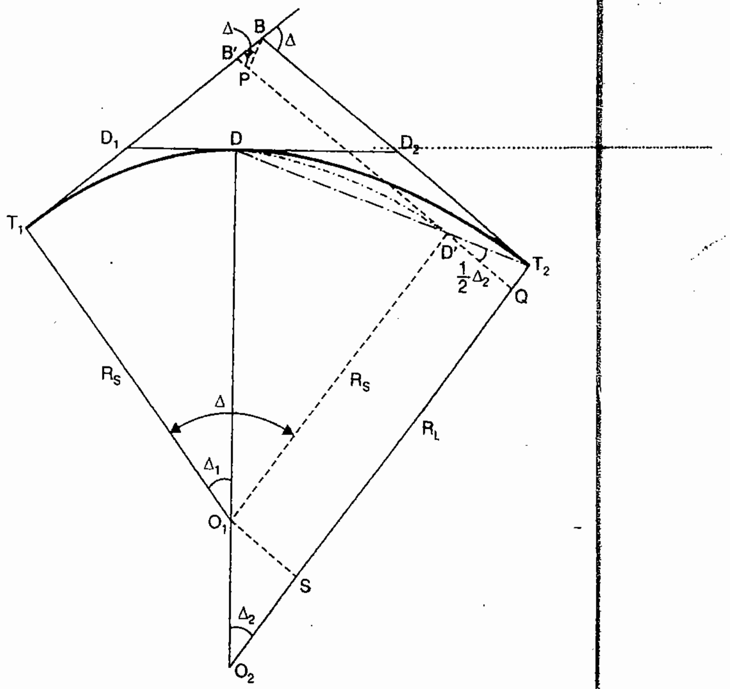


FIG 2.2. CALCULATION OF Δ_1, Δ_2 AND T_L

Then
$$T_1B' = B'D' = R_S \tan \tfrac{1}{2} \Delta \quad \text{...(1)}$$

Draw BP perpendicular to $B'D'$.

Prolong $B'D'$ to meet O_2T_2 in Q. Draw O_1S perpendicular to O_2T_2.

Then
$$T_2Q = BP = BB' \sin \Delta = (T_1B - T_1B') \sin \Delta = (T_S - R_S \tan\tfrac{1}{2} \Delta) \sin \Delta \quad \text{...(2)}$$

Also,
$$B'P = BB' \cos \Delta = (T_S - R_S \tan \tfrac{1}{2} \Delta) \cos \Delta \quad \text{...(3)}$$

Now,
$$O_2S = O_2T_2 - T_2Q - QS = R_L - BP - R_S \quad \text{...(4)}$$

From triangle O_1O_2S,
$$\cos \Delta_2 = \frac{O_2S}{R_L - R_S} \quad \text{...(5)}$$

$$\Delta_1 = \Delta - \Delta_2 \quad \text{...(6)}$$

and
$$O_1S = (R_L - R_S) \sin \Delta_2 \quad \text{...(7)}$$

$$T_L = BT_2 = PQ = QD' + B'D' - B'P = O_1S + B'D' - B'P \quad \text{...(8)}$$

Thus Δ_1, Δ_2 and T_L are determined from (5), (6) and (8) above.

Case (3) : *Given* : Δ, R_S, R_L and T_L

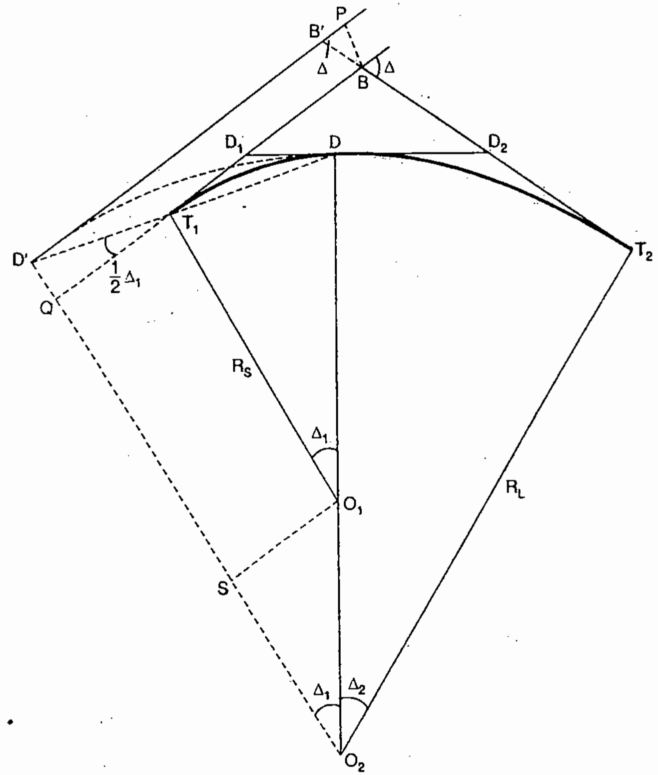


FIG. 2.3. CALCULATION OF Δ_1, Δ_2 AND T_S

Required : Δ_1, Δ_2 and T_S

In Fig. 2.3, prolong the long curve T_2D to a point D' until it has a central angle $D'O_2T_2 = \Delta$. Its tangent $B'D'$ will then be parallel to the tangent BT_1.

Then, $T_2B' = B'D' = R_L \tan \frac{1}{2}\Delta$ ...(1)

Braw BP perpendicular to $D'B'$.

Prolong BT_1 to meet $D'O_2$ in Q. Draw O_1S perpendicular to $D'O_2$.

Then, $D'Q = BP = BB' \sin \Delta = (T_2B' - T_2B) \sin \Delta$

$= (R_L \tan \frac{1}{2}\Delta - T_L) \sin \Delta$ ...(2)

Also, $B'P = BB' \cos \Delta = (R_L \tan \frac{1}{2}\Delta - T_L) \cos \Delta$ ...(3)

Now, $O_2S = O_2D' - D'Q - QS = R_L - BP - R_S$ ...(4)

From triangle O_1O_2S, $\cos \Delta_1 = \dfrac{O_2S}{R_L - R_S}$ ...(5)

$\Delta_2 = \Delta - \Delta_1$ ...(6)

$O_1S = (R_L - R_S) \sin \Delta_1$ ...(7)

and $T_S = T_1B = QB - QT_1 = D'P - O_1S$

$= D'B' + B'P - O_1S$ ...(8)

Thus, Δ_1, Δ_2 and T_S are determind from (5), (6) and (8) above.

Case (4) : Given Δ, T_S, T_L and R_S.

Required : Δ_1, Δ_2 and R_L.

Refer Fig. 2.2,

As in case (2), we have

$T_1B' = B'D' = R_S \tan \frac{1}{2}\Delta$ ...(1)

$T_2Q = (T_S - R_S \tan \frac{1}{2}\Delta) \sin \Delta$ ...(2)

and $B'P = B'B \cos \Delta = (T_S - R_S \tan \frac{1}{2}\Delta) \cos \Delta$ ...(3)

Now $O_1S = QD' = QP + PB' - B'D' = T_L + B'P - B'D'$ ...(4)

Join DD' and prolong it to pass through T_2.

Evidently. $\angle T_2D'Q = \frac{1}{2}\Delta_2$

$\therefore$ $\tan \frac{1}{2}\Delta_2 = \dfrac{T_2Q}{QD'} = \dfrac{BP}{QD'}$ ...(5)

$\Delta_1 = \Delta - \Delta_2$ ...(6)

$R_L - R_S = O_1O_2 = \dfrac{O_1S}{\sin \Delta_2}$

$\therefore$ $R_L = R_S + \dfrac{O_1S}{\sin \Delta_2}$ ...(7)

Thus, Δ_1, Δ_2 and R_L can be computed from (5), (6) and (7) above.

Case (5) : ***Given*** : T_S, T_L and R_L

Required : Δ_1, Δ_2 and R_S

Refer Fig. 2.3. As in case (3), we have

$T_2B' = B'D' = R_L \tan \frac{1}{2}\Delta$ ...(1)

$D'Q = BP = (R_L \tan \frac{1}{2}\Delta - T_L) \sin \Delta$ ...(2)

and $B'P = BB' \cos \Delta = (R_L \tan \frac{1}{2}\Delta - T_L) \cos \Delta$ ...(3)

Now $O_1S = QT_1 = QB - T_1B = D'P - T_1B = D'B' + B'P - T_S$ ...(4)

Join DT_1 and prolong it to pass through D'.

Evidently $\angle D'T_1Q = \frac{1}{2}\Delta_1$

$\therefore$ $\tan \frac{1}{2}\Delta_1 = \dfrac{D'Q}{QT_1} = \dfrac{BP}{O_1S}$ ...(5)

$\Delta_2 = \Delta - \Delta_1$ ...(6)

$R_L - R_S = O_1O_2 = \dfrac{O_1S}{\sin \Delta_1}$.

$R_S = R_L - \dfrac{O_1S}{\sin \Delta_1}$ ...(7)

Thus, Δ_1, Δ_2 and R_S can be computed from (5), (6) and (7).

2.3. SETTING OUT COMPOUND CURVE

The compound curve can be set by method of deflection angles. The first branch is set out by setting the theodolite at T_1 (P.C.) and the second branch is set out by setting the theodolite at the point D (P.C.C.). The procedure is as follows :

(1) After having known any four parts, calculate the rest of the three parts by the formulae developed in § 2.2.

(2) Knowing T_S and T_L, locate points T_1 and T_2 by linear measurements from the point of intersection.

(3) Calculate the length of curves l_S and l_L. Calculate the chainage of T_1, D and T_2 as usual.

(4) For the first curve, calculate the tangential angles etc., for setting out the curve by Rankine's method.

(5) Set the theodolite at T_1 and set out the first branch of the curve as already explaind.

(6) After having located the last point D (P.C.C) shift the theodolite to D and set it there. With the vernier set to $\left(360° - \dfrac{\Delta_1}{2}\right)$ reading, take a backsight on T_1 and plunge the telescope. The line of sight is thus oriented along T_1D produced and if the theodolite is now swung through $\dfrac{\Delta_1}{2}$, the line of sight will be directed along the common tangent DD_2. Thus the theodolite is correctly oriented at D.

(7) Calculate the tangential angles for the second branch and set out the curve by observations from D, till T_2 is reached.

(8) Check the observations by measuring the angle T_1DT_2, which should be equal to $\left(180° - \frac{\Delta_1 + \Delta_2}{2}\right)$ or $\left(180° - \frac{\Delta}{2}\right)$.

Example 2.1. *Two straights AB and BC are intersected by a line D_1D_2 (Fig. 2.1). The angles BD_1D_2 and BD_2D_1 are 40° 30′ and 36° 24′ respectively. The radius of the first arc is 600 metres and that of the second arc is 800 metres. If the chainage of intersection point B is 8248.1 metres, find the chainages of the tangent points and the point of compound curvature.*

Solution.

$$\angle BD_1D_2 = \Delta_1 = 40° 30'$$
$$\angle BD_2D_1 = \Delta_2 = 36° 24'$$
$$\therefore \quad \Delta = \Delta_1 + \Delta_2 = 40° 30' + 36° 24' = 76° 54'$$

For the first branch, the central angle $= \Delta_1 = 40° 30'$

$$t_S = T_1 D_1 = D_1D = R_S \tan\frac{\Delta_1}{2} = 600 \tan 20° 15' = 221.4 \text{ m}$$

For the second branch, the central angle $= \Delta_2 = 36° 24'$

$$\therefore \quad t_L = T_2 D_2 = D_2D = R_L \tan\frac{\Delta_2}{2} = 800 \tan 18° 12' = 263 \text{ m}$$
$$\therefore \quad D_1D_2 = D_1D + DD_2 = 221.4 + 263 = 484.4 \text{ m}$$

From triangle BD_1D_2, we have

$$\frac{BD_1}{\sin\Delta_2} = \frac{D_1D_2}{\sin\Delta}$$
$$BD_1 = D_1D_2 \frac{\sin\Delta_2}{\sin\Delta} = 484.4 \times \frac{\sin 36° 24'}{\sin 76° 54'} = 295.1 \text{ m}$$

Length of the first arc $= l_1 = \frac{\pi R_S \Delta_1}{180°} = \frac{\pi \times 600 \times 40° 30'}{180°} = 424.1$ m

Length of the second arc $= l_2 = \frac{\pi R_L \Delta_2}{180°} = \frac{\pi \times 800 \times 36° 24'}{180°} = 508.2$ m

$$T_S = BD_1 + T_1D_1 = 295.1 + 221.4 = 516.5 \text{ m}$$

Chainage of P.I.	= 8248.1 m
Subtract T_S	= 516.5
Chainage of T_1	= **7731.6**
Add length l_1	= 424.1
Chainage of P.C.C.	= **8155.7**
Add length l_2	= 508.2
Chainage of T_2	= **8663.9**

Example 2.2. *The following data refer to a compound circular curve which bears to the right :*

Total deflection angle 93°

Degree of first curve 4°

Degree of second curve 5°

Point of intersection at 45 + 61 (20 m units)

Determine in 20 metre units the running distance of the tangent points and the point of compound curvature, given that the latter point is 6 + 24 from the point of intersection at back angle of 290° 36′ from the first tangent.

Solution. (Fig. 2.1)

Let R_1 = Radius corresponding to 4° curve

and R_2 = Radius corresponding to 5° curve

From Eq. 1.3, $R = \frac{1146}{D}$ metres

$$\therefore \quad R_1 = \frac{1146}{4} = 286.5 \text{ m}$$

and $R_2 = \frac{1146}{5} = 229.2$ m

$$BD = 6 + 24 \text{ in } 20 \text{ m units} = 6.24 \times 20 = 124.8 \text{ m}$$
$$\angle T_1BD \text{ (external)} = 290° 36'$$
$$\therefore \quad \angle T_1BD \text{ (internal)} = 360° - 290° 36' = 69° 24'$$

From triangle T_1BD,

$$\frac{\sin BT_1D}{BD} = \frac{\sin T_1BD}{T_1D}$$

But $\angle BT_1D = \frac{\Delta_1}{2}$

and $T_1D = 2R_1 \sin\frac{\Delta_1}{2} = 573 \sin\frac{\Delta_1}{2}$

Hence $\frac{\sin\frac{\Delta_1}{2}}{124.8} = \frac{\sin 69° 24'}{573 \sin\frac{\Delta_1}{2}}$

or $\left(\sin\frac{\Delta_1}{2}\right)^2 = \frac{124.8}{573} \times \sin 69° 24'$

or $\sin\frac{\Delta_1}{2} = \left(\frac{124.8}{573} \times \sin 69° 24'\right)^{1/2}$

From which $\frac{\Delta_1}{2} = 26° 50'$, or $\Delta_1 = 53° 40'$

$$\Delta_2 = \Delta - \Delta_1 = 93° - 53° 40' = 39° 20'$$

$$t_1 = T_1D_1 = R_1 \tan \frac{\Delta_1}{2} = 286.5 \tan 26° 50' = 144.9 \text{ m}$$

$$t_2 = T_2D_2 = R_2 \tan \frac{\Delta_2}{2} = 229.2 \tan 19° 40' = 81.9 \text{ m}$$

$$T_1 = T_1B = t_1 + (t_1 + t_2) \frac{\sin \Delta_2}{\sin \Delta}$$

$$= 144.9 + (144.9 + 81.9) \frac{\sin 39° 20'}{\sin 93°} = 288.9 \text{ m}$$

Length of the first arc = l_1

$$= \frac{\Delta_1}{D_1} \times 20 = \frac{53° 40'}{4°} \times 20 = 268.3 \text{ m}$$

Length of the second arc = l_2

$$= \frac{\Delta_2}{D_2} \times 20 = \frac{39° 20'}{5°} \times 20 = 157.3 \text{ m}$$

Chainage of P.I = 912.2 m	= 45 + 610 (in 20 m units)	
Subtract T_1 = 288.9	= 14 + 445	
Chainage of P.C. = 623.3	= **31.165**	
Add l_1 = 268.3	= 13.415	
Chainage of P.C.C. = 891.6	= **44.580**	
Add l_2 = 157.3	= 7.865	
Chainage of P.T. = 1048.9	= **52.445**.	

Example 2.3. *A compound curve is to consist of an arc of 36 chains followed by one of 48 chains radius and is to connect two straights which yield a deflection angle of 84° 30′. At the intersection point the chainage, if continued along the first tangent, would be 86 + 48 and starting point of the curve is selected at chainage 47 + 50. Calculate the chainage at the point of junction of the two branches and at the end of the curve.*

Solution. (Fig. 2.2).

Here, R_S, R_L, Δ and T_S are given . In order to calculate the chainages of various points, we will have to first determine Δ_1 and Δ_2.

$$T_S = 86.48 - 47.50 = 38.98 \text{ chains.}$$

As in Fig. 2.2, prolong the shorter arc to a point D' so that its central angle is equal to $\Delta = 84° 30'$. The tangent $D'B'$ will then be parallel to intitial tangent BT_2. Draw BP perpendicular to $B'D'$.

Then $T_2Q = BP = BB' \sin \Delta$

$$= (T_1B - T_1B') \sin \Delta = (T_S - R_S \tan \tfrac{1}{2} \Delta) \sin \Delta$$

$$= (38.98 - 36 \tan 42° 15') \sin 84° 30' = 6.26 \text{ chains}$$

$$O_2S = O_2T_2 - T_2Q - QS = R_L - T_2Q - R_S$$

$$= 48 - 6.26 - 36 = 5.74 \text{ chains}$$

$$\cos \Delta_2 = \frac{O_2S}{R_L - R_S} = \frac{5.74}{48 - 36}$$

$\therefore$ $\Delta_2 = 61° 24$.

$\therefore$ $\Delta_1 = 84° 30' - 61° 24' = 23° 6'$

Length of the first arc $= l_1 = \frac{\pi R_S \Delta_1}{180°} = \frac{\pi \times 36 \times 23° 6'}{180°} = 14.52$ chains

Length of the second arc $= l_2 = \frac{\pi R_L \Delta_2}{180°} = \frac{\pi \times 48 \times 61° 24'}{180°} = 51.44$ chains

Chainage of P.C. =	47.50 chains
Add length of the first arc =	14.52
Chainage of P.C.C. =	**62.02**
Add length of the second arc =	51.44
Chainage of end of curve =	**113.46**

Example 2.4. *A compound curve is to connect two straights having a deflection angle of 90 °. As determined from the plan, the lengths of the two tangents are 350 metres and 400 metres repectively. Calculate the lengths of the two arcs if the radius of the first curve is be 300 metres.*

Solution. (Fig. 2.2)

Given $T_S = 350$ m

$T_L = 400$ m

$\Delta = 90°$

$R_S = 300$ m

Required to find Δ_1 , Δ_2 and R_L

$$T_1B' = B'D' = R_S \tan \tfrac{1}{2}\Delta = 300 \tan 45° = 300 \text{ m}$$

$$T_2Q = BP = (T_S - R_S \tan \tfrac{1}{2} \Delta) \sin \Delta = (350 - 300 \tan 45°) \sin 90° = 50 \text{ m}$$

$$B'P = BB' \cos 90° = \text{zero}$$

$$O_1S = QD' = QP + PB' - B'D' = T_L + 0 - 300 = 400 - 300 = 100 \text{ m}$$

$$\tan \tfrac{1}{2} \Delta_2 = \frac{T_2Q}{QD'} = \frac{BP}{QD'} = \frac{50}{100} = 0.5$$

$\therefore$ $\tfrac{1}{2}\Delta_2 = 26° 34'$

or $\Delta_2 = 53° 8'$

$$\Delta_1 = \Delta - \Delta_2 = 90° - 53° 8' = 36° 52'.$$

Also, $R_L - R_S = \dfrac{O_1S}{\sin \Delta_2}$

$\therefore \quad R_L = R_S + \dfrac{O_1S}{\sin \Delta_2} = 300 + \dfrac{100}{\sin 53° 8'} = \mathbf{425\ m}$

$\therefore$ Length of the first arc $= \dfrac{\pi R_S \Delta_1}{180°} = \dfrac{\pi(300)(36° 52')}{180°} = \mathbf{193.1\ m}$

Length of the second arc $= \dfrac{\pi R_L \Delta_2}{180°} = \dfrac{\pi(425)(53° 8')}{180°} = \mathbf{394.2\ m}$

Example 2.5. *A 200 m length of straight connects two circular curves which both deflect to the right. The radius of the first curve is 250 m and that of the second curve is 200 m. The central angle for the second curve is 27° 30′. The combined curve is to be replaced by a single circular curve between the same tangent points. Find the radius of the curve. Assume that the two tangent lengths of the earlier set are equal.*

Also, determine (a) central anlge of the new curve, (b) central angle of first curve of radius 250 m.

Solution.

Fig. 2.4 shows the two curves T_1A and BT_2 separated by the intervening straight AB of length 200 m. The data of the two curves are such that the tangent lengths T_1D and DT_2 are equal, where D is the point of intersection of the tangents. *This is*

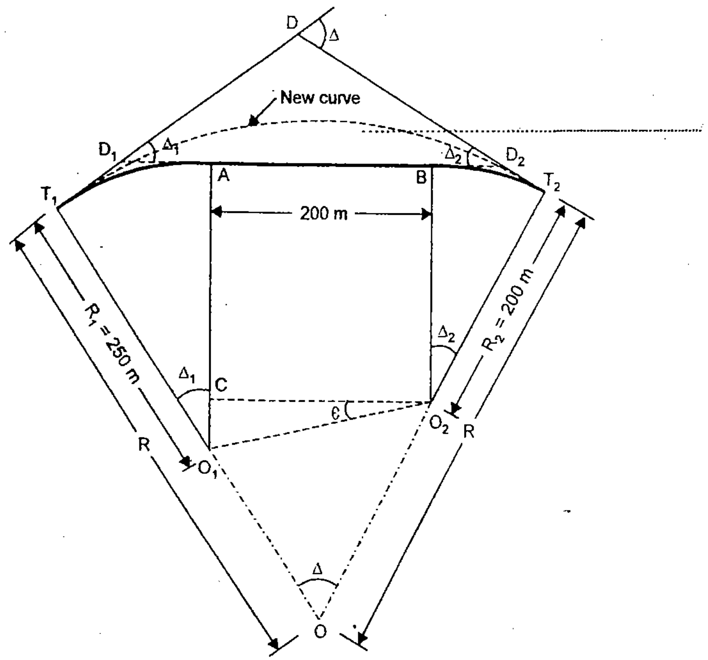


FIG. 2.4

an essential condition to replace the two curves and the straight by a single circular curve without the change in the direction of original tangents and also without shifting the original tangent points. Naturally, the central angle (Δ) for the new curve will be the original angle of intersection between the two tangents.

Let O be the centre of the new curve, and R be its radius. The centre will coincide with the point of intersection of the external radius vectors T_1O_1 and T_2O_2 (both produced back) of the two original curves.

Join O_1O_2. Draw O_2C perpendicular to O_1A.

Then $\quad O_1O_2 = \sqrt{O_1C^2 + O_2C^2}$,

But $\quad O_1C = O_1A - O_2B = 250 - 200 = 50$ m $\quad$ and $\quad O_2C = AB = 200$ m

$\therefore \quad O_1O_2 = \sqrt{(50)^2 + (200)^2} = 206.2$ m

$$\theta = \tan^{-1}\frac{O_1C}{O_2C} = \tan^{-1}\frac{50}{200} = 14° 2'$$

From triangle O_1OO_2,

$$\angle O_1O_2O = 180° - (\theta + 90° + \Delta_2) = 180° - (14° 2' + 90° + 27° 30') = 48° 28'$$

Now, $\quad O_1O = R - 250$ and $O_2O = R - 200$

Applying cosine rule,

$$O_1O^2 = (O_1O_2)^2 + (OO_2)^2 - 2O_1O_2 \, . \, OO_2 \cos 48° 28'$$

or $\quad (R - 250)^2 = (206.2)^2 + (R - 200)^2 - 2 \times (206.2)(R - 200) \cos 48° 28'$

From which $\quad R = \mathbf{430.6}$ m

In order to calculate central angle Δ of the new curve, consider triangle O_1O_2O, in which we have

$$\angle O_1O_2O = 48° 28'$$

$$O_1O_2 = 206.2 \text{ m}$$

$$O_1O = R - 250 = 430.6 - 250 = 180.6 \text{ m}$$

Applying the sine rule,

$$\frac{O_1O_2}{\sin \Delta} = \frac{O_1O}{\sin O_1O_2O}$$

$\therefore \quad \sin \Delta = \dfrac{O_1O_2}{O_1O} \times \sin O_1O_2O = \dfrac{206.2}{180.6} \times \sin 48° 28' = 0.8547$

$\therefore \quad \Delta = \mathbf{58° 43'}$

Now from triangle DD_1D_2,

$$\Delta = \Delta_1 + \Delta_2$$

$\therefore \quad \Delta_1 = \Delta - \Delta_2 = 58° 43' - 27° 30' = \mathbf{31° 13'}$.

REVERSE CURVES

2.4. ELEMENTS OF A REVERSE CURVE

A reverse curve consists of two simple curves of opposite direction that join at a common tangent point called the point of reverse curvature (P.R.C.). They are used when the straights are parallel or include a very small angle of intersection and are frequently encountered in mountaineous countries, in cities, and in the layout of railway spur tracks and cross-over. The use of reverse curve should be avoided on highways and main railway lines where speeds are high for the following reasons :

(1) Sudden change of cant is required from one side of P.R.C. to the other.

(2) There is no opportunity to elevate the outer bank at P.R.C.

(3) The sudden change of direction is uncomfortable to passengers and is objectionable.

(4) Steering is dangerous in the case of highways and the driver has to be very cautious.

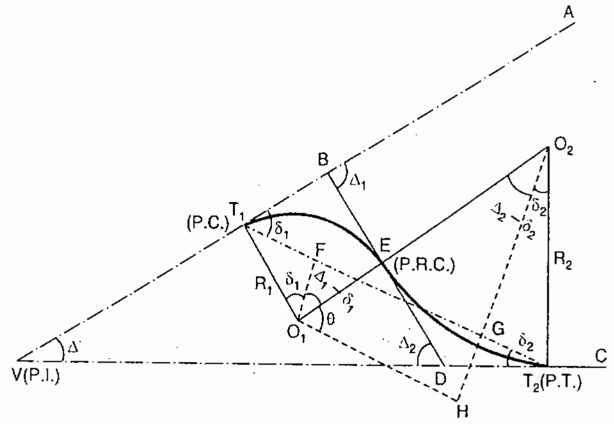


FIG. 2.5. REVERSE CURVE ($\Delta_1 > \Delta_2$)

It is definitely an advantage to separate to curves by either a short length of straight or a reversed spiral. The elements of a reverse curve are not directly determinate unless some condition or dimension is specified as, for example, equal radii ($R_1 = R_2$) or equal central angle ($\Delta_1 = \Delta_2$). Frequently, a common or equal radius is used for both parts of the curve in order to use largest radius possible.

Fig. 2.5 shows the general case of a reverse curve in which VA and VC are the two straights and T_1ET_2 is reverse curve. T_1 is the point of curvature (P.C.), E is the point of reverse curvature (P.R.C.) and T_2 is the point of tangency (P.T.). O_1 and O_2 are the centres of the two branches. BD is the common tangent.

Let R_1 = the smaller radius

R_2 = the greater radius

Δ_1 = central angle for the curve having smaller radius

Δ_2 = central angle for the curve having greater radius (Δ_1 is greater than Δ_2)

Δ = total deviation between the tangents

δ_1 = angle between tangent AV and the line T_1T_2 joining the tangent points

δ_2 = angle between tangent VC and the line T_2T_1 joining the tangent points.

Since E is the point of reverse curvature, the line O_1O_2 is perpendicular to the common tangent BD at E. Join T_1 and T_2 and drop perpendiculars O_1F and O_2G on it from O_1 and O_2 respectively. Through O_1, draw O_1H parallel to T_1T_2 to cut the line O_2G produced in H.

Since T_1B and BE are tangents to the first arc, $\angle ABE = \Delta_1$. Similarly, since ED and DT_2 are tangents to the second arc, $\angle EDV = \Delta_2$.

From triangle BVD, $\Delta_1 = \Delta + \Delta_2$

or $\Delta = \Delta_1 - \Delta_2$ ... (1) ...(2.4)

From triangle T_1VT_2, $\delta_1 = \Delta + \delta_2$

$\Delta = \delta_1 - \delta_2$ ... (2)

From (1) and (2), $\Delta_1 - \Delta_2 = \delta_1 - \delta_2$ ... (3) ...(2.5)

Since T_1O_1 is $\perp$ to T_1B and O_1F is $\perp$ to T_1T_2 we have

$\angle T_1O_1F = \angle BT_1F = \delta_1$

Similarly, $\angle T_2O_2G = \angle FT_2D = \delta_2$

Hence $\angle FO_1E = \Delta_1 - \delta_1$ and $\angle EO_2G = \Delta_2 - \delta_2$

Since O_1F and O_2G are parallel, we have

$\angle FO_1E = EO_2G$

or $(\Delta_1 - \delta_1) = (\Delta_2 - \delta_2)$ ... (3a)

which is the same as obtained in (3).

Again, $T_1F = R_1 \sin \delta_1$

$T_2G = R_2 \sin \delta_2$

and $FG = O_1H = O_1O_2 \sin (\Delta_2 - \delta_2) = (R_1 + R_2) \sin (\Delta_2 - \delta_2)$

Hence $T_1T_2 = T_1F + T_2G + FG$

or $T_1T_2 = R_1 \sin \delta_1 + R_2 \sin \delta_2 + (R_1 + R_2) \sin (\Delta_2 - \delta_2)$ ... (4) ...(2.6)

Again, $O_1F = HG = R_1 \cos \delta_1$.

$O_2G = R_2 \cos \delta_2$

$O_2H = O_1O_2 \cos (\Delta_2 - \delta_2) = (R_1 + R_2) \cos (\Delta_2 - \delta_2) = (R_1 + R_2) \cos (\Delta_1 - \delta_1)$

$O_2H = O_1F + O_2G$

or $(R_1 + R_2) \cos (\Delta_2 - \delta_2) = R_1 \cos \delta_1 + R_2 \cos \delta_2$

or $$\cos (\Delta_2 - \delta_2) = \cos (\Delta_1 - \delta_1) = \frac{R_1 \cos \delta_1 + R_2 \cos \delta_2}{R_1 + R_2} \quad \text{... (5) ...(2.7)}$$

In the above treatment, it has been assumed that Δ_1 is greater than Δ_2 so that $\Delta = \Delta_1 - \Delta_2$. In general, however, $\Delta = \pm (\Delta_1 - \Delta_2)$ according as the point of intersection occurs before or after the reverse curve.

2.5. RELATIONSHIPS BETWEEN VARIOUS PARTS OF A REVERSE CURVE

The various quantities involved in a reverse curve are Δ, Δ_1, Δ_2, δ_1, δ_2, R_1 and R_2. In order to co-relate these, three quantities and one condition equation (of either equal radius or equal central angle) must be known. We shall consider various cases of common occurrence.

CASE 1. NON-PARALLEL STRAIGHTS

Given. The central angles Δ_1 and Δ_2, $(\Delta_2 > \Delta_1)$ and the length of the common tangent.

Required. To find length of the common radius R and the chainages of T_1, E and T_2 if that of V is given.

Condition equation. $R_1 = R_2 = R$.

In Fig. 2.6, BD = common tangent of length d

$$O_1E = EO_2 = R$$

Other notations are the same as in Fig. 2.5.

Since T_1B and BE are tangents to the first arc, they are equal in length and $\angle VBE = \Delta_1$

Similarly, the tangents T_2D and DE are equal in length and $\angle EDC = \Delta_2$

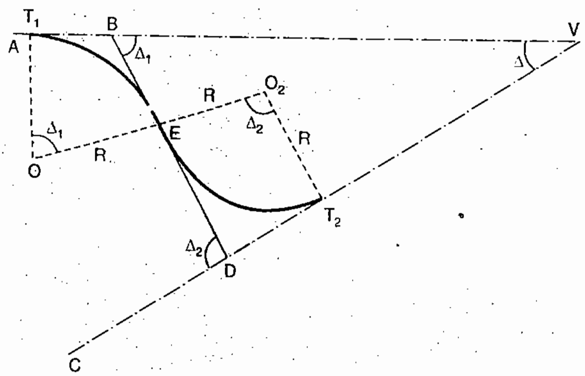


FIG. 2.6

$$\therefore \quad BT_1 = BE = R \tan \frac{\Delta_1}{2}$$

$$DT_2 = DE = R \tan \frac{\Delta_2}{2}$$

$$\therefore \quad BD = d = R \tan \frac{\Delta_1}{2} + R \tan \frac{\Delta_2}{2}$$

Hence $$R = \frac{d}{(\tan \frac{1}{2} \Delta_1 + \tan \frac{1}{2} \Delta_2)} \quad \text{...(2.8)}$$

Knowing R_1, Δ_1 and Δ_2, lengths of the two arcs can be calculated.

Again, $\Delta = \Delta_2 - \Delta_1$

From triangle BDV,

$$BV = BD \cdot \frac{\sin \Delta_2}{\sin \Delta} = d \frac{\sin \Delta_2}{\sin \Delta}$$

$$\therefore \quad T_1V = BT_1 + BV = R \tan \frac{\Delta_1}{2} + d \frac{\sin \Delta_2}{\sin \Delta}$$

Chainage of T_1 = chainage of $V - T_1V$.

Chainage of E = chainage of T_1 + length of first arc.

Chainage of T_2 = chainage of E + length of second arc.

The first branch of the curve can be set out from T_1 and the second branch from E by method of tangential angles.

CASE 2. NON-PARALLEL STRAIGHTS

Given. Length L of the line joining the tangents T_1 and T_2 and angles δ_1 and δ_2 which the line joining the tangent points makes with the two tangents.

Required. To find the common radius R.

Condition equation. $R_1 = R_2 = R$

In Fig. 2.7, let T_1 and T_2 be the two tangent points, the distance T_1T_2 being equal to L. The notations etc. are the same in Fig. 2.5. Draw O_1F and O_2G perpendicular to T_1T_2. Through O_1 draw O_1H parallel to T_1T_2, meeting O_2G produced in H. Let $\angle O_2O_1H = \theta$.

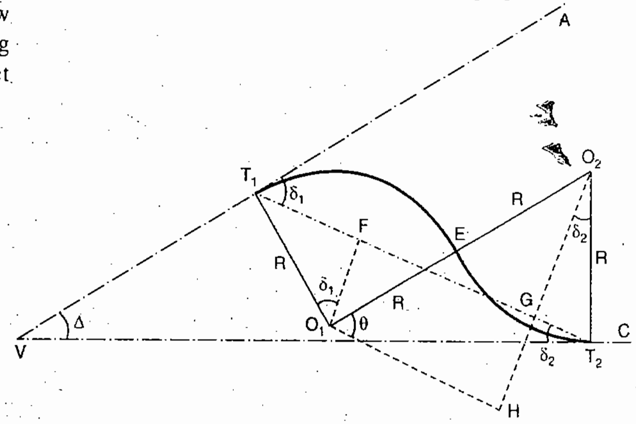


FIG. 2.7

Now $O_1F = R \cos \delta_1 = GH$

$O_2G = R \cos \delta_2$

$O_1O_2 = 2R$

$$\sin \theta = \frac{O_2H}{O_1O_2} = \frac{O_2G + GH}{O_1O_2}$$

$$= \frac{R \cos \delta_1 + R \cos \delta_2}{2R}$$

$$\theta = \sin^{-1} \frac{\cos \delta_1 + \cos \delta_2}{2} \quad \text{...(2.9)}$$

$T_1F = R \sin \delta_1$

$FG = O_1H = 2R \cos \theta$

and $GT_2 = R \sin \delta_2$

$$\therefore \quad T_1T_2 = T_1F + FG + GT_2 = L$$

or $$R \sin \delta_1 + 2R \cos \theta + R \sin \delta_2 = L$$

$$\therefore \quad R = \frac{L}{\sin \delta_1 + 2 \cos \theta + \sin \delta_2} \quad \text{...(2.10)}$$

where θ is given by the equation 2.9.

The central angle for the first branch $= \Delta_1 = \delta_1 + (90° - \theta)$

The central angle for the second branch $= \Delta_2 = \delta_2 + (90° - \theta)$

Knowing R, Δ_1 and Δ_2 the lengths of the arcs can be calculated.

CASE 3. NON-PARALLEL STRAIGHTS

Given. Length L of the line joining the tangent points T_1 and T_2, the angles δ_1 and δ_2 which the line joining the tangent points makes with the two tangents, *and any one of the two radii.*

Required. To find the other radius.

Refer Fig. 2.5 in which R_1 is smaller radius and R_2 is the greater radius.

$$T_1F = R_1 \sin \delta_1$$

$$T_2G = R_2 \sin \delta_2$$

$$FG = O_1H = \sqrt{(O_1O_2)^2 - (O_2H)^2} = \sqrt{(R_1 + R_2)^2 - (R_1 \cos \delta_1 + R_2 \cos \delta_2)^2}$$

Now $$T_1T_2 = L = T_1F + FG + T_2G$$

or $$L = R_1 \sin \delta_1 + \sqrt{(R_1 + R_2)^2 - (R_1 \cos \delta_1 + R_2 \cos \delta_2)^2} + R_2 \sin \delta_2$$

or $$\{L - (R_1 \sin \delta_1 + R_2 \sin \delta_2)\}^2 = \{(R_1 + R_2)^2 - (R_1 \cos \delta_1 + R_2 \cos \delta_2)^2\}$$

or $$L^2 + R_1^2 \sin^2 \delta_1 + R_2^2 \sin^2 \delta_2 + 2R_1 R_2 \sin \delta_1 \sin \delta_2 - 2L (R_1 \sin \delta_1 + R_2 \sin \delta_2)$$
$$= R_1^2 + R_2^2 + 2R_1 R_2 - (R_1^2 \cos^2 \delta_1 + R_2^2 \cos^2 \delta_2 + 2R_1 R_2 \cos \delta_1 \cos \delta_2)$$

or $$L^2 - 2L(R_1 \sin \delta_1 + R_2 \sin \delta_2) + R_1^2 (\sin^2 \delta_1 + \cos^2 \delta_1) + R_2^2 (\sin^2 \delta_2 + \cos^2 \delta_2)$$
$$= R_1^2 + R_2^2 + 2R_1 R_2 - 2R_1 R_2 \cos \delta_1 \cos \delta_2 - 2R_1 R_2 \sin \delta_1 \sin \delta_2$$

or $$L^2 - 2L (R_1 \sin \delta_1 + R_2 \sin \delta_2) = 2R_1 R_2 - 2R_1 R_2 \cos (\delta_1 - \delta_2)$$

or $$L^2 - 2L(R_1 \sin \delta_1 + R_2 \sin \delta_2) = 4R_1 R_2 \sin^2 \left(\frac{\delta_1 - \delta_2}{2} \right) \quad ...(2.11)$$

Knowing R_1 (or R_2), we can calculate R_2 (or R_1) from the above equation. The angle O_2O_1H (= θ) and hence Δ_1 and Δ_2 can then be calculated.

CASE 4. PARALLEL STRAIGHTS

Given. The two radii R_1 and R_2 and the central angles.

Required. To calculate various elements.

Condition Equation $\Delta_1 = \Delta_2$

In Fig. 2.8, let AT_1 and T_2C be two straights parallel to each other so that there is no point of intersection.

Let
- R_1 = smaller radius
- R_2 = larger radius
- Δ_1 = central angle corresponding to R_1
- Δ_2 = central angle corresponding to R_2
- L = distance T_1T_2
- v = perpendicular distance between the two straights
- h = distance between the perpendiculars at T_1 and T_2
- E = point of reverse curvature.

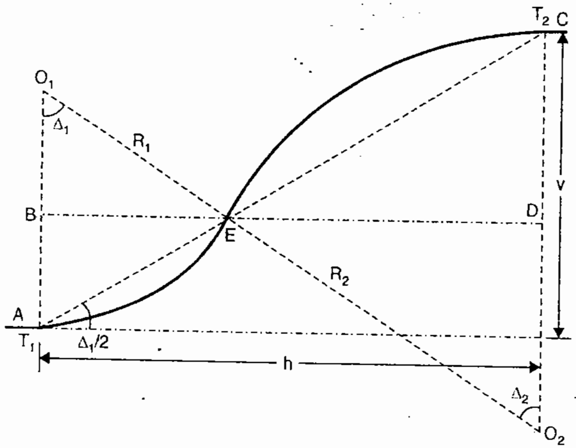


FIG. 2.8. REVERSE CURVE : PARALLEL TANGENTS.

Through E, draw a line BD parallel to the two tangents.

Since O_1T_1 and O_2T_2 are parallel to each other, we have

$$\Delta_1 = \Delta_2$$
$$T_1B = O_1T_1 - O_1B = R_1 - R_1 \cos \Delta_1 = R_1(1 - \cos \Delta_1) = R_1 \text{ versin } \Delta_1$$
$$T_2D = O_2T_2 - O_2D = R_2 - R_2 \cos \Delta_2 = R_2 - R_2 \cos \Delta_1$$
$$= R_2 (1 - \cos \Delta_1) = R_2 \text{ versin } \Delta_1$$
$$v = T_1B + DT_2 = R_1 \text{ versin } \Delta_1 + R_2 \text{ versin } \Delta_1$$
$$= (R_1 + R_2) \text{ versin } \Delta_1 = (R_1 + R_2)(1 - \cos \Delta_1) \quad ...(2.12)$$

Again, $$T_1E = 2 R_1 \sin \frac{\Delta_1}{2}$$
$$T_2E = 2R_2 \sin \frac{\Delta_2}{2} = 2 R_1 \sin \frac{\Delta_1}{2}$$

∴ $$T_1T_2 = L = T_1E + ET_2 = 2R_1 \sin \frac{\Delta_1}{2} + 2R_2 \sin \frac{\Delta_1}{2} = 2 (R_1 + R_2) \sin \frac{\Delta_1}{2} \quad ...(2.13)$$

But $$\sin \frac{\Delta_1}{2} = \frac{v}{L}$$

∴ $$L = 2(R_1 + R_2) \frac{v}{L}$$

From which, $$L = \sqrt{2v (R_1 + R_2)} \quad ...(2.14)$$
$$BE = R_1 \sin \Delta_1 \ ; \ ED = R_2 \sin \Delta_2 = R_2 \sin \Delta_1$$

∴ $$BD = h = (R_1 \sin \Delta_1 + R_2 \sin \Delta_1) \quad ...(2.15)$$
$$= (R_1 + R_2) \sin \Delta_1$$

Special case :

If $R_1 = R_2 = R$, we have
$$v = 2R(1 - \cos \Delta_1) \quad ...(2.12\ a)$$
$$L = 4 R \sin \frac{\Delta_1}{2} \quad ...(2.13\ a)$$
$$L = \sqrt{4Rv} \quad ...(2.14\ a)$$
$$h = 2R \sin \Delta_1 \quad ...(2.15\ a)$$

Example 2.6. *Two parallel railway lines are to be connected by a reverse curve, each section having the same radius. If the lines are 12 meters apart and the maximum distance between tangent points measured parallel to the straights is 48 metres, find the maximum allowable radius.*

If however, both the radii are to be different, calculate the radius of the second branch if that of the first branch is 60 metres. Also, calculate the lengths of both the branches.

Solution. (Fig. 2.8)

(*a*) *Given* : $h = 48$ m and $v = 12$ m

$$\tan \frac{\Delta_1}{2} = \frac{v}{h} = \frac{12}{48} = 0.25 \text{ m}$$

∴ $$\frac{\Delta_1}{2} = 14° 2' \text{ or } \Delta_1 = 28° 4'$$
$$\sin \Delta_1 = 0.47049$$

Now $$BE = R \sin \Delta_1 \text{ and } ED = R \sin \Delta_1$$

∴ $$BE + ED = h = R \sin \Delta_1 + R \sin \Delta_1 = 2R \sin \Delta_1$$

or $$R = \frac{h}{2 \sin \Delta_1} = \frac{48}{2 \times 0.47049} = \mathbf{51.1 \ m.}$$

(*b*) Let R_1 and R_2 be the radii.

As calculated above, $\Delta_1 = 28° 4'$ and $\sin \Delta_1 = 0.47079$

Now, $h = (R_1 + R_2) \sin \Delta_1$

$\therefore \quad (R_1 + R_2) = \dfrac{h}{\sin \Delta_1} = \dfrac{48}{0.47049} = 102.2 \quad \ldots(i)$

If $R_1 = 60$ m,

$R_2 = 102.2 - R_1 = 102.20 - 60 = \mathbf{42.2\ m.}$

Length of the first branch

$$= \frac{\pi R_1 \Delta_1}{180°} = \frac{\pi \times 60 \times 28° 4'}{180°} = 29.38 \text{ m}$$

Length of the second branch

$$= \frac{\pi R_2 \Delta_1}{180°} = \frac{\pi \times 42.2 \times 28° 4'}{180°} = \mathbf{20.67\ m.}$$

Example 2.7. *Two straights AB and CD interset at V. BD is the common tangent of length 200 metres. It is proposed to introduce a reverse curve consisting of two arcs of equal radii between them. The angles ABD and CDB are 150 ° 30′ and 43 ° 42′ respectively. Calculate (i) the common radius, (ii) the chainages of P.C., P.R.C. and P.T., if that of B is 9245.2 metres.*

Solution. (Fig. 2.6)

$\Delta_1 = \angle VBD = 180° - 150° 30' = 29° 30'$

$\Delta_2 = \angle BDC = 43° 42'$

$\Delta = \Delta_2 - \Delta_1 = 43° 42' - 29° 30' = 14° 12'$

Now $BD = 200 = BE + ED = R \tan \dfrac{\Delta_1}{2} + R \tan \dfrac{\Delta_2}{2}$

$\therefore \quad R = \dfrac{200}{\tan \frac{1}{2}\Delta_1 + \tan \frac{1}{2}\Delta_2} = \dfrac{200}{0.26328 + 0.40089} = 301.1$ m

$T_1B = 301.1 \tan 14° 45' = 79.3$ m

Length of the first branch $= \dfrac{\pi R \Delta_1}{180°} = \dfrac{\pi \times 301.1 \times 29° 5'}{180°} = 155$ m

Length of the 2nd branch $= \dfrac{\pi R \Delta_2}{180°} = \dfrac{\pi \times 301.1 \times 43° 12'}{180°} = 229.7$ m

Chainage of B	= 9245.2 m
Subtract T_1B	= 79.3 m
Chainage of T_1	= 9165.9 m
Add length of first curve	= 155.0 m
Chainage of P.R.C.	= 9320.9 m
Add length of second curve	= 229.7 m
Chainage of T_2	= 9550.6 m

Example 2.8. *Two straights AT_1 and CT_2 meet at V. It is proposed to introduce a reverse curve of common radius R, having T_1 and T_2 as tangent points. The angles AT_1T_2 and VT_2T_1 measured at T_1 and T_2 are 45° 30′ and 25° 30′ respectively. The distance T_1T_2 is equal to 800 metres. Determine the common radius and central angle for two arcs.*

Solution. (Fig. 2.7)

$\angle AT_1T_2 = \delta_1 = 45° 30'$

$\angle VT_2T_1 = \delta_2 = 25° 30'$

$\therefore \quad \Delta = \delta_1 - \delta_2 = 45° 30' - 25° 30' = 20°$

$\sin \theta = \dfrac{O_2H}{O_1O_2} = \dfrac{R \cos \delta_1 + R \cos \delta_2}{2R} = \dfrac{\cos \delta_1 + \cos \delta_2}{2}$

$\therefore \quad \theta = \sin^{-1} \dfrac{\cos 45° 30' + \cos 25° 30'}{2} = 53° 18'$

$\cos \theta = 0.59783$

Now $T_1T_2 = L = T_1F + FG + GT_2$

or $800 = R \sin \delta_1 + 2R \cos \theta + R \sin \delta_2$

$R = \dfrac{800}{\sin 45° 30' + 2 \cos 53° 18' + \sin 25° 30'} = \dfrac{800}{2.3395} = \mathbf{34.4\ m.}$

Now $\Delta_1 = \delta_1 + 90° - \theta = 45° 30' + 90° - 53° 18' = \mathbf{81° 12'}$

$\Delta_2 = \Delta_1 - \Delta = 81° 12' - 20° = 61° 12'$

(or $\Delta_2 = \delta_2 + 90° - \theta = 25° 30' + 90° - 53° 18' = 61° 12'$)

PROBLEMS

1. The following data refer to a compound circular curve which bears to the right :

Angle of intersection (or total deflection) = 59° 45′

Radius of 1st curve = 19.10 chains

Radius of 2nd curve = 12.74 chains.

Point of intersection = 164.25 chains.

Determine the running distances of the tangent point and the point of compound curvature, given that the latter point is 4.26 chains from the point of intersection at a back angle of 294° 32′ from the first tangent.

2. *AB* and *CD* are two straights such that *A* and *D* are on opposite sides of a common tangent *BC* ; and it is required to connect *AB* and *CD* with a reverse curve of radius *R*.

Given that angles *ABC* and *BCD* are respectively 148° 40′ and 139° 20′ and that *BC* is 16.28 chains , determine the common radius *R* and the chainage of the points of tangency and reverse curvature, the direction being from *A* to *D* and the chainage of *B* 145.20 chains. (*U.L.*)

3. The railway straights T_1AI and IBT_2 meeting in an inaccessible point *I* are to be connected by a compound circular curve such that the arc T_1C of radius 30 chains is equal in length to the arc CT_2 of radius 20 chains, *C* being the point of compound curvature. You are given the following data :

Line	W.C. Bearing	
T_1AI	55° 30′	Chainage of *A* 154.23 chains
IBT_2	114° 45′	*AB* = 12.63 chains
A	2° 36′	

(a) Prepare a sketch giving all the distances necessary for pegging T_1, C and T_2 initially.

(b) Submit in a tabular form complete notes for setting out the curve by tangential angles, pegging, through chainages (U.L.)

4. What do you understand by the following forms of curves and where are they generally used ?

1. Lemniscate 2. Compound curve 3. Reverse curve

If in a compound curve the directions of two straights and one radius are known, how will you find out analytically the radius of the other curve ?

5. A railway siding is to be curved through a right angle. In order to avoid buildings, the curve is to be compound, the radius of the two branches being 8 chains and 12 chains. The distance from the intersection point of the end straights to the tangent point at which the arc of 8 chains radius leaves straight is to be 10.08 chains. Obtain the second tangent length, or distance from the intersection point to the other end of the curve, and the length of the whole curve. (T.C.D.)

6. A compound railway curve ABC is to have the radius of arc AB 600 metres and that of BC 400 metres. The intersection point V of the straights is located, and the intersection angle is observed to be 35° 6′. If the arc AB is to have a length of 200 metres, calculate the tangent distances VA and VC.

7. A curve of 300 m radius has been pegged out to connect two railway tangents having deflection angle = 15° 26′, and the chainage of the initial tangent point has been found to be 3841.7 metres. On further examination of the ground, it is decided to alter the radius to 450 metres. Calculate the chainage of the new initial and final tangent points, and the distances between the new and original curves at their mid-point.

ANSWERS

1. Chainage of P.C. = 153.845 chains.
   Chainage of P.C.C. = 166.227 chains.
   Chainage of P.T. = 171.250 chains.
2. R = 25.00 chains ; chainage of 1st tangent point = 138.188 ; chainage of P.R.C. = 151.859 ; chainage of 2nd tangent point = 169.602.
3. IA, 7.823 chains ; IB, 6.697 chains
   AT_1 = 7.068 chains ; BT_2 = 5.658 chains
   chainage of C, 166.638 chains ; T_2 = 179.046 chains.
5. 11.51 chains ; 16.85 chains
6. 176.3 m ; 145.7 m.
7. 3821.4 m; 3942.6 m ; 1.37 m.

3

Curve Surveying III : Transition Curves

3.1. GENERAL REQUIREMENTS

A *transition* or easement curve is a curve of varying radius introduced between a straight and a circular curve, or between two branches of a compound curve or reverse curve. In case of a highway, in order to hold the vehicle in the centre of the lane, the driver is required to move his steering almost instantly to the position necessary for the curve at the moment he passes the P.C. In doing so, the sudden impact of centrifugal force coupled with the inertia of the vehicle would cause the vehicle to sway outwards, and if this exceeds a certain value the vehicle may overturn. In case of railways, the side thrust is wholly taken by the pressure exerted by the rails on the flanges of the wheels thus causing wear of the rail in the region of the tangent point. To avoid these effects, a curve of changing radius must be introduced between the straight and the circular curve. The functions of a transition curve are :

(1) To accomplish gradually the transition from the tangent to the circular curve, so that the curvature is increased gradually from zero to a specified value.

(2) To provide a medium for the gradual introduction or change of the required super-elevation.

A transition curve introduced between the tangent and the circular curve should fulfil the following conditions :

(1) It should be tangential to the straight.

(2) It should meet the circular curve tangentially.

(3) Its curvature should be zero at the origin on straight.

(4) Its curvature at the junction with the circular curve should be the same as that of the circular curve.

(5) The rate of increase of curvature along the transition should be the same as that of increase of cant or super-elevation.

(6) Its length should be such that full cant or super-elevation is attained at the junction with the circular curve.

Super-elevation

When a pavement or track is sloped upward towards the outside of a curve, it is said to be banked or super-elevated. Thus, 'super-elevation or 'cant' is the amount by which the outer end of the road or outer rail is raised above the inner one.

When a vehicle moves on a curve, there are two forces acting : (*i*) weight of the vehicle and (*ii*) the centrifugal force. Both the forces pass through the C.G. of the vehicle. Since the centrifugal force always acts perpendicular to the axis of rotation (which is vertical), its direction is horizontal acting away from the centre of the curve. The weight of the vehicle acts vertically. The resultant R of these two should be normal to the surface for equilibrium.

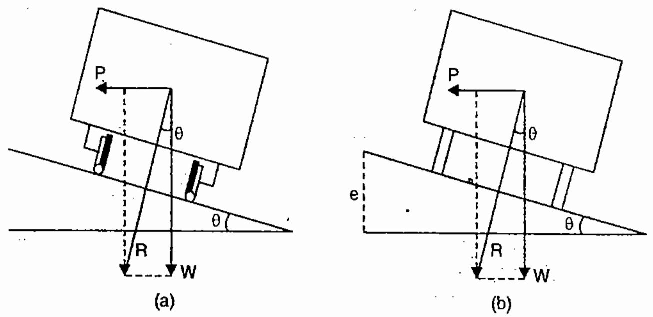


FIG. 3.1. SUPER-ELEVATION.

Let
W = Weight of the vehicle
P = Centrifugal force
v = Speed of the vehicle
g = Acceleration due to gravity
B = Width of the road
F = Distance between the centres of the rails
R = Radius of the curve
θ = Inclination of the road or rail surface.

From mechanics,
$$P = \frac{W v^2}{gR}$$
or
$$\frac{P}{W} = \frac{v^2}{gR} \quad ...(1)$$

If the resultant R is to be normal to the surface, its inclination with W will be the same as the inclination of the surface with the horizontal, *i.e.* θ. Hence

$$\tan\theta = \frac{P}{W} = \frac{v^2}{gR} \quad ...(2) \quad ...(3.1\ a)$$

If e is the cant or super-elevation, we have

$$e = B \tan\theta = \frac{Bv^2}{gR} \text{ on roads} \quad ...(3.1)$$

and
$$e = G \tan\theta = \frac{Gv^2}{gR} \text{ on railways.} \quad ...(3.2)$$

Equilibrium Cant and Cant Deficiency

In case of railways, if the cant is provided as given by equation 3.2, the load carried by both the wheels will be the same, the springs will be equally compressed and the passengers will not tend to lean in either direction. Such cant is known as the '*equilibrium cant'*. If the cant is provided less than this, more weight will be carried by the outer wheels, the outer springs will be more highly compressed than the inner and passengers will tend to lean outwards.

The track will, under these conditions, have *cant deficiency.*

The amount of cant is limited to 6″ (or 15 cm) on a standard gauge (4′ 8½″) having $G = 4'\ 11\frac{1}{2}''$ (= 1.5 m). Taking $G = 5'$ approximately, V' in m.p.h. and R' = radius in feet, the cant e' (inches) is given by

$$e' = \frac{5\left(\frac{V' \times 5280}{60 \times 60}\right)^2}{32 \times R'} \times 12 \text{ inches}$$

or
$$e' \approx \frac{4V'^2}{R'} \text{ inches} \quad ...(3.2\ a)$$

However, taking V in kilometers per hour, $G = 1.5$ m and R in meters, the cant e (cm) is given by

$$e = \frac{1.5 \times \left(\frac{V \times 1000}{60 \times 60}\right)^2}{9.81 \times R} \times 100 \text{ cm} \quad \text{or} \quad e = \frac{1.18\ V^2}{R} \text{ cm} \quad ...(3.2\ b)$$

Centrifugal Ratio

The ratio of the centrifugal force and the weight is called the *centrifugal ratio.*

Thus, centrifugal ratio $= \frac{P}{W} = \frac{W v^2}{gR} \cdot \frac{1}{W} = \frac{v^2}{gR}$ ...(3.3)

The maximum value of centrifugal ratio is taken equal to $\frac{1}{4}$ on roads and $\frac{1}{8}$ on railways.

Thus, for roads,
$$\frac{P}{W} = \frac{1}{4} = \frac{v^2}{gR}, \quad v = \sqrt{\frac{gR}{4}} \quad ...(3.4\ a)$$

For railways,
$$\frac{P}{W} = \frac{1}{8} = \frac{v^2}{gR} \quad ...(3.3)$$

or
$$v = \sqrt{\frac{gR}{8}} \quad ...(3.4\ b)$$

Equations 3.4 (*a*) and (*b*) decide the minimum radius of the curve for the vehicle to pass safely with the given speed v.

Super-elevation on Highways : Side Friction Factor

Side friction factor (f) is defined as the force transferred by friction parallel to the pavement per unit force normal to the pavement.

Consider Fig. 3.1 (b).

Let N = the sum of the forces normal to the pavement.

T = the sum of the forces parallel to the pavement transferred to it by friction.

f = side friction factor $= \frac{T}{N}$.

Resolving the forces P and W normal to the pavement,

$$N = P \sin\theta + W \cos\theta \qquad \ldots(i)$$

Resolving the forces P and W tangential to the pavement,

$$T = P \cos\theta - W \sin\theta \qquad \ldots(ii)$$

Now, $T = fN$

or $(P\cos\theta - W\sin\theta) = f(P\sin\theta + W\cos\theta)$

or $P(\cos\theta - f\sin\theta) = W(\sin\theta + f\cos\theta)$

or $$\frac{P}{W} = \frac{(\sin\theta + f\cos\theta)}{\cos\theta - f\sin\theta} = \frac{\tan\theta + f}{1 - f\tan\theta}$$

But $$\frac{P}{W} = \frac{v^2}{gR}$$

$\therefore$ $$\frac{v^2}{gR} = \frac{\tan\theta + f}{1 - f\tan\theta} \qquad \ldots(3.5)$$

Equation 3.5 represents an exact relationship between the various quantities involved and for the safe design of highways, the right hand side must be equal to or greater than the left hand side. The maximum value of f may be taken equal to 0.25 for average conditions.

It is evident from equation 3.5 that the centrifugal force is balanced by the sum of the effect of the super-elevation and the effect of friction. If the super-elevation is increased, more of the centrifugal force will be balanced by it and less friction will be required. It has yet not been agreed upon as to how much centrifugal force must be balanced by super-elevation and how much by side friction. There are two extreme methods :

(1) Method of maximum friction.

(2) Method of maximum super-elevation.

(1) Method of Maximum Friction

In this method, whole of the centrifugal force is balanced by the side friction till the maximum limit of the latter is reached. If the radius is still lesser, the rest of the centrifugal force is balanced by introducing the super-elevation. Thus, if R is the *minimum* radius for the standard veloctiy v on surface having no super-elevation or cant, side-slip will occur if the side thrust due to centrifugal force is greater than the adhesion between the tyres and the road surface. That is, if

$$\frac{Wv^2}{gR} > fW$$

or $$R = \frac{v^2}{fg} \qquad \ldots(3.6)$$

(The above equation could also be obtained by putting $\tan\theta = 0$ in Eq. 3.5)

If V' is in miles per hour, $g = 32.2$ ft/sec^2, and R' is in feet, we have

$$R' = \frac{V'^2}{14.97f}$$

Taking average value of $f = 0.25$, we get

$$R' = 0.267\ V'^2 \qquad \ldots(3.6\ a)$$

Similarly, if V is in kilometer per hour, $g = 981$ cm/sec^2, R in metres and $f = 0.25$, we get

$$R = 0.03143\ V^2 \qquad \ldots(3.6\ b)$$

If R is to be provided lesser than that given by equation 3.6, super-elevation of appropriate value will have to be introduced till equation 3.5 is satisfied.

(2) Method of Maximum Super-elevation

In this method, whole of the centrifugal force is balanced by the super-elevation only till the maximum limit of the latter is reached. If the radius provided is still lesser, friction would be relied on to balance the rest of the centrifugal force. If R is the minimum radius for the standard velocity v, we have

$$\tan\theta = \frac{v^2}{gR} \qquad \ldots(3.1\ a)$$

or $$R = \frac{v^2}{g\tan\theta} \qquad \ldots(3.7)$$

(The above equation could also be obtained by putting $f = 0$ in equation 3.5).

If R is to be provided lesser than that given by equation 3.7, friction would be relied on till equation 3.5 is satisfied.

3.2. LENGTH OF TRANSITION CURVE

The length of the transition curve should be such that the required super-elevation or cant is provided at a suitable rate. There are three methods for determining its length:

(a) First Method : By an Arbitrary Gradient

In this method, the super-elevation e is provided at an arbitray rate, say 1 in n. Then the length L of the transition curve is given by

$$L = ne \qquad \ldots(3.8)$$

The value of n may vary between 300 to 1200.

Let the rate of canting be 1 cm in n metres.

From equation 3.2 (b), $e = 1.18\frac{V^2}{R}$ cm where V in km/sec and R is in metres.

Substituting these values in equation 3.8, we get

$$L = 1.18 \frac{nV^2}{R} \text{ metres} \qquad \ldots(3.8\ a)$$

If, however, the rate of canting is 1 inch in n' ft and $e' = \frac{4V'^2}{R'}$ inches (equation 3.2 *a*), we have

$$L' = n'e' = \frac{4n'V'^2}{R} \text{ ft.} \qquad \ldots(3.8\ b)$$

(*b*) Second Method : By the Time Rate

In this method, the cant e is applied at an arbitrary time rate of r units per second.

Let L = the length of the transition curve in metres

e = amount of super-elevation in cm

v = speed of the vehicle in metres per second

r = time rate in cm/sec

V = speed in km/hour.

Time taken by a vehicle to pass over the transition curve $= t = \frac{L}{v}$ seconds

Super-elevation attained in this time $= t \times r$ cm $= \frac{L}{v} \cdot r$ cm

But this should be equal to e.

$$\therefore \quad \frac{L}{v} r = e, \qquad L = \frac{ev}{r} \text{ metres} \qquad \ldots(3.9)$$

Substituting the value of $e = 1.18 \frac{V^2}{R}$ cm

and of $v = \left(\frac{V \times 1000}{3600}\right)$ m/sec, in equation 3.9, we get

$$L = \frac{1}{r}\left(1.18 \frac{V^2}{R}\right)\left(\frac{V \times 1000}{3600}\right)$$

or

$$L = 0.327 \frac{V^3}{R\,r} \text{ metres}$$

In English units, let L' = length in ft.

R' = radius in ft

e' = cant in inches $= \frac{4V'^2}{R'}$ from equation 3.2 *a*.

V' = velocity in miles per hour

v' = velocity in ft/sec $= \left(\frac{V' \times 5280}{3600}\right)$

r' = rate in inches/sec.

Then

$$L' = \frac{e'V'}{r'} = \frac{1}{r'}\left(\frac{4V'^2}{R'}\right)\left(\frac{V' \times 5280}{3600}\right)$$

$$\therefore \quad L' = 5.86 \frac{V'^3}{R'r'} \qquad \ldots(3.9\ b)$$

(*c*) Third Method : By the Rate of Change of Radial Acceleration.

In this method, the length of the transition curve is decided on the basis of the *comfort* of the passengers. Mr. Shortt states that in his experience a rate of change of radial acceleration of 1 ft/sec²/sec (or 0.3 m/sec²/sec) will pass unnoticed.

Let L = length of transition curve in metres

α = rate of change of radial acceleration in m/sec³

v = maximum speed in m/sec

V = maximum speed in km/hour

The time taken to travel over the transition curve $= \frac{L}{v}$ sec

Acceleration attained in that time $= \alpha\, t = \alpha \frac{L}{v}$ m/sec²

By radial acceleration of the circular curve $= \frac{v^2}{R}$ m/sec²

$$\therefore \quad \frac{\alpha L}{v} = \frac{v^2}{R} \quad \text{or} \quad L = \frac{v^3}{\alpha R}$$

Taking $\alpha = 0.3$ m/sec³

and $v = \left(\frac{V \times 1000}{3600}\right)$ m/sec

We have

$$L = \frac{1}{0.3\,R}\left(\frac{V \times 1000}{3600}\right)^3 \quad \text{or} \quad L \approx \frac{V^3}{14\,R} \text{ metres} \qquad \ldots(3.10\ a)$$

Similarly taking $\alpha' = 1$ ft/sec³

$$v' = \left(\frac{V' \times 5280}{3600}\right) \text{ ft/sec}$$

We have

$$L' = \frac{1}{1 \times R'}\left(\frac{V' \times 5280}{3600}\right)^3$$

or

$$L' = \frac{3.16\,V'^3}{R'} \text{ ft.} \qquad \ldots(3.10\ b)$$

In methods (*b*) and (*c*), the length L is proportional to V^3 and both are preferable to the first. However, the third method is the most commonly used.

Now, *for roads*, $V = \left(\frac{R}{0.03143}\right)^{\frac{1}{2}}$ km/hour from Eq. 3.6 *b*

and $V' = \left(\frac{R'}{0.267}\right)^{\frac{1}{2}}$ mile/hour, from Eq. 3.6 *a*

Substituting the values of V and V' in Eqs. 3.10 *a* and 3.10 *b* respectively, we get

$$L = \left(\frac{R}{0.03143}\right)^{3/2} \frac{1}{14R} \approx 12.8\sqrt{R} \text{ metres...(for roads)} \qquad \ldots(3.11\ a)$$

and $$L' = \frac{3.16}{R'}\left(\frac{R'}{0.267}\right)^{3/2} \approx 23\sqrt{R'} \text{ ft.} \quad \text{...(for roads)} \qquad \text{...(3.11 } b)$$

3.3. THE IDEAL TRANSITION CURVE : THE CLOTHOID

As stated earlier, the centrifugal force acting on a vehicle is given by

$$P = \frac{W v^2}{g r}$$

where r is the radius of curvature at any point on the curve.

If the centrifugal force P is to increase at a constant rate, P must vary with time. Again if the speed of the vehicle is constant, the distance l along the transition curve measured from the tangent point must varv with time. Hence, we have

$$P \propto l \propto \frac{W v^2}{g r}$$

But W, v and g are all constants.

Hence $$l \propto \frac{1}{r} \qquad \text{...(3.12)}$$

or $$l\,r = \text{constant} = LR$$

where L = total length of the curve, upto its end

R = radius of the curve at its end (*i.e*, minimum radius)

Also the cant from the equilibrium point of view is given by

$$e = 1.18\frac{V^2}{r} \qquad \text{(Eq. 3.2 } b)$$

where r is the radius of the curve.

If e is to increase at a contant rate, it is proportional to l.

$\therefore$ $$e \propto l \propto 1.18\frac{V^2}{r} \quad \text{or} \quad l \propto \frac{1}{r} \qquad \text{...(3.12)}$$

or $$l\,r = \text{constant} = LR.$$

Thus, the fundamental requirement of a transition curve is that its radius of the curvature r at any point shall vary inversely as the distance (l) from the beginning of the curve. Such a curve is the *Clothoid or the Glover's sprial* and is known as the *ideal transition curve.*

Let T = tangent point = beginning of the transition cuve

TA = initial tangent

D = Point of junction of the transition and circular curve

B = any point on the curve at distance l along the curve

r = radius of the curve at any point B

ϕ = the inclination of the tangent to the transition curve at B to the initial tangent TA = deviation angle

Δ_s = spiral angle = the angle between the initial tangent and the tangent to the transition curve at the junction point D

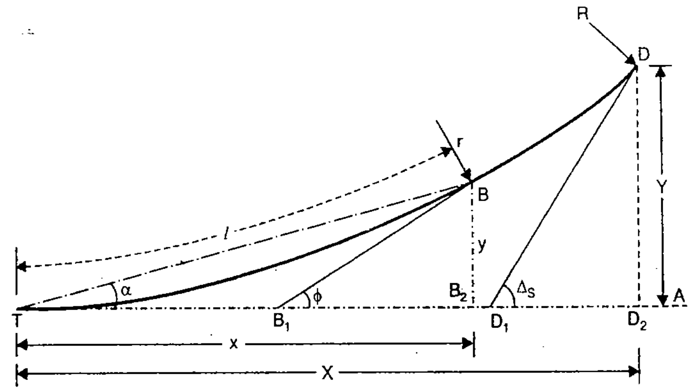


FIG. 3.2. THE IDEAL TRANSITION CURVE.

l = length of the curve from T to B

R = radius of the circular curve

L = total length of the transition curve

X = the x co-ordinate of D

Y = the y co-ordinate of D

$x = TB_2$ = x-co-ordinate of any point B on the transition curve

$y = BB_2$ = y-co-ordinate to any point B on the transition curve

We have $$l\,.\,r = L\,.\,R = \text{constant}$$

or $$\frac{1}{r} = \frac{l}{RL}$$

But $$\frac{1}{r} = \text{curvature} = \frac{d\phi}{dl}$$

$\therefore$ $$\frac{d\phi}{dl} = \frac{l}{RL}$$

or $$d\phi = \frac{l}{RL}\,.\,dl$$

Integrating, we get $$\phi = \frac{l^2}{2\,RL} + C$$

When $l = 0$, $\phi = 0$

$\therefore$ $C = 0$

Hence $$\phi = \frac{l^2}{2RL} \qquad \text{...(3.13)}$$

This is the intrinsic equation of the ideal transition curve.

Equation 3.13 can also be expressed in the form

$$l = \sqrt{2RL\phi} = K\sqrt{\phi} \qquad \text{...(3.13 } a)$$

where $K = \sqrt{2RL}$ ...(3.13 *b*)

When $l = L$, $\phi_d = \Delta_s = \dfrac{L^2}{2RL} = \dfrac{L}{2R}$ ...(3.13 *c*)

CARTESIAN CO-ORDINATES OF THE POINTS (Fig. 3.3)

In order to set out the curve by offsets from the tangents, the cartesian or rectangular co-ordinates referred to the tangent *TA* as the *x*-axis and a line perpendicular to it as *y*-axis, are necessary.

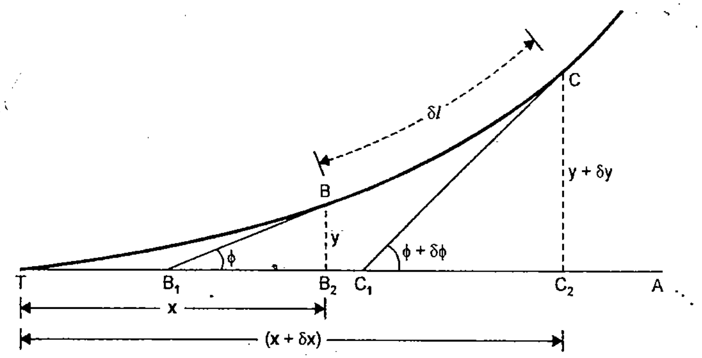


FIG. 3.3. THE CARTESIAN CO-ORDINATES.

Let *B* and *C* be two points δl apart on the curve

ϕ = angle between tangent BB_1 and the initial tangent *TA*

$(\phi + \delta\phi)$ = angle between tangent CC_1 and the initial tangent *TA*

x and *y* = co-ordinates of *B*

$(x + \delta x)$ and $(y + \delta y)$ = co-ordinates of *C*.

Now $\dfrac{dx}{dl} = \cos\phi$

or $dx = dl \cdot \cos\phi = dl\left(1 - \dfrac{\phi^2}{2\,!} + \dfrac{\phi^4}{4\,!} - \ldots\right)$

But $l = K\sqrt{\phi}$

or $dl = \dfrac{K}{2\phi^{1/2}}\, d\phi$

Substituting the value of *dl*, we get

$$dx = \frac{K}{2}\left(\phi^{-1/2} - \frac{\phi^{3/2}}{2\,!} + \frac{\phi^{7/2}}{4\,!} - \ldots\right) d\phi$$

Integrating the above,

$$x = K\left(\phi^{1/2} - \frac{\phi^{5/2}}{10} + \frac{\phi^{9/2}}{216} - \ldots\right) = K\phi^{1/2}\left(1 - \frac{\phi^2}{10} + \frac{\phi^4}{216} - \ldots\right) \quad \ldots(3.13)$$

or $x = l\left(1 - \dfrac{\phi^2}{10} + \dfrac{\phi^4}{216} - \ldots\right)$ ...(3.14)

But $\phi = \dfrac{l^2}{K^2}$

$\therefore$ $x = l\left(1 - \dfrac{l^4}{10K^4} + \dfrac{l^8}{216\,K^8} - \ldots\right)$ ...(3.14 *a*)

Putting $K = \sqrt{2RL}$, we get

$x = l\left(1 - \dfrac{l^4}{40R^2L^2} + \dfrac{l^8}{3456\,R^4L^4}\right)$ ...(3.14 *b*)

Similarly, $\dfrac{dy}{dl} = \sin\phi$

or $dy = dl\,\sin\phi = dl\left(\phi - \dfrac{\phi^3}{3\,!} + \dfrac{\phi^5}{5\,!} - \ldots\right)$

Substituting $dl = \dfrac{K}{2\phi^{1/2}}$, we get

$$dy = \frac{K}{2}\left(\phi^{1/2} - \frac{\phi^{5/2}}{6} + \frac{\phi^{9/2}}{120} - \ldots\right) d\phi$$

Integrating the above

$$y = K\left(\frac{\phi^{3/2}}{3} - \frac{\phi^{7/2}}{42} + \frac{\phi^{11/2}}{1320} - \ldots\right) = K\phi^{1/2}\frac{\phi}{3}\left(1 - \frac{\phi^2}{14} + \frac{\phi^4}{440} - \ldots\right)$$

Putting $K\phi^{1/2} = l$

$y = \dfrac{l^3}{3K^2}\left(1 - \dfrac{\phi^2}{14} + \dfrac{\phi^4}{440} - \ldots\right)$ ...(3.15)

But $\phi = \dfrac{l^2}{K^2}$

$\therefore$ $y = \dfrac{l^3}{3K^2}\left(1 - \dfrac{l^4}{14K^4} + \dfrac{l^3}{440\,K^8} - \ldots\right)$ ...(3.15 *a*)

Putting $K = \sqrt{2RL}$, we get

$y = \dfrac{l^3}{6RL}\left(1 - \dfrac{l^4}{56R^2L^2} + \dfrac{l^8}{7040\,R^4L^4} - \ldots\right)$ ...(3.15 *b*)

The above expressions for the cartesian co-ordinates *x* and *y* are not simple unless some approximations are made.

CALCULATION OF DEFLECTION ANGLES (Fig. 3.2)

Let α = polar deflection angle to the point *B*, *i.e.*, the angle between the chord *TB* and initial tangent *TA*.

Then
$$\tan\alpha = \frac{y}{x} = \frac{K\left(\dfrac{\phi^{3/2}}{3} - \dfrac{\phi^{7/2}}{42} + \dfrac{\phi^{11/12}}{1320} - \ldots\right)}{K\left(\phi^{1/2} - \dfrac{\phi^{5/2}}{10} + \dfrac{\phi^{9/2}}{216} - \ldots\right)} = \frac{\phi}{3} + \frac{\phi^3}{105} + \frac{\phi^5}{5997} - \ldots$$

This very closely resembles the expression

$$\tan\frac{\phi}{3} = \frac{\phi}{3} + \frac{\phi^3}{81} + \frac{\phi^5}{18225} \;; \qquad \text{Hence} \quad \tan\alpha \approx \tan\frac{\phi}{3}$$

Since ϕ is very small (usually a small friction of a radian)

$$\alpha = \frac{\phi}{3} \qquad ...(3.16)$$

$$= \frac{1}{3}\cdot\frac{l^2}{2RL} = \frac{l^2}{6RL} \text{ radians} \qquad ...(3.16\ a)$$

$$= \frac{l^2}{6RL}\cdot\frac{180}{\pi}\times 60 = \frac{1800\, l^2}{\pi RL} \text{ minutes} \qquad ...(3.16\ b)$$

Accurate Relation between α and φ

We have $\tan\alpha = \frac{\phi}{3} + \frac{\phi^3}{105} + \frac{\phi^5}{5997}$ .....

Since $\alpha = \tan\alpha - \frac{\tan^3\alpha}{3} + \frac{\tan^5\alpha}{5}$ ....

it can be shown that $\alpha = \frac{\phi}{3} - \frac{8\phi^3}{2835} - \frac{32\phi^5}{467775}$ ....

This can be expressed as

$$\alpha = \frac{\phi}{3} - \delta \qquad ...(3.16\ c)$$

where $\delta = 3.095 \times 10^{-3}\phi^3 + 2.285 \times 10^{-8}\phi^5$ ...(3.16 *d*)

(ϕ being in degrees and δ in seconds)

If it is required to find the value of α when ϕ is known, ϕ should be divided by 3 and a small correction δ, given in the Table 3.1 should be subtracted.

TABLE 3.1. VALUES OF δ

φ°	δ ′	δ ″	φ°	δ ′	δ ″	φ°	δ ′	δ ″
1	0	0	16	0	13	31	1	32
2	0	0	17	0	15	32	1	41
3	0	0	18	0	18	33	1	51
4	0	0	19	0	21	34	2	2
5	0	0	20	0	25	35	2	13
6	0	0	21	0	29	36	2	24
7	0	1	22	0	35	37	2	37
8	0	2	23	0	38	38	2	50
9	0	2	24	0	43	39	3	4
10	0	3	25	0	48	40	3	18
11	0	4	26	0	54	41	3	33
12	0	5	27	1	1	42	3	49
13	0	7	28	1	8	43	4	6
14	0	8	29	1	16	44	4	24
15	0	10	30	1	25	45	4	42

MODIFICATION OF THE IDEAL TRANSITION CURVE : THE CUBIC SPIRAL

Neglecting all the terms of equation 3.15 *b*, except the first one, we get

$$y = \frac{l^3}{6RL} \qquad ...(3.17)$$

which is the equation of the cubic spiral.

The approximation made here is

$$\sin\phi = \phi$$

or $\frac{dy}{dl} = \sin\phi = \phi = \frac{l^2}{2RL}$ (From Eq. 3.13)

$\therefore$ $dy = \frac{l^2}{2RL}\cdot dl$

Integrating, $y = \frac{l^3}{6RL} + c$

where c = constant of integration = 0, since $y = 0$ when $l = 0$

$\therefore$ $y = \frac{l^3}{6RL}$ ...(3.17)

The cubic spiral is set out by chords and offsets from the initial tangent. If, however, the curve is set out by deflection angles, we get

$$\alpha = \frac{\phi}{3} = \frac{l^2}{6RL} \text{ radians.}$$

THE CUBIC PARABOLA

Neglecting all the terms of equation 3.14 except the first one, we get

$$x = l \qquad ...(i)$$

Similarly, from equation 3.15 *b*, we have

$$y = \frac{l^3}{6RL} \qquad ...(ii)$$

From (*i*) and (*ii*), we get

$$y = \frac{x^3}{6RL} \qquad ...(3.18)$$

This is equation of the cubic parabola, which is also known as *Froude's transition curve*. The use of both the cartesian co-ordinates are made in setting out the curve.

The approximation $x = l$ corresponds to the assumption $\cos\phi = 1$. Thus, in cubic parabola two approximations are made, *viz*, $\cos\phi = 1$ and $\sin\phi = \phi$, while in cubic spiral, only one approximation *viz.*, $\sin\phi = \phi$, is made. Since the cosine series is less rapidly converging than the sine series, greater error is involved in the approximation $\cos\phi = 1$ than involved in the approximation $\sin\phi = \phi$. *Hence a cubic spiral is superior to a cubic parabola*. However, the cubic parabola is the most widely used transition curve owing to the ease with which it may be set out by rectangular co-ordinates.

Minimum Radius of Curvature of Cubic Parabola

The equation of the cubic parabola is

$$y = \frac{x^3}{6RL} = Mx^3, \quad \text{where } M = \frac{1}{6RL}$$

$$\therefore \quad \frac{dy}{dx} = 3Mx^2 = \tan\phi$$

$$\therefore \quad x = \sqrt{\frac{\tan\phi}{3M}} \quad \ldots(i)$$

Also $$\frac{d^2y}{dx^2} = 6Mx = \sqrt{\frac{36M^2\tan\phi}{3M}} = \sqrt{12M\tan\phi}$$

The radius of curvature r is given by

$$r = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}} = \frac{(1+\tan^2\phi)^{3/2}}{\sqrt{12M\cos\phi}} = \frac{\sec^3\phi}{\sqrt{12M\tan\phi}}$$

$$= \frac{1}{\sqrt{12M\sin\phi\cos^5\phi}} \quad \ldots(ii)$$

r has a minimum value when $\sin\phi\cos^5\phi$ is a maximum. Hence differentiating and equating to zero, we get

$$\cos^6\phi - 5\sin^2\phi\cos^4\phi = 0$$

or $$1 - 5\tan^2\phi = 0$$

$$\tan\phi = \frac{1}{\sqrt{5}}$$

which gives $$\phi = 24° \, 5' \, 41'' \quad \ldots(3.19)$$

corresponding to this, $$\alpha = \frac{\phi}{3} = 8° \, 1' \, 54'' \quad \ldots(3.19\ a)$$

Substituting the value ϕ in (ii), we get

$$r_{min.} = \frac{1}{\sqrt{12M\sin 24°\,5'\,41''\cos^5 24°\,5'\,41''}} = \frac{1}{1.762\sqrt{M}} = 1.39\sqrt{RL}$$

Hence the radius of curvature of the cubic parabola decreases from a value of *infinity* when $\phi = 0$ to a *minimum value* of $r = 1.39\sqrt{RL}$ when $\phi = 24°\,5'\,41''$. *Beyond this point the radius of curvature begins to increase again and so the curve is useless as a transition.*

Probably, the most widely used transition curve is the cubic parabola. It is almost identical with the clothoid and the lemniscate for deviation angles upto 12°. It has the great advantage that special tables are not required for setting it out. For small deviation angles, certain approximations made for calculations do not result in appreciable errors. However, for large deviation angles, these approximations may lead to larger errors. To use the curve for these large angles, it is necessary to express certain dimensions as infinite series, and hence the great advantage of simplicity in calculations is then lost. In such cases, the clothoid or lemniscate will give better results.

3.4. CHARACTERISTICS OF A TRANSITION CURVE

When the transition curves are introduced at the ends of a circular curve, it becomes necessary to accommodate them by shifting the main curve inwards in the new work. However, in amending old track the main curve is either sharpened or sharpened and shifted in order to accommodate the necessary shift.

In Fig. 3.4, let

TV = original tangent

BV' = the shift tangent parallel to the original tangent

$s = BA$ = shift of the circular curve

L = length of the transition curve

D = end of the transition curve and beginning of the circular curve

DD_1 = tangent common to both the transition and the circular curve at D

DB = extended portion of the circular curve (or the redundant circular curve)

$Y = D_2D$ offset of the junction point D

$X = TD_2 = x$ co-ordinate of the junction point D

R = radius of the circular curve

Δ_S = the spiral angle

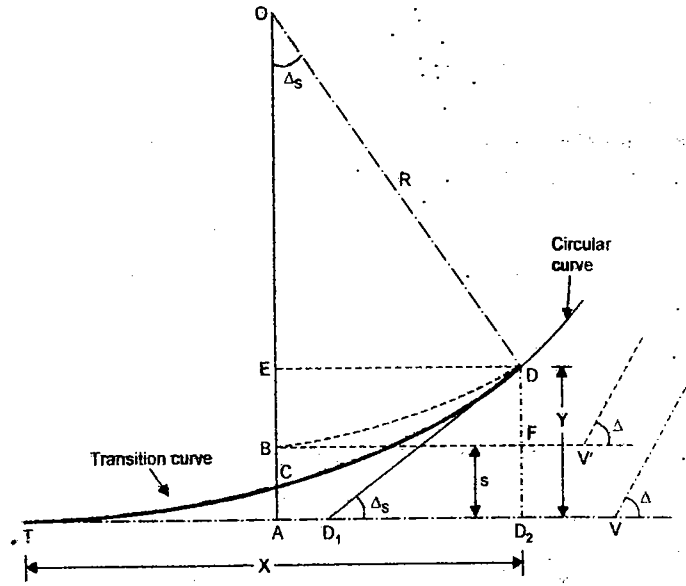


FIG. 3.4. CHARACTERISTICS OF A TRANSITION CURVE.

OB = perpendicular to the shift tangent at B

A = point of intersection of the perpendicular OB with the original tangent

DE = line perpendicular to OA

Since the tangent DD_1 makes an angle Δ_S with the original tangent, $\angle BOD = \Delta_S$.

Now, arc $\quad BD = R\Delta_S = R\dfrac{L}{2R} = \dfrac{L}{2}$, since $\Delta_S = \dfrac{L}{2R}$ from Eq. 3.13 c ...(i)

When CD is very nearly equal to BD, we have

$$CD = \frac{L}{2} \quad ...(3.20)$$

Hence the shift AB bisects the transition curve at C.

Again, $\quad s = BA = EA - EB = Y - (OB - OE) = Y - R(1 - \cos\Delta_S)$

$$= Y - 2R\sin^2\frac{\Delta_S}{2} = Y - 2R\frac{\Delta_S^2}{4}, \text{ where } \Delta_S \text{ is small.}$$

But $\quad EA = DD_2 = Y = \dfrac{L^3}{6RL} = \dfrac{L^2}{6R}$ and $\Delta_S = \dfrac{L}{2R}$

$\therefore \quad s = \dfrac{L^2}{6R} - \dfrac{2R}{4}\left(\dfrac{L}{2R}\right)^2$

or $\quad s = \dfrac{L^2}{6R} - \dfrac{L^2}{8R} = \dfrac{L^2}{24\,R}$ ...(3.21)

Also, $\quad CA = y$ co-ordinate of C when $l = \dfrac{L}{2}$

$$= \frac{l^3}{6RL} = \frac{\left(\frac{L}{2}\right)^3}{6RL} = \frac{L^2}{48R} = \frac{1}{2}s = \frac{1}{2}\cdot BA \quad ...(3.22)$$

Hence the transition curve bisects the shift.

Precise expression for shift (s)

From equation 3.15 (a), we have

$$Y = EA = K\left(\frac{\Delta_S^{3/2}}{3} - \frac{\Delta_S^{7/2}}{42} + \frac{\Delta_S^{11/12}}{1320} - \ldots\right)$$

Also, $\quad EB = R(1 - \cos\Delta_S)$ where $\Delta_S = L/2R$ radians

Now, $\quad s = EA - EB$

Substituting the values and expanding $\cos\Delta_S$, we get

$$s = \frac{L^2}{24R}\left(1 - \frac{\Delta_S^2}{48} + \frac{\Delta_S^4}{1320} - \ldots\right) \text{ where } \Delta_S \text{ is in radians.} \quad ...(3.21\ a)$$

When Δ_S is in degrees, the above expression reduces to

$$s = \frac{L^2}{24R}(1 - 1.08792 \times 10^{-5}\,\Delta_S^2 + 7.03 \times 10^{-11}\,\Delta_S^4 - \ldots) \quad ...(3.21\ b)$$

or

$$s = \frac{L^2}{24R}(1 - U) \quad ...(3.21\ c)$$

The values of U for different values of Δ_S are given in the Table 3.2.

TABLE 3.2. VALUES OF U FOR DIFFERENT VALUES OF Δ_S

Δ_S	0'	6'	12'	18'	24'	30'	36'	42'	48'	54'
0°	0.00000	000	000	000	000	000	000	001	001	001
1	0.00001	001	002	002	002	002	003	003	004	004
2	0.00004	005	005	006	006	007	007	008	009	009
3	0.00010	010	011	012	013	013	014	015	016	017
4	0.00017	018	020	020	021	022	023	024	025	026
5	0.00027	028	029	031	032	033	034	035	037	038
6	0.00039	040	042	043	045	046	047	049	050	052
7	0.00053	055	056	058	060	061	063	065	066	068
8	0.00070	071	073	075	077	079	080	082	084	086
9	0.00088	090	092	094	095	098	100	102	104	107
10	0.00109	111	113	115	118	120	122	125	127	129
11	0.00131	134	136	139	141	144	146	149	152	154
12	0.00157	159	162	164	167	170	173	175	178	181
13	0.00184	186	189	192	195	198	201	204	207	210
14	0.00213	216	219	222	225	229	232	235	238	241
15	0.00224	247	251	254	258	261	264	268	271	275
16	0.00278	282	285	289	292	296	299	203	307	310
17	0.00314	318	321	325	329	333	336	340	344	348
18	0.00352	356	260	364	368	372	376	380	384	388
19	0.00392	396	400	404	408	413	418	421	425	430
20	0.00434	438	442	447	451	456	460	465	469	474
21	0.00478	483	487	492	497	501	506	511	515	520
22	0.00525	530	535	539	544	549	554	559	563	568
23	0.00573	578	583	589	594	599	604	609	614	619
24	0.00624	629	634	640	645	650	665	661	666	672
25	0.00677	682	688	693	699	704	710	716	721	727
26	0,00732	738	744	749	755	761	766	772	778	784
27	0.00789	795	801	807	813	819	825	831	837	843
28	0.00849	855	861	867	873	879	885	891	898	904
29	0.00910	916	923	929	935	941	948	954	961	967
30	0.00973	980	986	993	999	$\overline{006}$	$\overline{013}$	$\overline{019}$	$\overline{026}$	$\overline{032}$
31	0.01039	046	052	059	066	073	079	086	093	100
32	0.01107	114	120	128	134	141	148	155	162	169
33	0.01177	184	191	198	205	212	219	227	234	241
34	0.01248	256	263	270	278	285	292	300	307	315
35	0.01322	330	337	344	352	360	368	375	383	390
36	0.01398	406	414	421	429	437	445	453	461	468
37	0.01476	484	492	500	408	516	524	532	540	548
38	0.01556	564	573	581	589	597	605	613	621	630
39	0.01639	647	655	644	672	681	689	697	706	714
40	0.01723	731	740	748	757	766	774	783	792	800
41	0.01809	818	827	835	844	853	862	871	880	889
42	0.01897	906	915	924	933	882	951	960	061	978
43	0.01988	997	$\overline{006}$	$\overline{015}$	$\overline{024}$	$\overline{034}$	$\overline{043}$	$\overline{052}$	$\overline{061}$	$\overline{071}$
44	0.02080	089	009	108	117	127	136	146	155	165
45	0.02174									

LENGTH OF LONG CHORD

To set out the curve on the ground, it is often necessary to know the length (C) of the long chord TD (Fig. 3.4), joining ends of the transition curve.

Evidently $\quad C = \sqrt{X^2 + Y^2}$

Substiuting the values of X and Y and using the series, we get

$$C = L(1 - 1.3539 \times 10^{-5} \Delta_S^2 + 6.5455 \times 10^{-11} \Delta_S^4 - 7.0665 \times 10^{-17} \Delta_S^6 + \ldots) \quad \ldots(3.21\ d)$$

or $$C = L(1 - M) \quad \ldots(3.21\ e)$$

The values of M for different values of Δ_S are given in Table 3.3.

TABLE 3.3. VALUES OF M FOR DIFFERENT VALUES OF Δ_S

Δ_S	0'	6'	12'	18'	24'	30'	36'	42'	48'	54'
0°	0.00000	000	000	000	000	000	000	000	001	001
1	0.00001	002	002	002	003	003	003	004	004	005
2	0.00005	006	007	007	007	008	009	010	010	011
3	0.00012	013	014	015	016	017	017	019	020	021
4	0.00022	023	024	025	026	027	028	030	032	033
5	0.00034	035	036	038	039	041	042	044	046	048
6	0.00049	051	052	054	056	057	059	061	062	065
7	0.00066	068	070	072	074	076	078	080	082	084
8	0.00087	089	092	094	096	098	100	103	105	107
9	0.00110	113	116	118	120	122	124	127	130	232
10	0.00135	137	140	143	146	149	152	155	158	161
11	0.00164	167	170	173	176	179	182	185	188	191
12	0.00195	198	201	204	207	211	214	218	221	225
13	0.00229	233	236	240	244	248	251	255	259	262
14	0.00265	269	273	277	280	285	289	293	297	301
15	0.00305	309	313	317	321	325	329	334	338	342
16	0.00347	351	355	359	364	360	373	378	382	386
17	0.00391	395	400	405	410	415	419	424	429	434
18	0.00439	444	449	454	459	464	469	474	479	484
19	0.00489	494	499	504	509	515	520	526	531	536
20	0.00541	546	552	558	562	568	573	579	584	590
21	0.00596	601	607	613	619	625	630	636	642	648
22	0.00653	659	665	671	677	683	689	695	701	707
23	0.00741	720	727	733	739	746	752	759	765	771
24	0.00778	784	791	798	804	811	817	824	831	837
25	0.00843	849	856	863	876	877	884	891	898	905
26	0.00912	919	926	933	940	948	955	962	969	979
27	0.00984	991	999	$\overline{006}$	$\overline{012}$	$\overline{020}$	$\overline{027}$	$\overline{035}$	$\overline{042}$	050
28	0.01058	065	073	080	088	096	103	111	119	127
29	0.01134	142	150	158	166	170	182	190	198	206
30	0.01214	222	230	238	246	255	263	270	278	286
31	0.01295	303	312	320	329	338	346	355	363	371
32	0.01379	387	396	405	414	423	432	441	450	459
33	0.01462	476	485	494	503	521	521	520	539	548
34	0.01556	565	575	584	593	603	612	621	630	640
35	0.01649	658	668	677	686	696	705	715	725	735
36	0.01844	753	763	773	783	793	803	813	823	833
37	0.01842	852	862	872	882	891	903	911	921	931
38	0.01941	951	972	962	982	993	$\overline{003}$	$\overline{014}$	$\overline{024}$	$\overline{034}$
39	0.02044	054	065	076	087	097	107	118	128	139
40	0.02149	160	171	182	193	203	214	225	236	247
41	0.02257	268	279	290	301	312	323	334	345	356
42	0.02368	379	391	402	414	425	436	448	459	471
43	0.02482	493	505	515	527	439	550	561	572	585
44	0.02596	608	620	632	644	656	667	679	691	703
45	0.02715									

THE TRUE SPIRAL

For the clothoid or the true spiral, the co-ordinate of any point B (Fig. 3.2 and 3.3) may be represented as

$$x = l\left(1 - \frac{\phi_S^2}{10}\right) = l\left(1 - \frac{l^4}{40R^2 L^2}\right) \quad \ldots(3.14\ b)$$

and $$y = \frac{l^3}{6RL}\left(1 - \frac{\phi_S^2}{14}\right) = \frac{l^3}{6RL}\left(1 - \frac{l^4}{56R^2 L^2}\right) \quad \ldots(3.15\ b)$$

The co-ordinates of the junction point (D) are given by

$$X = L\left(1 - \frac{\Delta_S^2}{10}\right) = L\left(1 - \frac{L^2}{40R^2}\right) = L\left(1 - \frac{3s}{5R}\right)$$

$$Y = \frac{L^2}{6R}\left(1 - \frac{\Delta_S^2}{14}\right) = \frac{L^2}{6R}\left(1 - \frac{L^2}{56R^2}\right)$$

The intrinsic equation is

$$\phi = \frac{l^2}{2RL} \quad \ldots(3.13)$$

$$\alpha = \frac{\phi}{3} = \frac{l^2}{6RL} \text{ radians} = \frac{1800 l^2}{\pi RL} \text{ minutes} = \frac{573\ l^2}{RL} \text{ minutes} \quad \ldots(3.16\ b)$$

$$\alpha_S = \text{polar deflection angle to the junction point} = \frac{573L}{R} \text{ minutes}$$

$$\Delta_S = 3\,\alpha_S = \frac{1719L}{R} \text{ minutes} = \frac{L}{2R} \text{ radians}$$

Total tangent length $TV = AV + TA$

$$AV = (R + s)\tan\frac{\Delta}{2}$$

$$TA = TD_2 - AD_2 = X - R\sin\Delta_S$$

But $$X = L\left(1 - \frac{\Delta_S^2}{10}\right) \text{ and } \Delta_S = \frac{L}{2R}$$

$$\therefore \quad TA = L\left(1 - \frac{\Delta_S^2}{10}\right) - R\left(\Delta_S - \frac{\Delta_S^3}{6}\right) = L\left(1 - \frac{L^2}{40R^2}\right) - R\left(\frac{L}{2R} - \frac{L^3}{48\,R^3}\right)$$

$$= \frac{L}{2}\left(1 - \frac{L^2}{120R^2}\right) = \frac{L}{2}\left(1 - \frac{s}{5R}\right)$$

Hence the total tangent length

$$= (R + s)\tan\frac{\Delta}{2} + \frac{L}{2}\left(1 - \frac{s}{5R}\right) \quad \ldots(3.23)$$

In the above expression, the amount $s \tan\frac{\Delta}{2}$ is called as the *shift increment* and $(X - R\sin\Delta_S)$ or $\frac{L}{2}\left(1 - \frac{s}{5R}\right)$ is called as the *spiral extension.*

THE CUBIC SPIRAL

The co-ordinates of any point B are represented by

$$y = \frac{l^3}{6RL} \quad \text{where } l \text{ is measured along the curve.}$$

For the junction point D, $l = L$

$$\therefore \quad Y = \frac{L^3}{6RL} = \frac{l^2}{6R}$$

The intrinsic equation of the curve is

$$\phi = \frac{l^2}{2RL}$$

$$\alpha = \frac{\phi}{3} = \frac{l^2}{6RL} \text{ radians} = \frac{1800\, l^2}{\pi RL} \text{ minutes} = \frac{573\, l^2}{RL} \text{ minutes}$$

$$\alpha_S = \frac{573\, L}{R} \text{ minutes}$$

Total tangent length $TV = AV + TA = (R + s) \tan \frac{\Delta}{2} + \frac{L}{2}\left(1 - \frac{s}{5R}\right)$

If, however Δ_S is very small, the total tangent length may be taken approximetly equal to $(R + s) \tan \frac{\Delta}{2} + \frac{L}{2}$.

THE CUBIC PARABOLA

The co-ordinates of any point B are respresented by

$$y = \frac{x^3}{6RL} \quad \text{where } x \text{ and } y \text{ are cartesian co-ordinates}$$

$$\tan \alpha = \frac{y}{x} = \frac{x^2}{6RL} = \frac{l^2}{6RL}$$

$$\alpha = \frac{l^2}{6RL} \text{ radians} = \frac{1800\, l^2}{\pi LR} \text{ minutes} = \frac{573\, l^2}{RL} \text{ minutes}$$

$$\alpha_S = \frac{1800\, L}{\pi R} = \frac{573\, L}{R} \text{ minutes}$$

$$\Delta_S = \frac{L}{2R} \text{ radians} = \frac{1719\, L}{R} \text{ minutes}$$

The co-ordinates of the junction point D are

$$X = L$$

and $$Y = \frac{L^2}{6R} = 4s$$

Total tangent length $= AV + TA = (R + s) \tan \frac{\Delta}{2} + (X - R \sin \Delta_S)$

But $X = L$ and $\sin \Delta_S \simeq \Delta_S = \frac{L}{2R}$ radians.

$\therefore$ Total tangent length $= (R + s) \tan \frac{\Delta}{2} + \left(L - R \frac{L}{2R}\right) = (R + s) \tan \frac{\Delta}{2} + \frac{L}{2}$ ...(3.24)

Length of the Combined Curve

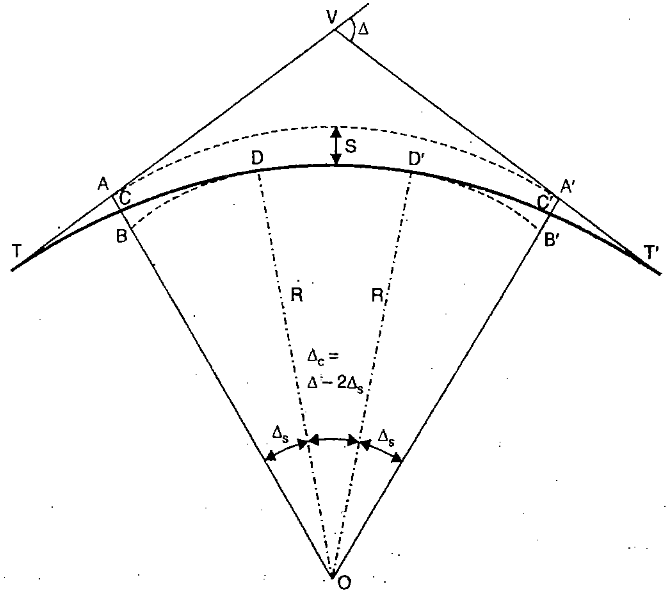


FIG. 3.5. THE COMBINED CURVE

In Fig. 3.5, DD' is the circular curve, and TD and $D'T'$ are the two transition curves at the two ends of the circular curve. If Δ is the total deflection angle between the original tangents and Δ_S is the spiral angle for each transition curve, we have

$$\Delta_C = \Delta - 2\Delta_S \quad \text{where} \quad \Delta_C = \text{central angle for the circular curve.}$$

Hence, the length of the circular curve

$$= \frac{\pi R \Delta_C}{180^\circ} = \frac{\pi R (\Delta - 2\Delta_S)}{180^\circ}$$

Total length of the combined curve

$$= \frac{\pi R (\Delta - 2\Delta_S)}{180^\circ} + 2L \quad ...(3.25)$$

The total length of the combined curve can also be approximately found by considering the circular arc $BDD'B'$ having the total central angle Δ.

$$\text{Total length} = \frac{\pi R \Delta}{180^\circ} + \frac{L}{2} + \frac{L}{2} = \frac{\pi R \Delta}{180^\circ} + L \quad ...(3.25\ a)$$

3.5. COMPUTATIONS AND SETTING OUT

In order to make the computations for various quantities of the transition and circular curves, the data necessary are : (*i*) the deflection angle Δ between the original tangents, (*ii*) the radius R of the circular curve, (*iii*) the length L of the transition curve and (*iv*) the chainage of the point of the intersection (V).

The computations are done in the following steps :

(1) Calculate the spiral angle Δ_s by the equation

$$\Delta_s = \frac{L}{2R} \text{ radians.}$$

(2) Calculate the shift s of the circular curve by the relation

$$s = \frac{L^2}{24R}$$

(3) Calculate the total length of the tangent from equation 3.23 or 3.24 depending whether it is a spiral or cubic parabola.

(4) Calculate length of the circular curve.

(5) From the chainage of the P.I., subtract the length of the tangent to get the chainage of the point T.

(6) To the chainage of T, add the length of the transition curve to get the chainage of the junction point (D) of the transition curve with the circular curve.

(7) Determine the chainage of the other junction point (D') of the circular arc with the transition curve, by adding the length of the circular curve to the chainage of D.

(8) Determine the chainage of the point T' by adding the length L of the transition curve to the chainage of D'.

(9) If it is required to peg the points on through chainages, calculate the length of the sub-chords and full chords of the transition curves and the circular curve. The peg interval for the transition curve may be 10 metres, while that for the circular curve it may be 20 metres.

(10) If the curves are to be set out by a theodolite, calculate the deflection angles for transition curve from the expression

$$\alpha = \frac{573\, l^2}{RL} \text{ minutes}$$

and the delecflion angles (referred to the tangent at D) for the circular curve from the expression

$$\delta = 1719 \frac{C}{R} \text{ minutes}$$

The toal tangential angles Δ_n for the circular curve must be equal to $\frac{1}{2}(\Delta - 2\Delta_S)$.

(11) If, however, the curves are to be set out by linear methods, calculate the offsets from the following formulae :

For the true spiral

$$y = \frac{l^3}{6RL}\left(1 - \frac{\phi^2}{14}\right) \quad \text{or} \quad = \frac{l^3}{6RL}\left(1 - \frac{l^4}{56\, R^2 L^2}\right)$$

y being measured perpendicular to the tangent TV and l measured along the curve.

For the cubic spiral

$$y = \frac{l^3}{6\, RL}$$

y being measured perpendicular to the tangent TV and l measured *along* the curve.

For the cubic parabola

$$y = \frac{x^3}{6\, RL}$$

x being measured along the tangent TV and y perpendicular to it.

For the circular curve

$$O_n = \frac{b_n(b_{n-1} + b_n)}{2\, R}$$

where O is the offset from the chords (produced).

SETTING OUT THE COMBINED CURVE BY DEFLECTION ANGLES (Fig. 3.6)

(1) Locate the tangent point T by measuring back the tangent length from the P.I. (V). Similarly, locate the other tangent point T' by measuring along the forward tangent the length from the P.I.

Alternatively, the position of T can be found by first locating A from the measurement

$VA = (R + s) \tan \frac{1}{2} \Delta$.

At A, set a perpendicular $AC = \frac{1}{2} s$. From C, swing an arc of length $\frac{1}{2} L$ to intersect the initial tangent T.

(2) Set the theodolite at T and direct the line of sight towards V when the reading is zero.

(3) Release the vernier plate and set the vernier to the first deflection angle (α_1) thus directing the line of sight to the first peg on the transition curve.

(4) With the zero of the tape pinned at T, swing the length of the tape equal to the length of the first chord till the arrow held at that distance along the tape is bisected by the line of sight. The first peg is thus fixed.

(5) Set the angle α_2 on the circle so that the line of sight is directed to the second point. With the zero of the tape pinned at T hold an arrow at a distance equal to the length of the second chord and swing it till bisected by the line of sight, thus fixing the second point.

(6) Repeat the procedure until the last point D is set out. For every point, the chord distance is measured from the point T and *not* from the previous point as is done in a circular curve. Check

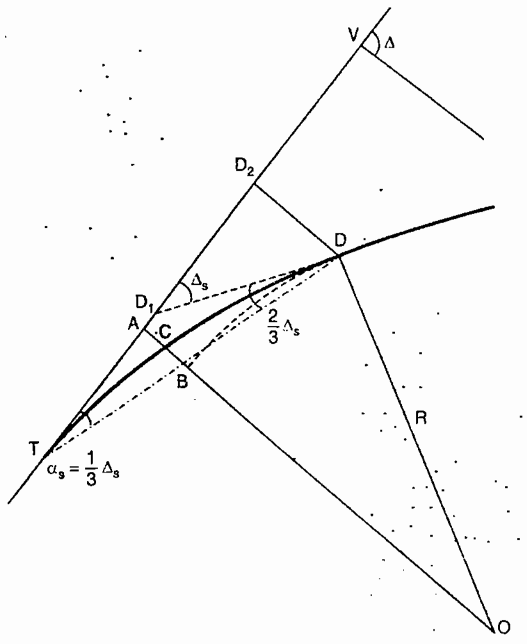


FIG. 3.6. SETTING OUT THE COMBINED CURVE

the position of D by measuring the offset $DD_2 = \frac{L^2}{6R} = 4s$.

(7) To set out the circular curve, shift the theodolite to the junction point D. To orient the theodolite with reference to the common tangent DD_1, direct the line of sight towards DT with the reading equal to $(360° - \frac{2}{3}\Delta_S)$ for a right hand curve. Since $\angle DTV = \frac{1}{3}\angle DD_1V = \frac{1}{3}\Delta_S$, we have $\angle D_1DT = \frac{2}{3}\Delta_S$. When the theodolite is rotated in azimuth by an angle $\frac{2}{3}\Delta_S$ (till zero reading is obtained on the circle), the line of sight will be directed along DD_1. On transiting the theodolite now, the line of sight is directed along the tangent D_1D with reference to which the deflection angles of the circular curve have been calculated. When the line of sight is thus correctly oriented, the reading on the circle will be zero. To locate the first peg on the circular curve, the first deflection angle Δ_1 is set out on the curve as usual.

Set out the circular curve in the usual way till the junction point D' is reached, the position of which may be checked by measuring the offset $(= 4s)$ to the second tangent at the point.

(8) Set out the other transition curve from T' as before.

SETTING OUT BY TANGENT OFFSETS (Fig. 3.3)

(1) Locate the tangents point T as explained above and obtain its chainages.

(2) Calculate the offset y from the expression

$$y = \frac{l^3}{6RL}.$$

(3) Locate each peg by swinging the chord length from the preceding peg until required offset is obtained.

Cubic Parabola

(1) Locate the tangent point T as explained.

(2) Choose convenient values of co-ordinates x and calculate the corresponding values of y from the equation

$$y = \frac{x^3}{6RL}.$$

(3) Measure the abscissae (x) along the tangent TV and locate the points on the curve by setting out the respective offsets (y).

SETTING OUT BY FIXED ANGLES OF EQUAL CHORDS

Sometimes, it is not necessary to drive the pegs at even chainages along the transition curve. In that case, the calculations are simplified by using a fixed set of angles and calculating the corresponding length of chords required.

From equation 3.13, we have

$$\phi = \frac{l^2}{2RL} \qquad \ldots(i)$$

At $\qquad l = L, \ \phi = \Delta_S$

$$\therefore \qquad \Delta_S = \frac{L^2}{2RL} \qquad \ldots(ii)$$

Dividing (i) and (ii), we have

$$\frac{\phi}{\Delta_S} = \frac{l^2}{L^2}$$

or

$$l = L\sqrt{\frac{\phi}{\Delta_S}} \qquad \ldots(3.26)$$

Let the first deviation angle be equal to ϕ_1 and the corresponding value of the chord $= l = c$. If after setting out n chords (each of length c), the deviation angle is ϕ_n, we have

$$l = nc = L\sqrt{\frac{\phi_n}{\Delta_S}}$$

or

$$\phi_n = \frac{\Delta_S\, n^2c^2}{L^2}$$

$$\therefore \qquad \phi_1 = \frac{\Delta_S\, c^2}{L^2}$$

Hence $\qquad \phi_n = n^2\,\phi_1 \qquad \ldots(3.27)$

Thus, the length c for the first deviation angle can be calculated from the equation 3.26. For the equal chords, the subsequent deviation angles can calculated from equation 3.27. For example,

$$\phi_2 = n^2\phi_1 = (2)^2\,\phi_1 = 4\,\phi_1$$
$$\phi_3 = (3)^2\phi_1 = 9\,\phi_1$$
$$\phi_4 = (4)^2\,\phi_1 = 16\,\phi_1$$

and so on.

The last deviation angle will be Δ_S corresponding to a total length of $L = mc + c'$, where m = total number of chords each of length c and c' is the last sub-chord. Since the polar deflection angle $\alpha = \frac{1}{3}\phi$, we have

$$\alpha_1 = \frac{1}{3}\phi_1$$
$$\alpha_2 = \frac{1}{3}\phi_2 = \frac{4}{3}\phi_1$$
$$\alpha_3 = \frac{1}{3}\phi_3 = \frac{9}{3}\phi_1$$
$$\alpha_4 = \frac{1}{3}\phi_4 = \frac{16}{3}\phi_1$$

and so on.

With the help of Table 3.1, Table 3.4 can be prepared giving the polar deflection angles for equal chords, the chords length being selected so that first angle is $\alpha_1 = 1'$ (or $\phi_1 = 3'$).

TABLE 3.4
VALUES OF α FOR EQUAL CHORDS

Chord No.	Polar Deflection angle (α)			Chord No.	Polar Deflection angle (α)		
	°	′	″		°	′	″
1	0	1	0	16	4	15	54
2	0	4	0	17	4	48	51
3	0	9	0	18	5	23	47
4	0	16	0	19	6	0	47
5	0	25	0	20	6	39	35
6	0	36	0	21	7	20	27
7	0	49	0	22	8	3	15
8	1	4	0	23	8	47	58
9	1	21	0	24	9	34	46
10	1	40	0	25	10	23	26
11	2	1	0	26	11	14	0
12	2	24	0	27	12	6	30
13	2	48	58	28	13	0	53
14	3	15	57	29	13	57	10
15	3	44	56	30	14	55	28

3.6. SPIRALLING COMPOUND CURVES

Transition curves are also sometimes introduced between two branches of compound curves in railway so that the radius of curvature may not change abruptly from R_1 and R_2. Such spiralling is shown in Fig. 3.7. The computations etc., are done in the following manner:

(1) For a design speed v, calculate the super-elevations (e_1 and e_2) corresponding to the two radii from the expression

$$e = \frac{v^2}{gR} G$$

(2) Calculate the lengths (L_1 and L_2) of the transition curves from the relation $L = ne$.

However, the lengths L_1 and L_2 can also be fixed either on the comfort basis or fixed arbitrarily :

(3) Calculate the shifts (s_1 and s_2) for both the branches by the relation

$$s = \frac{L^2}{24 R}$$

The distance $F_1 F_3$ between the tangents of the shifted curves $= s_1 - s_2$. The transition curve at the common point of tangency (E) is bisected at F_2, and F_2 is midway between F_1 and F_3.

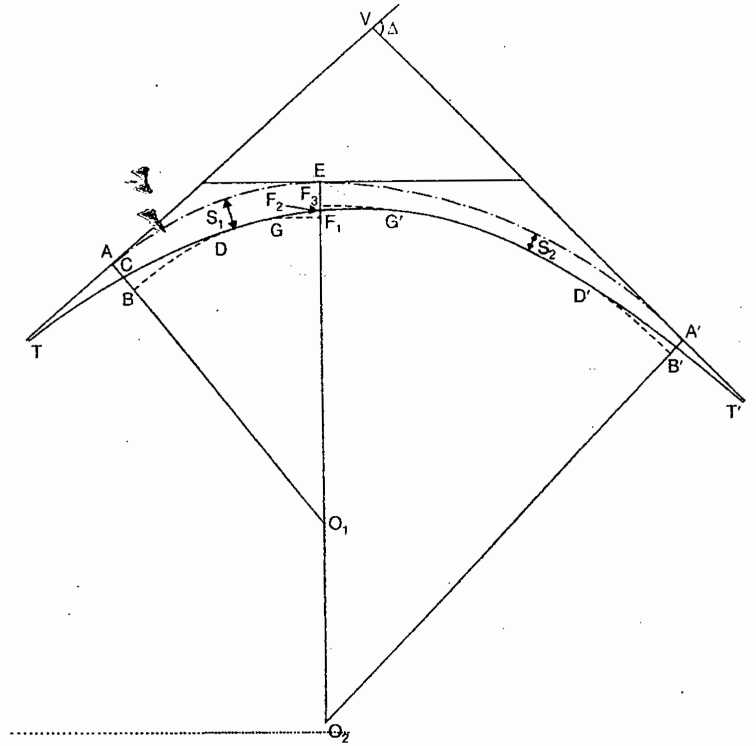


FIG. 3.7. SPIRALLING A COMPOUND CURVE

(4) The length (L') of the transition curve required at the common point of tangency is calculated from the equation.

$$L' = n(e_1 - e_2)$$

However, L' can also be fixed either arbitrarily or on the comfort basis.

(5) Obtain the chainages of the point T as explained earlier.

Chainage of D = chainage of $T + L_1$.

Chainage of G = (chainage of D + length of first circular arc) $- \frac{L'}{2}$.

Chainage of F_2 = chainage of $G + \frac{L'}{2}$

Chainage of G' = chainage of $F_2 + \frac{L'}{2}$

Chainage of D' = chainage of G' + length of second circular arc $- \frac{L'}{2}$

Chainage of T' = chainage of $D' + L_2$.

(6) The first and the last transition curves and the two branches of the circular curves can be set out as explained earler.

(7) The offsets for the intermediate or common transition curve can be approximately calculated from the equation

$$y = \frac{4(s_1 - s_2)}{L^3} x^3.$$

(8) Locate the points G and G' (*i.e.*, the points in which the intermediate transition curve meet the two arcs) by setting out $\frac{L'}{2}$ from F_2 in each direction.

3.7. SPIRALLING REVERSE CURVES

In the case of reverse curve, the amount and *direction* of curvature changes from one value to the other and hence a reverse transition curve, as shown in Fig. 3.8, should be inserted between the two branches. The procedure for calculations is similar to that for a compound curve.

Let e_1 and e_2 be the required super-elevations calculated from the expression.

$$e = \frac{v^2}{gR} G$$

Greatest change of cant = $(e_1 + e_2)$

If n = rate of canting, the length L' of the reverse transition curve is given by

$$L' = n(e_1 + e_2).$$

Half of this will be provided to each side of P.R.C.

The distance EG between the tangents of the two shifted arcs = $GF + FE = s_1 + s_2$.

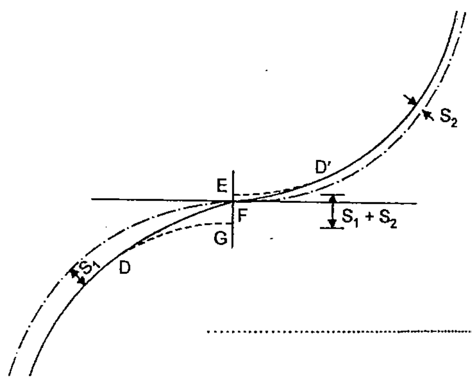


FIG. 3.8. SPIRALLING REVERSE CURVE.

The offsets to the transition curve may be calculated from the expression :

$$y = \frac{4(s_1 + s_2)}{L'^3} x^3.$$

3.8. BERNOULLI'S LEMNISCATE CURVE

Bernoulli's Lemniscate is commonly used in road work where it is required to have the *curve transitional throughout* having no intermediate circular curve. Since the curve is symmetrical and transitional throughout, the super-elevation or cant continuously increases till the apex is reached. This may, sometimes, be objectionable specially in railways. However, on highways, it is used in perference to the spiral for the following reasons :

(1) Its radius of curvature decreases more gradually.

(2) Its rate of increase of curvature diminishes towards the transition curve — thus fulfiling an essential condition.

(3) It corresponds to an autogenous curve of an automobile (*i.e.*, the path actually traced by an automobile when turning freely).

In Fig. 3.9, curve OA is the lemniscate, OB the clothoid and OC the cubic parabola. For small deviation angles (upto say 12°) there is little difference between the three, but for large angles, the cubic parabola leaves the other two curves ; its radius of curvature reaches a minimum value when $\phi \approx 24° 6'$ and starts to increase again. The clothoid and lemniscate are almost identical upto deviation angle of 60°, but after that the radius of curvature of lemniscate is greater than that of the clothoid. At deviation angle of 135°, the radius of curvature of the lemniscate is minimum and at a greater deviation angle it begins to increase again.

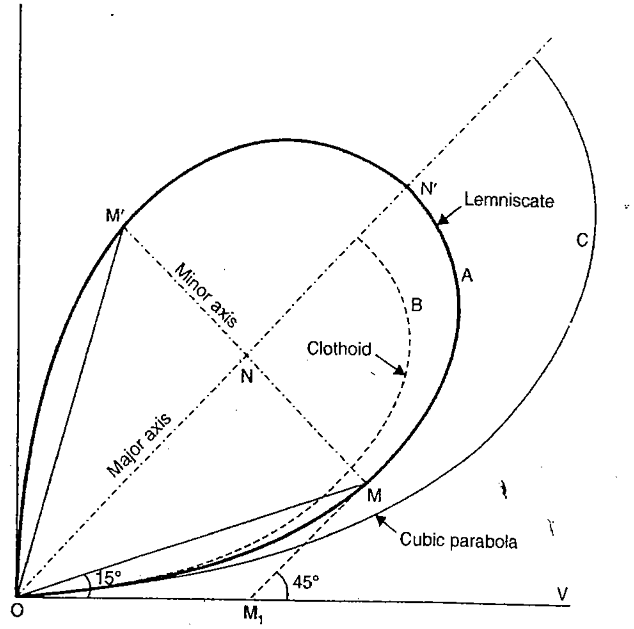


FIG. 3.9. VARIOUS TRANSITION CURVES.

Fig. 3.10 shows half the lemniscate curve in the first quadrant.

OV = initial tangent

OA = the major axis of the curve (or the polar ray making a polar deflection angle of 45° with the tangent)

P = any point on the curve

PP_1 = tangent to the curve at P

ϕ = angle between the tangent to the curve at P and the initial tangent = deviation angle

b = length OP of the polar ray

α = polar deflection angle

θ = angle between the polar ray PO and the tangent PP_1 to the curve at P.

The polar equation of Bernoulli's Lemniscate is

$$b = K \sqrt{\sin 2\alpha} \qquad \text{...(3.28)}$$

From the properties of polar co-ordinates,

$$\tan \theta = b \frac{d\alpha}{db}$$

But $\frac{db}{d\alpha} = \frac{K \cos 2\alpha}{\sqrt{\sin 2\alpha}}$ from Eq. 3.28.

$\therefore \quad \tan\theta = K\sqrt{\sin 2\alpha}\,\dfrac{\sqrt{\sin 2\alpha}}{K\cos 2\alpha}$

or $\quad \tan\theta = \tan 2\alpha$

$\therefore \quad \theta = 2\alpha$ ...[3.29 (*a*)]

Hence $\quad \phi = \alpha + \theta = 3\alpha$ ...(3.29)

Thus, for the lemniscate curve, deviation angle is exactly equal to three times the polar deflection angle. For clothoid or cubic parabola, this relation is approximately true.

Equation 3.29 is the most important property of the lemniscate.

The radius of curvature *r* at any point is given by the usual formulae for polar co-ordinate, *i.e.*,

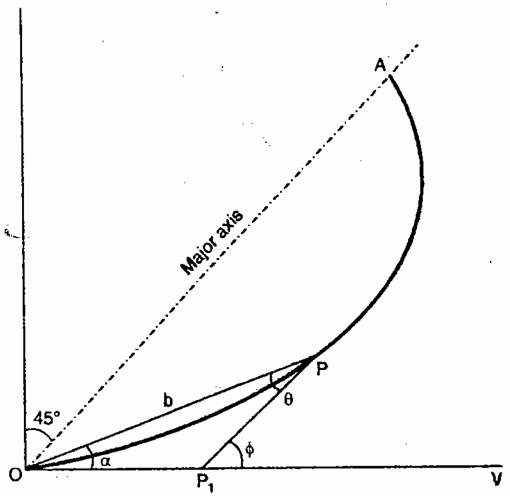


FIG. 3.10. BERNOULLI'S LEMNISCATE.

$$r = \frac{\left[b^2 + \left(\dfrac{db}{d\alpha}\right)^2\right]^{3/2}}{\left[b^2 + 2\left(\dfrac{db}{d\alpha}\right)^2 - b\dfrac{d^2b}{d\alpha^2}\right]}$$

Substituting $\quad \dfrac{db}{d\alpha} = \dfrac{K\cos 2\alpha}{\sqrt{\sin 2\alpha}}$

and $\quad \dfrac{d^2b}{d\alpha^2} = -\dfrac{K}{(\sin 2\alpha)^{3/2}}(1 + \sin^2 2\alpha)$

We get

$$r = \frac{\left[K^2\sin 2\alpha + \dfrac{K^2\cos^2 2\alpha}{\sin 2\alpha}\right]^{3/2}}{\left[K^2\sin 2\alpha + 2\dfrac{K^2\cos^2 2\alpha}{\sin 2\alpha} + K\sqrt{\sin 2\alpha}\,\dfrac{K}{(\sin 2\alpha)^{3/2}}(1+\sin^2 2\alpha)\right]}$$

or $\quad r = \dfrac{K}{3\sqrt{\sin 2\alpha}}$ ...(3.30)

Substituting $\quad K = \dfrac{b}{\sqrt{\sin 2\alpha}}$, we get

$\quad r = \dfrac{b}{3\sin 2\alpha}$ ...(3.31)

From equation 3.30, $\quad K = 3r\sqrt{\sin 2\alpha}$

Substituting the value of $\sqrt{\sin 2\alpha} = \dfrac{b}{K}$, we get

$K = 3r\dfrac{b}{K}$ or $K = \sqrt{3br}$ ...(3.32)

If *l* is the length of the curve corresponding to a deviation angle ϕ, we have

$$\frac{dl}{d\phi} = r = \frac{K}{3\sqrt{\sin 2\alpha}}$$

Integrating this, we get

$$l = \frac{K}{\sqrt{2}}\left(2\tan^{1/2}\alpha - \frac{1}{5}\tan^{5/2}\alpha + \frac{1}{12}\tan^{9/2}\alpha - \frac{5}{104}\tan^{13/2}\alpha + \ldots\right) \quad \ldots(3.33)$$

The series of Eqn. 3.33 does not converge very rapidly and is not convenient to compute. However, Prof. F.G. Royal Dawson has suggested the following empirical formula for the length of the lemniscate as a road transition curve :

$$l = \frac{2K\alpha}{\sqrt{\sin\alpha}}\,.\cos k\alpha = 6\,r\alpha\sqrt{\cos\alpha}\,.\cos k\alpha \quad \ldots[3.33\ (a)]$$

In the above expression, *k* is a co-efficient whose approximate value for different values of α are given in Table 3.5.

TABLE 3.5. VALUES OF k

α	*k*	α	*k*
5°	0.190	30°	0.173
10°	0.187	35°	0.168
15°	0.184	40°	0.163
20°	0.181	45°	0.159
25°	0.177		

For small angles, we can write

$l = 6\,r\alpha$, where α is in radians ...[3.33 (*b*)]

or $\quad l = \dfrac{r\alpha}{9.55}$ where α is in degrees.

In Fig. 3.9, *ON'* is the polar ray for α = 45° and is the major axis. To locate the minor axis, draw the polar ray *OM* at 15° with *OV*. Draw MM_1 tangent, to the curve. Thus, $\angle MM_1V = 3 \times 15° = 45°$ and hence MM_1 and *ON'* are parallel. From *M*, draw *MNM'* perpendicular to *ON'*. *MM'* is then the minor axis and triangle *OMM'* is equilateral.

$$MM' = OM = K\sqrt{\sin 30°} = \frac{K}{\sqrt{2}}$$

$$ON' = K\sqrt{\sin 90°} = K$$

$\therefore \quad MM' = \dfrac{ON'}{\sqrt{2}}$

or $\quad \dfrac{MM'}{ON'} = \dfrac{\text{minor axis}}{\text{major axis}} = \dfrac{1}{\sqrt{2}} = \dfrac{1}{1.4142}$

Now $\quad r = \dfrac{K}{3\sqrt{\sin 2\alpha}}$, decreasing with the increasing value of α

Now $\quad \alpha_{max} = 45°$ at *N'*, we have

$$r_{min} = \frac{K}{3} = \frac{1}{3}(ON') = \frac{1}{3} \text{ major axis}$$

Length $\quad OMN' = 1.31115\ ON' = 1.31115\ K.$

Lemnisate Curve used Transitional Throughout

Fig. 3.11 shows the lemniscate curve used transitional throught out. Let T_1 and T_2= tangent points.

M = apex of the curve
V = P.I.
Δ = total deflection angle of the tangents
VM = the apex distance
AMB = common tangent to the two branches of the lemniscate
$\angle VAM = \phi$ for the polar ray T_1M
α_n = polar deflection angle for T_1M.

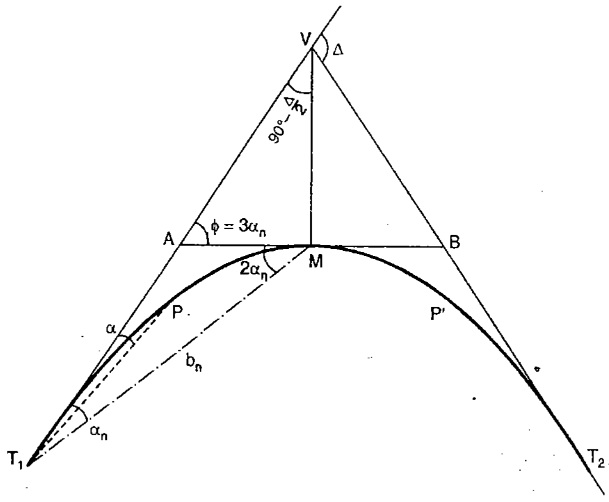


FIG. 3.11. LEMNISCATE CURVE TRANSITIONAL THROUGHOUT.

Curve T_1M and MT_2 are two lemniscates symmetrical about VM. The curve is transitional throughout having no circular curve between the two branches. VM is the bisector of $\angle AVB$ and is common normal to the common tangent AMB.

$$\angle AVB = (180° - \Delta)$$

$\therefore$
$$\angle AVM = \frac{1}{2}(180° - \Delta) = 90° - \frac{\Delta}{2}$$

$$\angle VMA = 90°$$

$$\angle VAM = \phi_n = 180° - \frac{1}{2}(180° - \Delta) = 90° - \frac{\Delta}{2}$$

But
$$\phi_n = 3\ \alpha_n$$

$\therefore$
$$\alpha_n = \frac{1}{3}\phi_n = \frac{1}{3}\frac{\Delta}{2} = \frac{\Delta}{6} \quad \ldots(3.34)$$

Hence, for the curve to be transitional throughout, the maximum polar deflection angle must be equal to $\frac{1}{6}th$ of the deflection angle between the initial tangents.

Now, consider triangle T_1VM

$$\angle T_1VM = \frac{1}{2}(180° - \Delta) = 90° - \frac{\Delta}{2}$$

$$\angle T_1MV = 90° + 2\alpha_n = 90° + \frac{\Delta}{3}.$$

$$\angle VT_1M = \alpha_n = \frac{\Delta}{6}$$

Thus, all the three angles are known.

(*i*) *If the apex distance OM and the angle* Δ *are given*, the other two sides T_1V and T_1M can be calculated by sine rule. Knowing the tangent length $T_1V(= VT_2)$ the tangent points T_1 and T_2 can be located and the curve can be set out.

(*ii*) *If the minimum radius at end (M) and the angle* Δ *are given*, the length of the polar ray T_1M can be calculated from the equation 3.31, *i.e.*

$$b_n = 3\ r \sin 2\ \alpha_n$$

Knowing $T_1M = b_n$, the lengths T_1V and VM can be calculated by the sine rule. The tangent points T_1 and T_2 can then be located and the two branches can be set out by theodolite from T_1 and T_2.

For setting out the curve by deflection angles, a table giving various values of α and b may be prepared by assuming successive values of α and then calculating the values of b from the relation.

$$b = K\sqrt{\sin 2\ \alpha}.$$

Lemniscate as Transition Curve at the Ends of Circular Curve:

We have seen that for the lemniscate to be transitional throughout, the polar deflection angle should be $\frac{1}{6}$th of the deflection angle between the tangents. If, however, α_n is lesser than $\frac{1}{6}\Delta$, it is necessary to introduce circular curve between the two lemniscate curves.

Fig. 3.12 shows lemniscate curve TD, used as a transition at the beginning of a circular curve DD', D being the junction point where the lemniscate meets the curve tangentially.

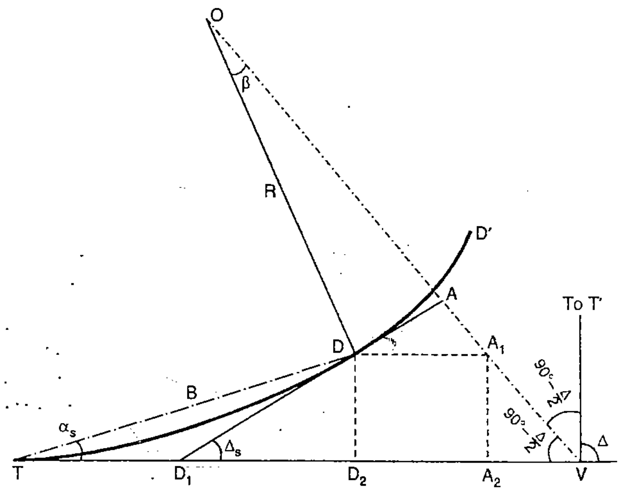


FIG. 3.12

Let $\angle DTV = \alpha_S$ = total polar deflection angle

$\angle DD_1V = \Delta_S$ = total deviation angle.

Draw DD_1 tangent to the curve at D. Join V and the centre O of the circular curve. Due to the symmetry of the transition at both the ends of the circular curve, OV will bisect the angle TVT', where TV and VT' are the initial tangents. Draw DA_1 parallel to the tangent TV and DD_2 and A_1A_2 perpendicular to it.

Calculation of tangent length. In order to set out the curve, it is necessary to calculate the tangent length TV and hence to locate the tangent point T.

Now $$TV = TD_2 + D_2A_2 + A_2V \quad \ldots(i)$$

$$TD_2 = B \cos \alpha_S \quad \ldots(ii)$$

where B = length of extreme polar ray when $\alpha_S = \frac{\Delta_S}{3}$

$$\angle T'VO = \angle OVT = \tfrac{1}{2}(180° - \Delta) = 90° - \frac{\Delta}{2}$$

$$\therefore \quad \angle OAD = \Delta_S + 90° - \frac{\Delta}{2}$$

$$\therefore \quad \angle AOD = \beta = 90° - \angle OAD = 90° - \left(\Delta_S + 90° - \frac{\Delta}{2}\right) = \frac{\Delta}{2} - \Delta_S$$

From triangle ODA_1,

$$\frac{DA_1}{DO} = \frac{\sin A_1OD}{\sin OA_1D} = \frac{\sin \beta}{\sin(90° - \Delta/2)}$$

$$\therefore \quad DA_1 = \frac{R \sin \beta}{\sin(90° - \Delta/2)} = \frac{R \sin(\Delta/2 - \Delta_S)}{\cos \Delta/2} = R\left(\cos \Delta_S \tan \frac{\Delta}{2} - \sin \Delta_S\right)$$

$$D_2A_2 = DA_1 = R\left(\cos \Delta_S \tan \frac{\Delta}{2} - \sin \Delta_S\right) \quad \ldots(iii)$$

$$A_2V = A_1A_2 \cot\left(90° - \frac{\Delta}{2}\right) = DD_2 \cot\left(90° - \frac{\Delta}{2}\right) = B \sin \alpha_S \tan \frac{\Delta}{2} \quad \ldots(iv)$$

Adding (ii), (iii) and (iv), we get the total tangent length

$$TV = B \cos \alpha_S + R\left(\cos \Delta_S \tan \frac{\Delta}{2} - \sin \Delta_S\right) + B \sin \alpha_S \tan \frac{\Delta}{2}.$$

Example 3.1. *A transition curve is required for a circular curve of 200 metre radius, the gauge being 1.5 m and maximum super-elevation restricted to 15 cm. The transition is to be designed for a velocity such that no lateral pressure is imposed on the rails and the rate of gain of radial acceleration is 30 cm/sec³. Calculate the required length of the transition curve and the design speed.*

Solution.

On the basis of radial acceleration, the length of the transition curve is given by

$$L = \frac{v^3}{\alpha R}$$

where $\alpha = 0.30$ m/sec² ; $R = 200$ m ; v = velocity in m/sec

$$\therefore \quad L = \frac{v^3}{0.3 \times 200} = \frac{v^3}{60} \quad \ldots(i)$$

The velocity v is determined from the requirement of no lateral pressure on a super-elevation of 15 cm for $G = 15$ m.

$$\tan \theta = \frac{15}{150} = \frac{v^2}{gR}$$

$$v = \left(\frac{15}{150} \times gR\right)^{1/2} = \left(\frac{1}{10} \times 9.81 \times 200\right)^{1/2}$$

$= 14$ m/sec or **50.4 km/hour.**

Substituting the value of v in (i), we get

$$L = \frac{v^3}{60} = \frac{(14)^3}{60} \approx \mathbf{46\ m.}$$

Example 3.2. *A road bend which deflects 80° is to be designed for a maximum speed of 100 km per hour, a maximum centrifugal ratio of 1/4 and a maximum rate to the change of acceleration of 30 cm/sec³, the curve consisting of a circular arc combined with two cubic spirals. Calculate (a) the radius of the circular arc, (b) the requisite length of transition (c) the total length of the composite curve, and (d) the chainages of the beginning and end of transition curve, and of the junctions of the transition curves with the circular arc if the chainage of the P.I. is 42862 metres.*

Solution.

$V = 100$ kmph

$$\therefore \quad v = \frac{100 \times 1000}{60 \times 60} = 27.78 \text{ m/sec.}$$

$$\text{Centrifugal ratio} = \frac{v^2}{gR} = \frac{1}{4} \text{ (given)}$$

$$\therefore \quad R = \frac{4v^2}{g} = \frac{4(27.78)^2}{9.81} = 314.68 \approx 315 \text{ m}$$

The length of the transition curve is given by

$$L = \frac{v^3}{\alpha R} = \frac{(27.78)^3}{0.3 \times 315} = 226.9 \text{ m} \approx \mathbf{227\ m.}$$

$$\Delta_S = \frac{L}{2R} \text{ radians} = 1719 \frac{L}{R} \text{ min} = 1719 \frac{227}{315} = 20° 38' 48''$$

Central angle for the circular curve $\Delta_C = \Delta - 2\Delta_S$

$$= 80° - 41° 17' 36'' = 38° 42' 24''$$

$$\text{Length of the circular curve} = \frac{\pi R \Delta_C}{180°}$$

$$= \frac{\pi \times 315 \times 38° 42' 24''}{180°} = 212.8 \text{ m}$$

Total length of the composite curve = 212.8 + (2 × 227) = **666.8 m**

Shift $s = \frac{L^2}{24R} = \frac{(227)^2}{24(315)} = 6.82$ m

Total tangent length $= (R+s)\tan\Delta + \frac{L}{2} = (315 + 6.82)\tan 80° + \frac{227}{2} = 1938.6$ m.

Chainage of P.I.	= 42862.0
Deduct tangent length	= 1938.6
Chainage of T_1	= 40923.4
Add length of transition curve	= 227.0
Chainage of junction	= 41150.4
Add length of circular curve	= 212.8
Chainage of the other junction	= 41363.2
Add length of transition	= 227.0
Chainage of T_2	= 41590.2

Example 3.3. *Two Clothoid spirals for a road transition between two straights meet at a common tangent point . If the deflection angle between the straights is 30°, the chainage of P.I. 6387 metres and the maximum speed 120 km per hour, calculate the chainage of the tangent points and the point of compound curvature. The curve may be designed on the basis of comfort condition of centrifugal ratio (safety condition).*

Take $\alpha = 0.4\ m/sec^2/sec$.

Solution.

(*a*) ***For the comfort condition***

$$L = \frac{v^3}{\alpha R} \qquad \text{But} \qquad L = 2R\Delta_S$$

$$\therefore \quad 2R\Delta_S = \frac{v^3}{\alpha R}$$

or

$$R = \sqrt{\frac{v^3}{2\alpha\Delta_S}}$$

Here $2\Delta_S = 30°$

$$\Delta_S = \frac{30}{2} \times \frac{\pi}{180} = \frac{\pi}{12} \text{ radians}$$

$V = 120$ km per hour

$$v = \frac{120 \times 1000}{3600} = \frac{100}{3} \text{ m/sec} \; ; \; \alpha = 0.4 \text{ m/sec}^3$$

$$R = \sqrt{\left(\frac{100}{3}\right)^3 \times \frac{1}{2 \times 0.4}\frac{12}{\pi}} = 420.5 \text{ m}$$

$$L = 2R\Delta_S = 2 \times 420.5\frac{\pi}{12} = 220.2 \text{ m}.$$

(*b*) ***For the centrifugal ratio***

$$\frac{v^3}{gR} = \frac{1}{4}$$

or

$$R = \frac{4v^2}{g} = \frac{4}{9.81}\left(\frac{100}{3}\right)^2 = 453.1 \text{ m}$$

$$L = 2R\Delta_S = 237.3 \text{ m}$$

The curve may, therefore, be designed on the safety condition having $L = 237.3$ m and $R = 453.1$ m = minimum radius of curvature for safety condition.

If X and Y are the co-ordinates at the end of the first clothoid, the tangent length is equal to $X + Y\tan\frac{\Delta_S}{2}$.

But
$$X = L\left(1 - \frac{L^2}{40R^2}\right) = 237.3\left\{1 - \frac{(237.3)^2}{40(453.1)^2}\right\} = 235.66 \text{ m}$$

and
$$Y = \frac{L^2}{6R}\left(1 - \frac{L^2}{56R^2}\right) = \frac{(237.3)^2}{6 \times 453.1}\left\{1 - \frac{(237.3)^2}{56(453.1)^2}\right\} = 20.7 \text{ m}.$$

$\therefore$ Total tangent length $= 235.66 + 20.7\tan\frac{150}{2} = 238.3$ m

Chainage of P.I.	= 6387 metres
Subtract tangent length	= 238.3
Chainage of P.C.	= 6148.7
Add length of transition curve	= 237.3
Chainage of junction with the other clothoid	= 6386.0
Add length of transition curve	= 237.3
Chainage of P.T.	= 6623.3 metres.

Example 3.4. *A circular curve of 1000 m radius deflects through an angle of 40°. This curve is to be replaced by one of smaller radius so as to admit transition 200 m long at each end. The deviation of the new curve from the old at their mid-point is 1 m towards the intersection point.*

Determine the amended radius assuming that the shift can be calculated with sufficient accuracy on the old radius. Calculate the lengths of track to be lifted and of new track to be laid.

Solution.

In Fig. 3.13, let $T_1E_1T_1'$ be the old curve with radius R_1 and centre O_1. Let TET' be the new curve with TD and $D'T'$ as the transitions and DED' as the circular arc of radius R and centre O. Evidently AC is the shift of the new curve.

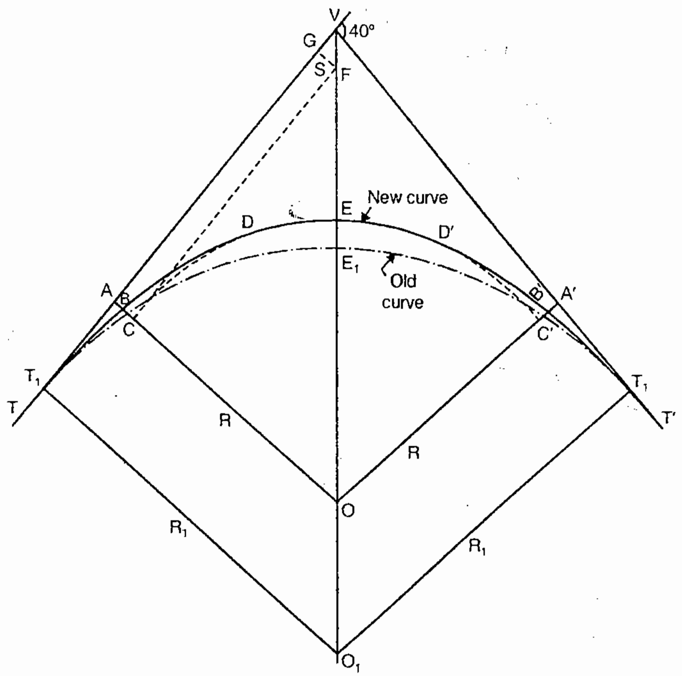


FIG. 3.13. OLD CURVE SHARPENED TO ADMIT TRANSITIONS.

For the old curve :

Tangent length $T_1V = R_1 \tan \frac{\Delta}{2} = 1000 \tan 20° = 363.97$ m

$$VE_1 = R_1\left(\sec \frac{\Delta}{2} - 1\right) = 1000 \times 0.06418 = 64.18 \text{ m}$$

and length of the curve $T_1E_1T_1' = \frac{\pi R_1 \Delta}{180°} = \frac{\pi \times 1000 \times 40}{180°} = 698.14$ m.

For the new curve :

Shift (using old radius) $= \frac{L^2}{24R_1} = \frac{(200)^2}{24 \times 1000} = 1.67$ m

Draw a line CF parallel to AV. CF will be tangential to the redundant circular arc CDE.

$$VF = GF \sec \frac{\Delta}{2} = s \sec \frac{\Delta}{2} = 1.67 \sec 20° = 1.77 \text{ m}$$

Now $FE = VE_1 - E_1E - VF = 64.18 - 1.0 - 1.77 = 61.41$ m

But $FE = R\left(\sec \frac{\Delta}{2} - 1\right) = R (\sec 20° - 1) = 0.06418\ R$

$\therefore \quad R = \frac{FE}{0.06418} = \frac{61.41}{0.06418} = \mathbf{956.8\ m}$

Length of new track to be laid :

The length of the new track to be laid will evidently be equal to the total length of the combined curve.

Total length of combined curve $= \frac{\pi R\, \Delta_C}{180°} + \frac{L}{2} + \frac{L}{2}$

$$= \frac{\pi R\, \Delta_C}{180°} + L \qquad \ldots [3.25\ (a)]$$

$$= \frac{\pi\, (956.8)\, 40°}{180°} + 200 = \mathbf{867.98\ m.}$$

Length of old track to be lifted :

Shift (using new radius) $= \frac{L^2}{24\,R} = \frac{(200)^2}{24 \times 956.8} = 1.75$ m

$$AV = (R + s) \tan \frac{\Delta}{2} = (956.8 + 1.75) \tan 20° = 348.89.$$

Since the shift bisects the transition curve, we have

$$TA \approx TB = \frac{L}{2} = 100 \text{ m}$$

$\therefore \quad TV = TA + AV = 100 + 343.89 = 448.89$

$TT_1 = TV - T_1V = 448.89 - 363.97 = 84.92$

Length of old track to be lifted $= 2(TT_1 + T_1E_1)$

$= 2TT_1 +$ Length of old curve

$= (2 \times 84.92) + 698.14$

$= \mathbf{867.98}.$

Example 3.5. *On a proposed railway, two straights intersect at chainage 68 + 35 chains in 20 m units with a deflection of 40° 30′ (Right). It is proposed to put in a circular arc of 20 chains radius with transition curves 3 chains long at each end. The circular curve is to be set out with pegs at 1 chain intervals and the transition curve with pegs at $\frac{1}{2}$ chain intervals of through chainage. Make all the necessary calculations for setting out the combined curve by theodolite.*

Solution (Fig. 3.5)

$$\Delta = 40°\ 30' \ ;\ R = 20 \text{ chains} \ ;\ L = 3 \text{ chains}$$

Shift $s = \frac{L^2}{24\,R} = \frac{3 \times 3}{24 \times 20} = 0.0187$ chains $= 0.374$ metre.

Having cubic parabola as the transition curve, we have

Total tangent length $TV = (R + s) \tan \frac{\Delta}{2} + \frac{L}{2} = (20.0187) \tan 20°\ 15' + 1.5 = 8.882$ chains.

Spiral angle, $\Delta_S = \frac{L}{2R} \times \frac{180°}{\pi} = \frac{3}{2 \times 20} \times \frac{180°}{\pi} = 4° 17'.8.$

Central angle for the circular arc = $\Delta_C = \Delta - 2\Delta_S$

$= 40° 30' - 8° 35'.6 = 31° 54'.4$

Length of the circular arc = $\frac{\pi R \Delta_C}{180°} = \frac{\pi(20)\, 31° 54'.4}{180°} = 11.136$ chains

Length of the combined curve	= 11.136 + (2 × 3) = 17.136 chains
Chainage of P.I.	= 68 + 350 chains
Subtract tangent length	= 8 + 882
Chainage of the beginning of the transition curve	= 59 + 468
Add length of transition curve	= 3 + 000
Chainage of the junction of the transition curve with the circular curve	= 62 + 468
Add length of circular curve	= 11 + 136
Chainage of the junction of the circular curve with the transition curve	= 73 + 604
Add length of transition curve	= 3 + 000
Chainage of end the transition curve	= 76 + 604 chains

Deflection angle (α) for the first transition curve

$$\alpha = \frac{1800\, l^2}{\pi RL} = \frac{1800\, l^2}{\pi \times 20 \times 3} = \frac{30}{\pi} l^2 \text{ minutes}$$

The various values of α are tabulated below :

Point	Chainage	l (Chains)	α °	α ′	α ″
T	59 + 468	–		–	
1	59 + 500	0.032	0	0	0.6
2	60 + 000	0.532	0	2	42
3	60 + 500	1.032	0	10	10
4	61 + 000	1.532	0	22	25
5	61 + 500	2.032	0	39	26
6	62 + 000	2.532	1	1	13
D	62 + 468	3.000	1	25	56

Check : $\alpha_d = \alpha_S = \frac{1}{3}\Delta_S = 1° 25' 56''.$

Deflection angles for the circular curve :

$$\delta = 1718.9 \frac{c}{R} \text{ min.}$$

Length of first sub-chord	= (63 + 000) – (62 + 468) = 0.532 chains
Length of regular chord	= 1 chain
Length of the last sub-chord	= (73 + 604) – (73) = 0.604

The deflection angles are tabulated below :

Point	Chainage	δ °	δ ′	δ ″	Δ °	Δ ′	Δ ″
D	62 + 468		—			—	
1	63 + 000	0	45	44	0	45	44
2	64 + 0	1	25	57	2	11	41
3	65 + 0	1	25	57	3	37	38
4	66 + 0	1	25	57	5	03	35
5	67 + 0	1	25	57	6	29	32
6	68 + 0	1	25	57	7	55	29
7	69 + 0	1	25	57	9	21	26
8	70 + 0	1	25	57	10	47	23
9	71 + 0	1	25	57	12	13	20
10	72 + 0	1	25	57	13	39	17
11	73 + 0	1	25	57	15	05	14
D′	73 + 604	0	51	55	15	57	09

Check : $\Delta_D' = \frac{1}{2}\Delta_C = \frac{1}{2}(31° 54' 4'') = 15° 57' 12''.$

Deflection angles for the second transition curve :

The second transition curve will be set out from the point of tangency (T').

$$\alpha = \frac{1800\, l^2}{\pi RL} = \frac{1800\, l^2}{\pi \times 20 \times 3} = \frac{30}{\pi} l^2 \text{ minutes}$$

Point	Chainage	l (Chains)	α °	α ′	α ″
D′	73 + 604	3.000	1	25	56
1	74 + 000	2.604	1	04	45
2	74 + 500	2.104	0	42	16
3	75 + 000	1.604	0	24	33
4	75 + 500	1.104	0	11	38
5	76 + 000	0.604	0	03	29
6	76 + 500	0.104	0	00	10
T′	76 + 604	0.000	0	00	00

Example 3.6. *In a road curve between two straights having deflection angle of 108°, Bernoulli's Lemniscate is used as a curve transitional throughout. Make necessary calculations for setting out the curve if the apex distance is 20 metres.*

Solution. (Fig. 3.11)

Given: $\Delta = 108°$; $VM = 20$ m

$$\angle AVM = 90° - \frac{\Delta}{2} = 90° - \frac{108°}{2} = 36°$$

$$\angle VAM = \phi = \frac{\Delta}{2} = 54°$$

$$\angle VT_1M = \alpha_n = \tfrac{1}{3}\phi = \frac{54°}{3} = 18°$$

$$\angle T_1MV = 90° + 2\alpha_n = 90° + 36° = 126°$$

From triangle T_1VM, $VM = 20$ m

$$\therefore \quad T_1V = VM \cdot \frac{\sin T_1M V}{\sin VT_1M} = 20 \frac{\sin 126°}{\sin 18°} = 52.36 \text{ m}$$

$$T_1M = VM \cdot \frac{\sin T_1VM}{\sin VT_1 M} = 20 \frac{\sin 36°}{\sin 18°} = 38.05 \text{ m}$$

$$b_n = T_1M = 38.05 \text{ m}$$

From equation $b = K\sqrt{\sin 2\alpha}$, we have

$$K = \frac{b_n}{\sqrt{\sin 2\alpha_n}} = \frac{38.05}{\sqrt{\sin 36°}} = 49.63$$

The polar equation of the curve is therefore,

$$b = 49.63\sqrt{\sin 2\alpha}$$

Values of b for various values of α can be calculated from the above formula and tabulated below :

α	b *(in metres)*
15′	4.13
30′	6.56
1°	9.27
2°	14.11
4°	18.52
6°	22.63
8°	26.01
10°	29.03
12°	31.65
14°	34.01
16°	36.13
18°	38.05

The tangent points T_1 and T_2 can be located by measuring distance $VT_1 = VT_2 = 52.36$ m from V. Half the curve can be set out by observations from T_1 and the other half from the other tangent point T_2.

Example 3.7. *The deflection angle between two tangents is 60°. Bernoulli's lemniscate is used transitional throughout having the minimum radius of curvature equal to 100 metres. Make necessary calculations for setting out the curve.*

Solution. (Fig. 3.11)

$$\Delta = 60°$$

$$\angle AVM = 90° - \frac{\Delta}{2} = 90° - 30° = 60°$$

$$\angle VAM = \phi = \frac{\Delta}{2} = 30°$$

$$\angle VT_1M = \alpha_n = \tfrac{1}{3}\phi = 10°$$

$$\angle T_1MV = 90° + 2\alpha_n = 90° + 20° = 110°$$

From equation 3.30, we have

$$K = 3\, r\sqrt{\sin 2\alpha}$$

At $\quad r = R = 100, \quad \alpha = \alpha_n = 10°$

$$\therefore \quad K = 3 \times 100\sqrt{\sin 20°} = 175.5$$

Hence the equation of the curve is

$$\therefore \quad b = K\sqrt{\sin 2\alpha} = 175.5\sqrt{\sin 2\alpha}$$

$$\therefore \quad b_n = T_1M = 175.5\sqrt{\sin 20°} = 102.6 \text{ m.}$$

To calculate the tangent length T_1V, consider the triangle T_1VB. Thus,

$$T_1V = T_1M \frac{\sin 110°}{\sin 60°} = 102.6 \frac{\sin 110°}{\sin 60°} = 111.3 \text{ m}$$

The tangent points T_1 and T_2 can be located by measurements from V. The two branches of the curve can be set out from T_1 and T_2, using the tabulated values of α and b. For various values of α, b can calculated from the equation $b = 175.5\sqrt{\sin 2\alpha}$ exactly in the same manner as in the previous example.

PROBLEMS

1. What is a transition (or easement) curve ? Why it is used ? Define 'shift' of a curve. Draw two tangents and show a circular curve and two transition curves connecting the tangents, marking the 'shift' on your sketch. How may the transition curve be set out ?

2. Derive an expression for the length and shift of a transition curve required for a first-class railway track.

3. (*a*) What is meant by 'shift' of a curve. Derive an expression for the same.

b) Explain the various methods of determining the length of a transition curve.

4. Show that a cubic parabola is suitable for a railway transition curve, and explain clearly how the clothoid becomes a cubic parabola when set out normally with the theodolite and chain, the intrinsic equation of the curve being $\lambda = m\sqrt{\phi}$, where m is constant and λ and ϕ are the co-ordinates

of point with respect to an origin assumed at the point of tangency of the spiral with the main tangent. (*U.L.*)

5. The deflection angle between two straights forming the tangents of a highway curve is 48°. The curve is to consist of central circular arc with two equal transition spirals and the following conditions are to be satisfied:

(*a*) Radius of circular arc : 140 m

(*b*) External distance not greater than 16 m

(*c*) Tangent length not greater than 106 m.

It is proposed to adopt a spiral length of 80 m. Ascertain whether this length is suitable.

6. A transition curve is required for a circular curve of 400 m radius, the gauge being 1.5 m between rail centre and maximum super-elevation restricted to 12 cm. The transition is to be designed for a velocity such that no lateral pressure is imposed on the rails and the rate of gain of the radial acceleration is 30 cm/sec^3. Calculate the required length of transition curve and the design speed.

7. Two straights on the centre line of a proposed railway intersect at 1270.8 metres, the deflection angle being 38° 24′. It is proposed to put in a circular curve of 320 m radius with cubic parabolic transition curve 36 m long at each end. The combined curve is to be set out by the method of deflection angles with pegs at every 10 m through chainage on the transition curves and with pegs at every 20 m through chainage on the circular curve. Tabulate the data relative to the first two stations on the first transition curve and the junctions of the transition curve with circular etc.

ANSWERS

5. Tangent length = 103.24 m. , External distance = 15.54 m.
6. 46.4 m.
7. Deflection angles 3′ 47″ ; 17′26″ ; 1° 4′ 27″ ; 15° 58′ 48″.

4

Curve Surveying IV : Vertical Curves

4.1. GENERAL

A vertical curve is used to join two intersecting grade lines of rail-roads, highways or other routes to smooth out the changes in vertical motion. An abrupt change in the rate of the grade could otherwise subject a vehicle passing over it to an impact that would be either injurious or dangerous. The vertical curve, thus, contributes to the safety, comfort and appearance. Either a circular arc or a parabola may be used for this purpose, but for simplicity of calculation work, the latter is preferred and is invariably used. The parabolic curve also produces the best riding qualities, since the rate of change in grade is uniform throughout in a parabola having a vertical axis. This is proved as under.

The general equation of a parabola with a vertical axis can be written as

$$y = ax^2 + bx \qquad \ldots(i)$$

The slope of this curve at any point in given by $\frac{dy}{dx} = 2ax + b$ ...(*ii*)

The rate of change of slope or rate of change of grade (*r*) is given by

$$\frac{d^2y}{dx^2} = r = 2a = \text{constant} \qquad \ldots(iii)$$

Thus, the grade changes *uniformly throughout the curve,* which is a desired condition.

The Grade

The grade or gradient of a rail-road or highway is expressed in two ways :

(*i*) As a percentage : *e.g.* 2% or 3%

(*ii*) As 1 vertical in *n* horizontal (1 in *n*) : *e.g.* 1 in 100 or 1 in 400.

A grade is said to be *upgrade or* + ve grade when elevations along it increase, while it is said to be a downgrade or – ve grade when the elevations decrease along the direction of motion.

Rate of change of grade (*r*) **:** Equation (*ii*) gives the grade at any point on the curve. The gradient changes from point to point on the curve, but the rate of change of grade, given by equation, (*iii*) is constant in a parabola. For first class railways, the rate of change of gradient is recommended as 0.06% per 20 m station at summits and 0.03% for 20 m station at sags. Twice these values may be adopted for second class railways.

For example, in a summit vertical curve, let the gradient in the beginning of the curve be 1%. Then if the rate of change of grade is 0.05% (say) per 20 m station, the gradient at different stations will be as under :

Station	*Distance from beginning (m)*	*Gradient*
0	0	1%
1	20	0.95%
2	40	0.90%
3	60	0.85%
4	80	0.80%
5	100	0.75%
etc.	etc.	etc.

4.2. TYPES OF VERTICAL CURVES :

Vertical curves may be of the following six types :

(1) An upgrade $(+g_1\%)$ ***followed by a downgrade*** $(-g_2\%)$ ***(Fig. 4.1).***

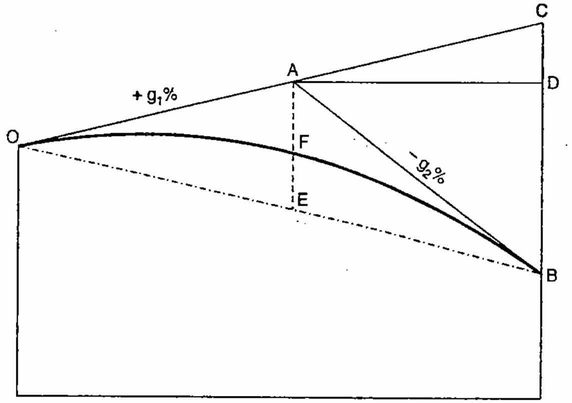


FIG. 4.1. SUMMIT OR CONVEX.

(2) A downgrade $(-g_1\%)$ ***followed by an upgrade*** $(+g_2\%)$ ***(Fig. 4.2)***

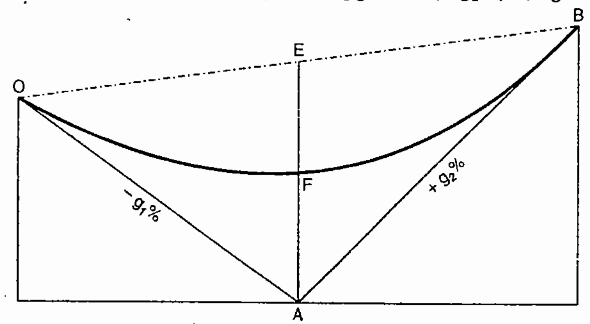


FIG. 4.2. SAG OR CONCAVE $(g_2 > g_1)$.

(3) An upgrade $(+g_1\%)$ ***followed by another upgrade*** $(+g_2\%)$ $g_2 > g_1$. ***(Fig. 4.3).***

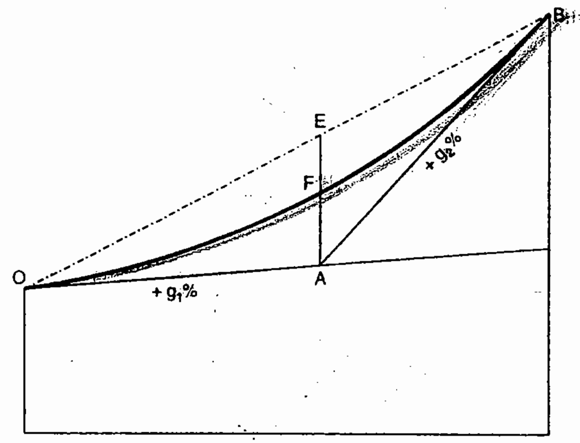


FIG. 4.3. SAG OR CONCAVE $(g_2 > g_1)$

(4) An upgrade $(+g_1\%)$ ***followed by another upgrade*** $(+g_2\%)$ $g_1 > g_2$ ***(Fig. 4.4).***

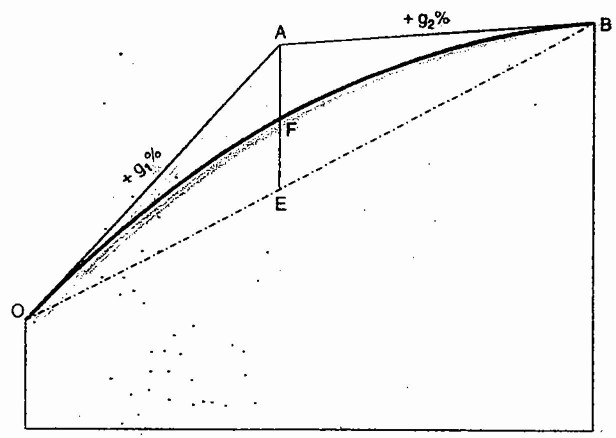


FIG. 4.4. SUMMIT OR CONVEX $(g_1 > g_2)$

(5) A downgrade ($-g_1$%) followed by another downgrade ($-g_2$%) : $g_2 > g_1$ (Fig. 4.5)

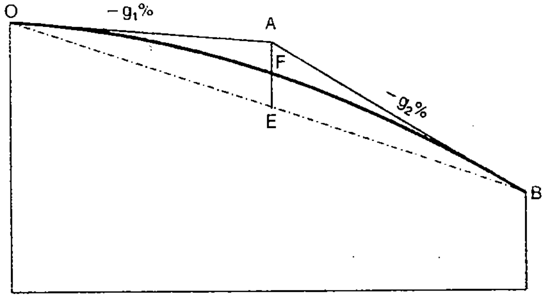


FIG. 4.5. SUMMIT OR CONVEX ($g_2 > g_1$).

(6) A downgrade ($-g_1$%) followed by another downgrade ($-g_2$%) : $g_1 > g_2$ (Fig. 4.6).

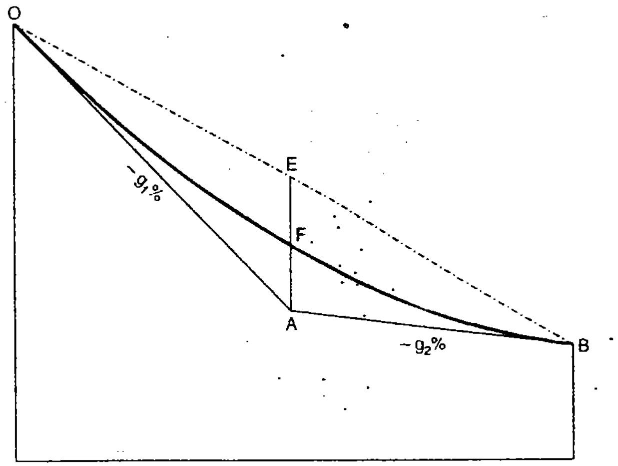


FIG. 4.6. SAG OF CONCAVE ($g_1 > g_2$).

4.3. LENGTH OF VERTICAL CURVE

The length of the vertical curve can be obtained by dividing the algebraic difference of the two grades by the rate of change of grade, due regard being paid to the sign of the grade. Thus,

$$\text{Length of curve } (L) = \frac{\text{Total change of grade}}{\text{Rate of change of grade}} = \frac{g_1 - g_2}{r} \text{ chains} \quad ...(4.1)$$

where $g_1 - g_2$ = Algebraic difference of the two grades (%)

r = Rate of change of grade (%) per chain

While substituting the numerical values of g_1 and g_2, due regard should be paid to the sign of the grade.

For example, if $g_1 = +1.2\%$ and $g_2 = -0.8\%$

and $r = 0.1\%$ per 20 m chain

$$L = \frac{g_1 - g_2}{r} = \frac{(+1.2) - (-0.8)}{0.1} = \frac{1.2 + 0.8}{0.1} = 20 \text{ chains} = 400 \text{ m}$$

In general practice, nearest number of L in chain lengths is adopted and $\frac{1}{2}L$ is set out to the either side of the apex. In case of highways, however, the minimum length of the curve is determined from the consideration of sight distance as discussed in § 4.5.

4.4. COMPUTATIONS AND SETTING OUT A VERTICAL CURVE

In vertical curves, all distances along the curve are measured *horizontally* and all offsets from the tangents to the curve are measured *vertically*. The length of the curve is thus its horizontal projection, without appreciable error since the curve is quite flat.

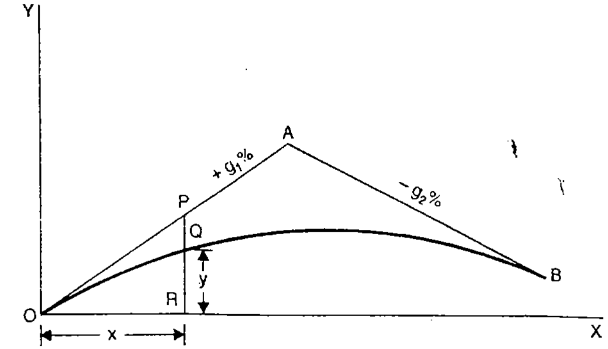


FIG. 4.7. THE PARABOLA.

In Fig. 4.7, let

OX and OY = The axes of the rectangular ordinates passing through the beginning (O) of the vertical curve

OA = Tangent having $+g_1$% slope

AB = Tangent having $-g_2$% slope

Q = Any point on the curve having co-ordinates (x, y)

Draw PQR, a vertical line through Q.

The equation of the parabola can be written as

$$y = ax^2 + bx \qquad \therefore \quad \frac{dy}{dx} = 2ax + b$$

At $x = 0, \quad \frac{dy}{dx} = +g_1$

$\therefore \quad g_1 = 2a(0) + b \quad$ or $\quad b = g_1$

Hence, the equation of the parbola is

$$y = ax^2 + g_1 x \quad ...(4.2)$$

Let $PQ = h$ = vertical distance between the tangent and the corresponding point Q on the curve
= Tangent correction

$\therefore \quad PQ = PR - QR$

But $PR = g_1 x$ and $QR = y$

$\therefore \quad PQ = h = g_1 x - y$

But $g_1 x - y = -ax^2$, from equation 4.2

Hence $h = g_1 x - y = -ax^2$

or $$h = Cx^2 \qquad \text{...(4.3 } a)$$

or $$h = kN^2 \qquad \text{...(4.3)}$$

where N is counted from O at the beginning of the curve.

Thus, the difference in elevation between a vertical curve and a tangent to it varies as the square of its horizontal distance from the point of tangency. This difference in elevation is also known as the *tangent correction.*

The offsets are measured vertically downwards, though they should be measured parallel to the axis of the parabola for a true curve. Due to unequal values of g_1 and g_2, the axis is slightly tilted. Hence by making the offsets vertical (and not parallel to the tilted axis), the curve will be slightly distorted from its parabolic form. However, the distortion is negligible for all practical purposes.

The value of k in equation 4.3 can be found by considering Fig. 4.8 as follows.

In Fig. 4.8, produce OA to C, a point vertically above B. Through A, draw AD horizontal to meet BC in D.

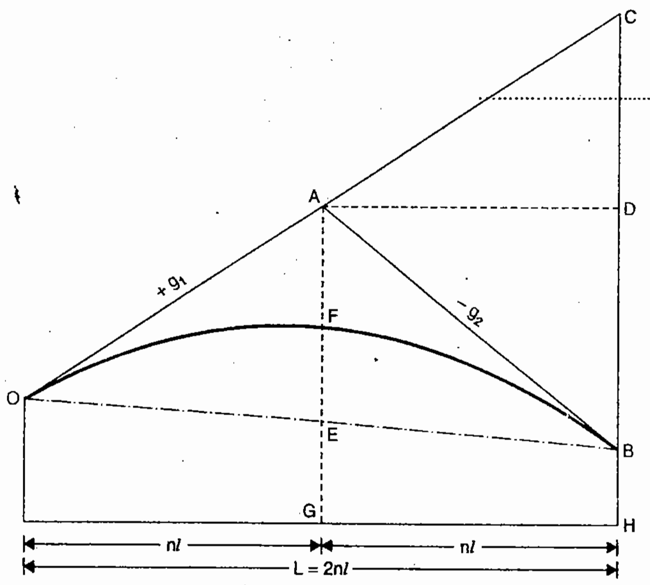


FIG. 4.8.

Let $2n$ = Total number of equal chords, each of length l, on each side of the apex

g_1 and g_2 = Grades of the two tangents

e_1 and e_2 = Corresponding rises or falls *per chord length l* (figures plus or minus as they represent rises or falls)

$$OA = AC$$

$$CD = ne_1$$

and $$BD = -ne_2$$

$\therefore \quad CB = CD + DB = n(e_1 - e_2)$, algebraically.

From equation 4.3,

$$CB = kN^2, \quad \text{where } N = 2n$$
$$= k(2n)^2$$

$\therefore \quad 4kn^2 = n(e_1 - e_2)$

or $$k = \frac{e_1 - e_2}{4n} \qquad \text{...(4.4)}$$

In the above equation, proper care for the signs must be taken while substituting the numerical values of e_1 and e_2.

Elevation by Tangent Correction

Knowing the value of k, the tangent corrections for various values of N can be calculated from equation 4.3 and the elevations of various points on the curve can be computed in the following steps :

(1) Let the elevation and chainage of the apex A be known.

Let the length of the curve on either side of the station be n chords of equal length l.

Then chainage of point of tangency (O) = Chainage of $A - nl$ and,

chainage of point of tangency (B) = Chainage of $A + nl$.

(2) Knowing the grades g_1 and g_2 and the elevation of the apex A, the elevation of O and B can be calculated as under :

Elevation of O = Elevation of $A \mp ne_1$ (use minus sign if e_1 is positive and plus sign if it is negative)

Elevation of B = Elevation of $A \pm ne_2$ (use plus sign if e_2 is positive and minus sign if it is negative)

If, however O is taken as the datum, elevation of F can be found as under :

Elevation of $A = ne_1$

Elevation of B = Elevation of $A + ne_2 = ne_1 + ne_2$

Elevation of $E = \frac{1}{2}$ (Elev. of O + Elev. of B) $= \frac{n}{2}(e_1 + e_2)$

Since $OE = EB$, AE is a diameter of the parabola,

and $AF = FE$.

Elevation of F = Elevation of $E + n^2 k$

The elevation of F can also be found by subtracting algebraically n^2k from the elevation of A. Thus.

Elevation of F = Elevation of $A - n^2 k = ne_1 - n^2k$.

(3) Compute the tangent corrections from the expression

$$h = kN^2$$

Thus
$$h_1 = 1\ k$$
$$h_2 = 4\ k$$
$$h_3 = 9\ k$$
$$\dots \quad \dots$$
$$h_N = (2n)^2\ k.$$

(4) Compute the elevation of the corresponding stations on the tangent OAC. Thus:

Elevation of tangent at any station (n')=elevation of point of tangency $(O) + n'e_1$

where n' is the *number* of that station from O.

(5) Find the elevations of the corresponding stations on the curve by adding algebraically the tangent corrections to the elevations of the corresponding stations.

If the value of k is positive, the tangent corrections are to be subtracted from grade elevations ; if it is negative, tangent corrections are additive.

The result may be tabulated as under :

Station	*Chainage*	*Tangent or grade elevation*	*Tangent correction*	*Elevation of the curve*	*Remarks*
....	....	....			

Elevation by Chord Gradients

The chord gradient is the difference in elevation between the two ends of a chord joining two adjacent stations. Thus, in this method, the successive differences in elevation between the points on the curve are calculated and the elevation of each point is determined by adding the chord gradient to the elevation of the preceding point.

Consider two adjacent points P and Q of vertical curve having OA and BA as the initial tangents meeting at A. Through O, draw a horizontal line OQ_2. Through P and Q draw vertical line P_1PP_2 and Q_1QQ_2 shown in Fig. 4.9. Through P, draw a horizontal PQ_3.

Let e_1 and e_2 be the rises (or falls) of the tangents per chord length l.

$\therefore$ $P_1P_2 = e_1$ if P is the first station

P_1P = tangent correction, given by $h = kN^2 = 1\ k$

Difference in elevation between P and O

$= PP_2 = P_1P_2 - P_1P = e_1 - k$

where $k = \dfrac{e_1 - e_2}{4n}$

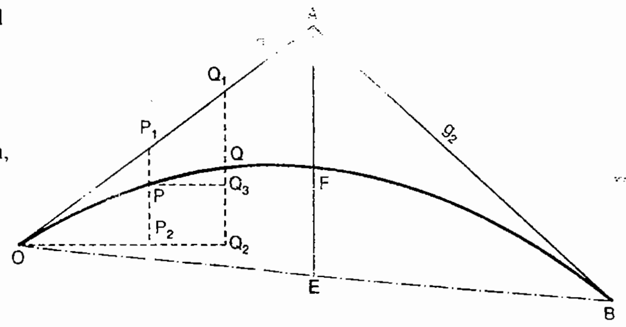


FIG. 4.9. METHOD OF CHORD GRADIENTS.

$\therefore$ *First chord gradient* $= e_1 - k$

Similarly, $Q_1Q_2 = 2\ e_1$

$Q_1Q = (2)^2\ k = 4k$

$Q_3Q_2 = PP_2 = e_1 - k$

Difference in elevation between Q and $P = QQ_3$

$$= Q_1Q_2 - Q_1Q - Q_3Q_2 = 2\ e_1 - 4k - (e_1 - k) = e_1 - 3\ k$$

$\therefore$ *Second chord gradient* $= e_1 - 3k$ ...(2)

Hence ***Nth chord gradient*** $= e_1 - (2N - 1)k$ ...(4.5)

Knowing the chord gradient for different points, their elevations can be easily calculated.

Thus, elevation of 1st station = Elevation of tangent point + First chord gradient

Elevation of 2nd station = Elevation of 1st station + Second chord gradient etc. etc.

Example 4.1. *A parabolic vertical curve is to be set out connecting two unifrom grades of + 0.8% and – 0.9%. The chainage and reduced level of point of intersection are 1664 metres and 238.755 m respectively. The rate of change of grade is 0.05% per chain of 20 m. Calculate the reduced levels of the various station pegs.*

Solution. (Fig. 4.8)

Total change of grade $= g_1 - g_2$

$= (+0.8) - (-0.9) = +1.7\%$

Rate of change of grade $= r = 0.05\%$ per chain

Length of the vertical curve $= \dfrac{1.7}{0.05} = 34$ chains

Length of the curve on either side of the apex

= 17 chains = 340 m

Chainage of the point of intersection = 1664 m

Chainage of the first tangent point = 1664 – 340 = 1324 m

Chainage of the second tangent point = 1664 + 340 = 2004 m

R.L. of point of intersection (A) $= 238.755$ m

e_1 per chord length of 20 m $= \frac{g_1}{100} \times 20 = +\frac{0.8}{5}$

R.L. of the beginning (O) of the curve $= 238.755 - 17e_1$

$= 238.755 - \frac{17 \times 0.8}{5}$ m

$= 236.035$

e_2 per chord length of 20 m $= \frac{g_2}{100} \times 20$

$= \frac{g_2}{5} = \frac{-0.9}{5}$ m

R.L. of the end (B) of the curve $= 238.755 + 17e_2$

$= 238.755 - \frac{17 \times 0.9}{5} = 235.695$ m

R.L. of E $= \frac{1}{2}$(R.L. of O + R.L. of B)

$= \frac{1}{2}(236.035 + 235.695) = 235.865$

Since F is midway betweeen A and E,

R.L. of vertex F $= \frac{1}{2}$(R.L. of A + R.L. of E)

$= \frac{1}{2}(238.755+235.865) = 237.310$

Difference in elevation between A and F $= 238.755 - 237.310 = 1.445$ m

Check : From equation 4.3

$$AF = kN^2$$

$$k = \frac{e_1 - e_2}{4n} = \frac{\frac{0.8}{5} - \left(-\frac{0.9}{5}\right)}{4 \times 17} = \frac{0.8 + 0.9}{5 \times 4 \times 17} = \frac{1.7}{340} = \frac{1}{200}$$

$\therefore$ $AF = kN^2 = \frac{1}{200}(17)^2 = \frac{289}{200} = \mathbf{1.445}$ **m.**

The tangent correction at any point is calculated from the expression

$$h = kN^2$$

or $$h = \frac{N^2}{200}$$

For the first station having chainage = 1344 m and $N = 1$

$\therefore$ $h_1 = \frac{1}{200}$ m $= 0.005$ m

Tangent elevation of first point = R.L. of $O + e_1$

$= 236.035 + 0.16 = 236.195$

R.L. of first station on the curve = Tangent elevation – Tangent correction

$= 236.195 - 0.005 = 236.190$ m

Similarly, for the second station having chainage = 1364 and $N = 2$

$$h_2 = \frac{(2)^2}{200} = \frac{4}{200} = 0.02 \text{ m}$$

Tangent elevation on second point = R.L. of $O + 2e_1 = 236.035 + 0.32 = 236.355$

R.L. of second station of the curve $= 236.355 - 0.02 = 236.335$ m

The elevation of other points can similarly be calculated and tabulated as below :

Station	*Chainage*	*Tangent Elevation*	*Tangent Correction (–ve)*	*Curve Elevation*	*Remarks*
0	1324	236.035	0.000	236.035	Beginning of the curve
1	1344	236.195	0.005	236.190	
2	1364	236.355	0.020	236.335	
3	1384	236.515	0.045	236.470	
4	1404	236.675	0.080	236.595	
5	1424	236.835	0.125	236.710	
6	1444	236.995	0.180	236.815	
7	1464	237.155	0.245	236.910	
8	1484	237.315	0.320	236.995	
9	1504	237.475	0.405	237.070	
10	1524	237.635	0.500	237.135	
11	1544	237.795	0.605	237.190	
12	1564	237.955	0.720	237.235	
13	1584	238.115	0.845	237.270	
14	1604	238.275	0.980	237.295	
15	1624	238.435	1.125	237.310	
16	1644	238.595	1.280	237.315*	*Highest point (Apex A of the curve)
17	1664	238.755	1.445	237.310	
18	1684	238.915	1.620	237.295	
19	1704	239.075	1.805	237.270	
20	1724	239.235	2.000	237.235	
21	1744	239.395	2.205	237.190	
22	1764	239.555	2.420	237.135	
23	1784	239.715	2.645	237.070	
24	1804	239.875	2.880	236.995	
25	1824	240.035	3.125	236.910	
26	1844	240.195	3.380	236.815	
27	1864	240.355	3.645	236.710	

(*Table continued on next page*)

Station	Chainage	Tangent Elevation	Tangent Correction (–ve)	Curve Elevation	Remarks
28	1884	240.515	3.920	236.595	
29	1904	240.675	4.205	236.470	
30	1924	240.835	4.500	236.335	
31	1944	240.995	4.805	236.190	
32	1964	241.155	5.120	236.035	
33	1984	241.315	5.445	235.870	
34	2004	242.475	5.780	235.695	End of the curve

Example 4.2. *A – 1.0 percent grade meets a + 2.0 percent grade at station 470 of elevation 328.605 metres. A vertical curve of length 120 metres is to be used. The peges are to be fixed at 10 metres interval. Calculate the elevations of the points on the curve by (a) tangent corrections and (b) by chord gradients.*

If the pege are to be driven with their tops at the formation of the curve, calculate the staff readings required. given that height of collimation is 330.890.

Solution.

(a) *Tangent correction*

Total number of stations in 10 m unit $= \frac{120}{10} = 12$

Number of stations to each side of apex $= n = 6$

Change of elevation of first tangent per chord length of 10 m

$$\therefore = e_1 = \frac{g_1}{100} \times 10 = \frac{-1.0}{100} \times 10 = -0.10 \text{ m}$$

Change of elevation of second tangent per chord length of 10 m

$$= e_2 = \frac{g_2}{100} \times 10 = \frac{+2.0}{100} \times 10 = +0.20 \text{ m}$$

Elevation of point of intersection $= 328.605$ m

Elevation of the beginning of curve $= 328.605 - ne_1$

$= 328.605 - (6)(-0.10)$

$= 329.205$ m

Elevation of the end of curve $= 328.605 + ne_2$

$= 328.605 + (6)(0.2)$

$= 329.805$ m

The tangent correction with respect to the first tangent is given by

$$h = kN^2$$

where $k = \frac{e_1 - e_2}{4n} = \frac{(-0.10) - (0.20)}{4 \times 6} = \frac{-0.3}{24} = -\frac{1}{80}$

Hence $h = -\frac{N^2}{80}$

Since the sign of k is negative, h will be additive to the tangent elevations to get the elevations on the curve.

For the first point, tangent elevation = elevation of the beginning of the curve + e_1

$= 329.205 - 0.10 = 329.105$

Tangent correction $= \frac{1}{80} = 0.0125$ m ≈ 0.010 m

(Since the readings can be taken upto an accuracy of the multiples of 0.005 m).

Elevation of first point $= 329.105 + 0.010 = 329.115$ m

Similarly, for the second point, tangent elevation

$= 329.205 - 0.2 = 329.005$

Tangent correction $= \frac{(2)^2}{80} = 0.050$ m

Elevation of second point $= 329.005 + 0.050 = 329.055$ m.

The values for other points along with the required staff reading are tabulated below. The required staff readings for the pegs are obtained by subtracting the elevations of the points from the height of collimation.

Station	Chainage	Tangent elevation	Tangent correction (+ ve)	Curve elevation	Ht. of collimation	Staff reading	Remarks
0	410	329.205	0	329.205	330.890	1.685	Beginning of the curve
1	420	329.105	0.010	329.115		1.775	
2	430	329.005	0.050	329.055		1.835	
3	440	328.905	0.115	329.020		1.870	
4	450	328.805	0.200	329.005		1.885	
5	460	328.705	0.315	329.020		1.870	
6	470	328.605	0.450	329.055		1.835	Vertex of the curve
7	480	328.505	0.615	329.120		1.770	
8	490	328.405	0.800	329.205		1.685	
9	500	328.305	1.015	329.320		1.570	
10	510	328.205	1.250	329.455		1.435	
11	520	328.105	1.515	329.620		1.270	
B	530	328.005	1.800	329.805		1.085	End of the curve

Check:

Elevation of mid-point of OB $= \frac{1}{2}(329.205 + 329.805) = 329.505$ m

Elevation of the vertex $= \frac{1}{2}(329.505 + 328.605) = 329.055$ m

(b) Chords gradients

The chord gradient for any point is given by equation 4.5 *i.e.*,

Nth chord gradient $= e_1 - (2N - 1)\,k$

Here, $e_1 = -0.1$, $k = -\frac{1}{80}$

(1) For the first point, chord gradient

$= -0.1 - (2-1)\left(-\frac{1}{80}\right) = -0.1 + \frac{1}{80} \approx -0.090.$

Elevation of first point = elevation of O + chord gradient

$= 329.205 - 0.090 = 329.115.$

(2) For the second point, chord gradient

$= -0.1 - (4-1)\left(-\frac{1}{80}\right) = -0.1 + \frac{3}{80} = -0.060$

Elevation of second point = 329.115 − 0.060 = 329.055

(3) For the third point, chord gradient

$= -0.1 - (6-1)\left(-\frac{1}{80}\right) = -0.1 + \frac{5}{80} = -0.040$

Elevation of third point = 329.055 − 0.040 = 329.015.

(4) For the fourth point, chord gradient

$= -0.1 - (8-1)\left(-\frac{1}{80}\right) = -0.1 + \frac{7}{80} = -0.010$

Elevation of fourth point = 329.015 − 0.010 = 329.005.

(5) For the fifth point, chord gradient

$= -0.1 - (10-1)\left(-\frac{1}{80}\right) = -0.1 + \frac{9}{80} = +0.015$

Elevation of fifth point = 329.005 + 0.015 = 329.020

(6) For the sixth point chord gradient

$= -0.1 - (12-1)\left(-\frac{1}{80}\right)$

$= -0.1 + \frac{11}{80} = +0.035$

Elevation of sixth point = 329.020 + 0.035 = 329.055

(7) For the seventh point, chord gradient

$= -0.1 - (14-1)\left(-\frac{1}{80}\right) = -0.1 + \frac{13}{80} = +0.065$

Elevation of seventh point = 329.055 + 0.065 = 329.120

(8) For the eighth point, chord gradient

$= -0.1 - (16-1)\left(-\frac{1}{80}\right) = -0.1 + \frac{15}{80} = +0.085$

Elevation of eighth point = 329.120 + 0.085 = 329.205

(9) For the ninth point, chord gradient

$= -0.1 - (18-1)\left(-\frac{1}{80}\right) = -0.1 + \frac{17}{80} = +0.115$

Elevation of ninth point = 329.205 + 0.115 = 329.320

(10) For the tenth point, chord gradient

$= -0.1 - (20-1)\left(-\frac{1}{80}\right) = -0.1 + \left(\frac{19}{80}\right) = +0.135$

Elevation of tenth point = 329.320 + 0.135 = 329.455

(11) For the eleventh point, chord gradient

$= -0.1 - (22-1)\left(-\frac{1}{80}\right) = -0.1 + \frac{21}{80} = +0.165$

Elevation of eleventh point = 329.455 + 0.165 = 329.620

(12) For point B, chord gradient

$= -0.1 - (24-1)\left(-\frac{1}{80}\right) = -0.1 + \frac{23}{80} = +0.185$

Elevation of B = 329.620 + 0.185 = 329.805.

Knowing the elevations of the points on the curve, the staff readings for various pegs can be calculated and tabulated as done earlier.

4.5 SIGHT DISTANCE

For any given value of the difference in the tangent grades, the length of vertical curve must be long enough to provide at least the minimum required sight distance throughout the vertical curve. *Sight distance is the length of roadway ahead visible to the driver.* The *stopping sight distance* is the total distance travelled during the three time intervals: (1) the time for the driver to perceive the hazard, (2) the time to react, and (3) the time to stop the vehicle after the brakes are applied. The required minimum length of sight distances recommended by the A.A.S.H.O. are given below :

Design speed	***Stopping sight distance***
30 mph (48 km/hour)	200 ft (61 m)
40 mph (64 km/hour)	275 ft (84 m)
50 mph (80 km/hour)	350 ft (107 m)
60 mph (96 km/hour)	475 ft (145 m)
70 mph (113 km/hour)	600 ft (184 m)

The expressions for sight distance (S) on vertical curves will now be derived for two cases :

(*i*) When the sight distance S is entirely on the curve ($S < L$) and

(*ii*) When the sight distance overlaps the curve and extends on to the tangent ($S > L$.)

Let h_1 = height of driver's eye above the roadway

h_2 = height of object or hazard on the travelled road.

Case 1. $S < L$

Fig. 4.10 shows a vertical curve FCG having grades g_1 and g_2. Let A be the algebraic difference in grades in percent

i.e, $A = g_1 - g_2$

In Fig. 4.10, BCD is the line of sight of driver, C being the point where the line of sight is tangential to the curve.

From equation 4.3 (*a*), we have

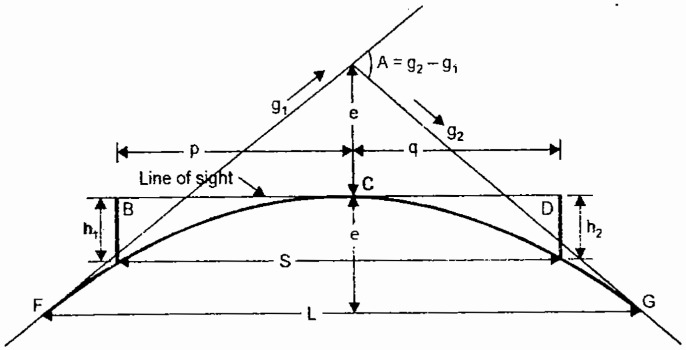


FIG. 4.10. SIGHT DISTANCE ($S < L$)

$$h = C x^2$$

From Fig. 4.10, when $x = L$, $h = \frac{(g_1 - g_2)}{100} \frac{L}{2}$

$$\therefore \quad \frac{(g_1 - g_2)}{100} \frac{L}{2} = CL^2$$

or
$$C = \frac{g_1 - g_2}{200L} \quad ...(2)$$

At C,
$$h = e = \frac{g_1 - g_2}{200L}\left(\frac{L}{2}\right)^2 = \frac{g_1 - g_2}{800} L = \frac{AL}{800} \quad ...(4.6)$$

where $A = g_1 - g_2$ = algebraic difference in grades in percent.

Now, from Fig. 4.10, we have

$$h_1 = Cp^2 \text{ and } h_2 = Cq^2$$

$$\therefore \quad S = p + q = \sqrt{\frac{h_1}{C}} + \sqrt{\frac{h_2}{C}} = \frac{1}{\sqrt{C}}(\sqrt{h_1} + \sqrt{h_2}) \quad ...(4.7\ a)$$

$$S = \frac{14.14\sqrt{L}}{\sqrt{g_1 - g_2}}(\sqrt{h_1} + \sqrt{h_2}) \quad ...(4.7)$$

To calculate the length L of the curve in terms of S, square Eq. 4.7. Thus,

$$S^2 = \frac{200\,L}{g_1 - g_2}(\sqrt{h_1} + \sqrt{h_2})^2$$

or
$$L = \frac{S^2(g_1 - g_2)}{200(\sqrt{h_1} + \sqrt{h_2})^2} \quad ...(4.8)$$

The unit of L will be the same as the units of S and h.

Taking $h_1 = h_2 = h$ for *passing condition*, we have

$$L = \frac{S^2(g_1 - g_2)}{800\,h} \quad ...[4.8\ (a)]$$

Taking $h_1 = 4.5$ ft and $h_2 = \frac{1}{2}$ ft (*stopping condition*),

we get
$$L = \frac{S^2(g_1 - g_2)}{1460} \text{ ft} \quad ...[4.8\ (b)]$$

Taking $h_1 = 1.37$ m and $h_2 = 0.10$ m, we get

$$L = \frac{S^2(g_1 - g_2)}{297} \text{ metres.} \quad ...[4.8\ (c)]$$

Example :

Let $g_1 = 1\%$; $g_2 = -1.5\%$

$h_1 = 1.37$ m ; $h_2 = 0.10$ m

and $S = 200$ metres (min.)

$$\therefore \quad L = \frac{(200)^2(1 + 1.5)}{297} = 336 \text{ metres.}$$

Case 2. $S > L$

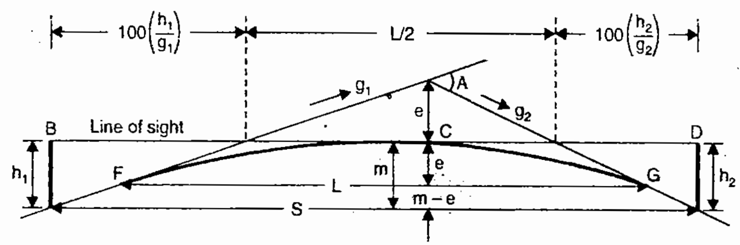


FIG. 4.11. SIGHT DISTANCE ($S > L$).

Fig. 4.11 shows the condition when the sight distance S is greater than L. Assuming scalar values for g_1 and g_2, we have

$$S = \tfrac{1}{2} L + 100\left(\frac{h_1}{g_1} + \frac{h_2}{g_2}\right) \quad ...(1)$$

For the value of A making S a minimum, the rate of change in g_2 will be equal and opposite to the rate of change in g_1.

Setting the first derivative of S to zero, we get

$$\frac{h_1}{(g_1)^2} - \frac{h_2}{(g_2)^2} = 0$$

or
$$g_2 = \sqrt{\frac{h_2}{h_1}} \times g_1$$

$$\therefore \quad A \text{ (scaler value)} = g_2 + g_1 = \sqrt{\frac{h_2}{h_1}}\, g_1 + g_1$$

or
$$A = \frac{\sqrt{h_1} + \sqrt{h_2}}{\sqrt{h_1}} \times g_1$$

or
$$g_1 = \frac{\sqrt{h_1}}{\sqrt{h_1} + \sqrt{h_2}} \times A \quad ...(2)$$

and
$$g_2 = \frac{\sqrt{h_2}}{\sqrt{h_1} + \sqrt{h_2}} \times A \qquad \ldots(3)$$

Substituting the values of g_1 and g_2 in (1), we get
$$S = \tfrac{1}{2} L + \frac{100\,(\sqrt{h_1} + \sqrt{h_2})^2}{A} \qquad \ldots(4.9)$$

or
$$L = 2S - \frac{200\,(\sqrt{h_1} + \sqrt{h_2})^2}{A} \qquad \ldots(4.10)$$

4.6. SIGHT DISTANCE AT UNDERPASS STRUCTURES

Let us now find the minimum length of vertical curve which will provide a specified sight distance for underpass structures. Let us assume that the *PI* of the tangents to the parabolic vertical curve is located vertically under the critical edge of the overhead structure restricting the view ahead. Here again, we will consider two cases : (*i*) $S > L$ and (*ii*) $S < L$.

Let A = algebraic difference of grades = $g_1 - g_2$

S = sight distance

C = vertical clearance at critical edge of underpass

h_1 = vertical height of driver's eye above road

h_2 = vertical height of sighted object.

Case 1. $S > L$ (Fig. 4.12)

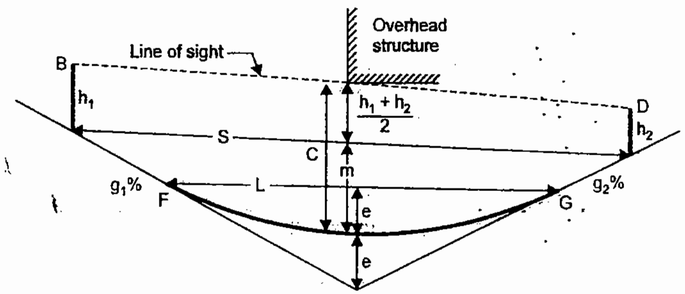


FIG. 4.12. $S > L$.

From similar triangles,
$$\frac{S}{L} = \frac{e + m}{2e} = \frac{1}{2} + \frac{m}{2e} \qquad \ldots(4.11)$$

where
$$e = \frac{L\,.\,A}{8} \text{ (Eq. 4.6) and } m = C - \frac{h_1 + h_2}{2}$$

If C = 4.2 m (= 14 ft)

h_1 = 1.8 m (= 6.0 ft)

h_2 = 0.45 m (= 1.5 ft)

we get
$$m = 4.2 - \frac{1.8 + 0.45}{2} = 3.075 \text{ m}$$

Substituting the evalues of m and e in Eq. 4.11, we get
$$S = \tfrac{1}{2} L + \frac{12.3}{A} \qquad \ldots(4.12)$$

From which
$$L = 2S - \frac{24.6}{A} \qquad \ldots(4.13)$$

Case 2. $S < L$ (Fig. 4.13)

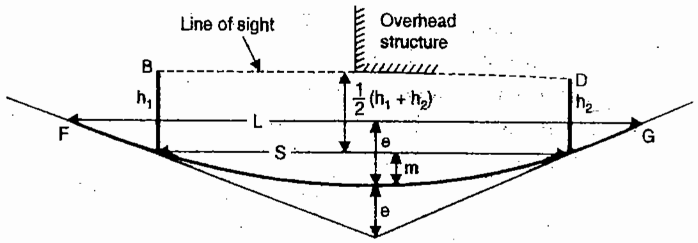


FIG. 4.13. $S < L$.

Every flat parabola may be assumed to be closely a circle. Let R be the average radius of the vertical curve.

Δ_L = central angle subtended by L (radian)

Δ_S = central angle subtended by S (radian)

Both Δ_L and Δ_S are assumed to be small.

$$e \text{ for parabola} = \frac{L\,A}{8} \text{ [Eq. 4.6]} \qquad \ldots(1)$$

$$e \text{ (for corresponding circle)} = R \tan \frac{\Delta_L}{2} \tan \frac{\Delta_L}{4} \approx \frac{R\,\Delta_L^2}{8} \text{ (approx).} \qquad \ldots(2)$$

Equating (1) and (2), we get
$$\Delta_L^2 = \frac{LA}{R} \qquad \ldots(3)$$

Again, let us assume that m for the parabola is equivalent to m for the circle.
$$\therefore \quad m = R\,.\,\frac{\text{vers}\,\Delta_S}{2} = R \sin \frac{\Delta_S}{2} \tan \frac{\Delta_S}{4} \approx \frac{R\,\Delta_S^2}{8} \qquad \ldots(4)$$

Combining (3), and (4), we get
$$\left(\frac{\Delta_L}{\Delta_S}\right)^2 = \frac{L\,A}{8\,m} \qquad \ldots(5)$$

Now $L = R\,\Delta_L$ and $S = R\,\Delta_S$ (approx.)
$$\therefore \quad \frac{\Delta_L}{\Delta_S} = \frac{L}{S}$$

or $$\left(\frac{\Delta_L}{\Delta_S}\right)^2 = \frac{L^2}{S^2} \quad ...(6)$$

From Eqs. (5) and (6), we get

$$L = \frac{S^2 . A}{8\,m} \quad ...(4.14)$$

Taking $C = 4.2$ m (14 ft)

$h_1 = 1.8$ m (6.0 ft)

$h_2 = 0.45$ m (1.5 ft)

we get $$m = C - \frac{h_1 + h_2}{2} = 4.2 - \frac{1.8 + 0.45}{2} = 3.075$$

Hence from Eq. 4.14,

$$L = \frac{S^2 A}{24.6} \quad ...(4.15)$$

and $$S = \sqrt{\frac{24.6\,L}{A}} \quad ...(4.16)$$

Note. When the critical edge of the overhead stucture is not directly above the vertex of the vertical curve, Eqs. 4.12 and 4.16 are still valid, provided that the edge is not more than 60 m from the vertex.

PROBLEMS

1. A rising gradient of 1 in 50 on a proposed road meets a falling gradient of 1 to 40 and the reduced level of the intersection point is found from a longitudinal section to be 207.54 feet above O.D. These gradients are to be connected by a parabolic vertical curve, and a shifting distance of 1000 ft is to be provided over the summit, assuming that the line of sight is 3 ft 9 in. above the road surface. Calculate the necessary levels for setting out the vertical curve.

ANSWERS

1. Length of curve = 1500 ft.

5

Trigonometrical Levelling

5.1. INTRODUCTION

Trigonometrical levelling is the process of determining the differences of elevations of stations from observed vertical angles and known distances, which are assumed to be either horizontal or geodetic lengths at mean sea level. The vertical angles may be measured by means of an accurate theodolite and the horizontal distances may either be *measured* (in the case of plane surveying) or *computed* (in the case of geodetic observations).

We shall discuss the trigonometrical levelling under two heads:

(1) Observations for height and distances, and (2) Geodetical observations.

In the first case, the principles of the plane surveying will be used. It is assumed that the distances between the points observed are not large so that either the effect of curvature and refraction may be neglected or proper correction may be applied *linearly* to the calculated difference in elevation. Under this head fall the various methods of angular levelling for determining the elevations of particular points such as the top of chimney, or church spire etc.

In the geodetical observations of trigonometrical levelling, the distance between the points measured is geodetic and is large. The ordinary principles of plane surveying are not applicable. The corrections for curvature and refraction are applied in *angular measure* directly to the observed angles.

HEIGHTS AND DISTANCES

In order to get the difference in elevation between the instrument station and the object under observation, we shall consider the following cases :

Case 1 : Base of the object accessible.

Case 2 : Base of the object inaccessible : instrument stations in the same vertical plane as the elevated object.

Case 3 : Base of object inaccessible : instrument stations not in the same vertical plane as the elevated object.

5.2. BASE OF THE OBJECT ACCESSIBLE

Let it be assumed that the horizontal distance between the instrument and the object can be measured accurately (Fig. 5.1).

In Fig. 5.1, let P = instrument station

Q = point to be observed

A = centre of the instrument

Q' = projection of Q on horizontal plane through A

$D = AQ'$ = horizontal distance between P and Q

h' = height of the instrument at P

$h = QQ'$

S = Reading on staff kept at B.M., with line of sight horizontal

α = angle of elevation from A to Q

From triangle AQQ',

$$h = D \tan \alpha \quad \ldots(5.1)$$

R.L. of Q = R.L. of instrument axis + $D \tan \alpha$

If the R.L. of P is known,

R.L. of Q = R.L. of $P + h' + D \tan \alpha$.

If the reading on the staff kept at the B.M. is S with the line of sight horizontal,

R.L. of Q = R.L. of B.M. $+ S + D \tan \alpha$

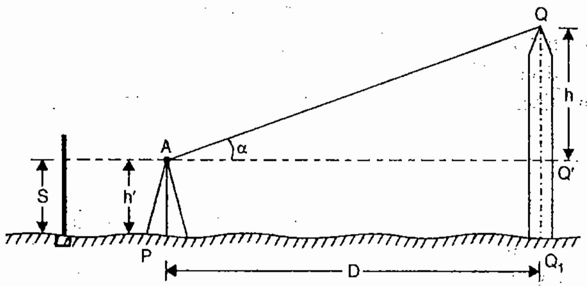


FIG. 5.1. BASE ACCESSIBLE.

The method is usually employed when the distance D is small. However, if D is large, the combined correction for curvature and refraction can be applied.

In order to get the sign of the combined correction due to curvature and refraction, consider Fig. 5.2. $PP''P'$ is the vertical (or plumb) line through P and $QQ'Q''$ is the vertical line through Q. P' is the projection of P on the horizontal line through Q, while P'' is the projection of P on the level line through Q. Similarly, Q' and Q'' are the projections of Q on horizontal and level lines respectively through P.

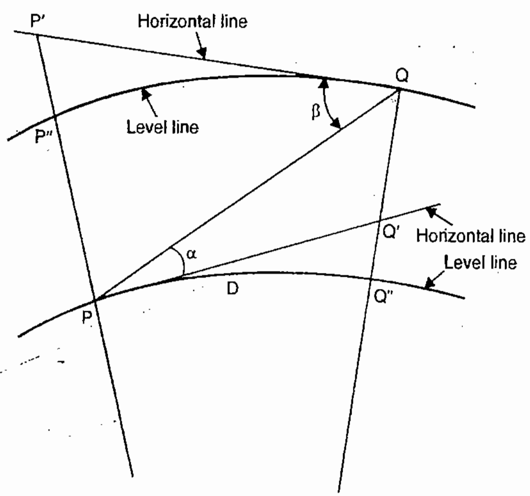


FIG. 5.2

If the distance between P and Q is not *very* large, we can take $PQ' = PQ'' = D = QP'' = QP'$

and $\angle QQ'P = \angle QP'P = 90°$ (approximately)

Then $QQ' = D \tan \alpha$

But the true difference in elevation between P and Q is QQ''. Hence the combined correction for curvature and refraction = $Q'Q''$ which should be added to QQ' to get the true difference in elevation QQ''.

Similarly, if the observation is made from Q, we get

$$PP' = D \tan \beta$$

The true difference in elevation is PP''. The combined correction for curvature and refraction = $P'P''$ which should be subtracted from PP' to get the true difference in elevation PP''.

Hence we conclude that if the combined correction of curvature and refraction is to be applied *linearly*, its sign is positive for angles of elevation and negative for angles of depression. As in levelling (Vol.I), the combined correction for curvature and refraction in linear measure is given by

$$C = 0.06728\, D^2 \text{ metres, when } D \text{ is in kilometres.}$$

Thus, in Fig. 5.1,

R.L. of Q = R.L. of B.M. $+ S + D \tan \alpha + C$.

Indirect Levelling. The above principle can be applied for running a line of indirect levels between two points P and Q whose difference of level is required (Fig. 5.3).

In order to find the difference in elevation between P and Q, the instrument is set at a number of places, Q_1, O_2, O_3 etc., with A, B, C etc., as the turning points as shown in Fig. 5.3. From each instrument station, observations are taken to both the points on either side of it, the instrument being set midway between them. Thus, in Fig. 5.4, let O_1 be the first position of the instrument set midway between P and A. If α_1 and β_1 are the angles observed from to P and A, we get

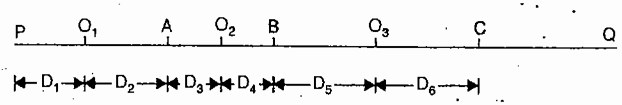


FIG. 5.3.

$$PP' = D_1 \tan \alpha_1$$

and $$AA' = D_2 \tan \beta_1$$

The difference in elevation between A and P $= H_1 = PP'' + A''A$

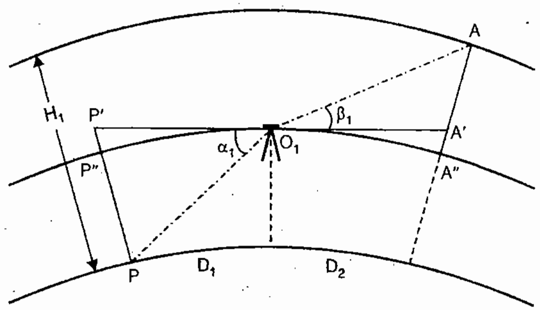


FIG. 5.4

$$= (PP' - P'P'') + (AA' + A'A'')$$
$$= (D_1 \tan\alpha_1 - P'P'') + (D_2 \tan\beta_1 + A'A'')$$

If $D_1 = D_2 = D$, $P'P''$ and $A'A''$ will be equal.

Hence, $H_1 = D(\tan\alpha_1 + \tan\beta_1)$

The instrument is then shifted to O_2, midway between A and B, and the angles α_2 and β_2 are observed. Then the difference in elevation between B and A is

$$H_2 = D'(\tan\alpha_2 + \tan\beta_2)$$

where $D' = D_3 = D_4$

The process is continued till Q is reached.

5.3. BASE OF THE OBJECT INACCESSIBLE : INSTRUMENT STATIONS IN THE SAME VERTICAL PLANE WITH THE ELEVATED OBJECT

If the horizontal distance between the instrument and the object cannot be measured due to obstacles, etc., two instrument stations are used so that they are in the same vertical plane as the elevated object (Fig. 5.5).

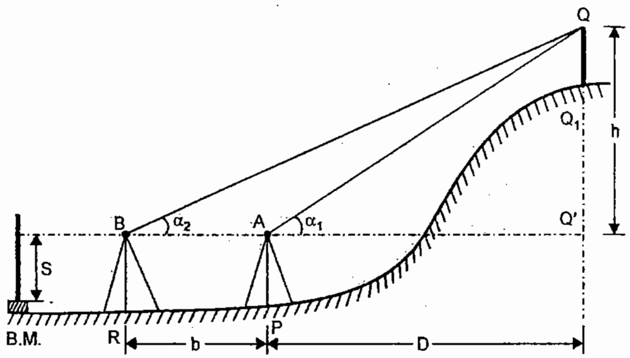


FIG. 5.5. INSTRUMENT AXES AT THE SAME LEVEL.

Procedure :

1. Set up the theodolite at P and level it accurately with respect to the altitude bubble.
2. Direct the telescope towards Q and bisect it accurately. Clamp both the plates. Read the vertical angle α_1.
3. Transit the telescope so that the line of sight is reversed. Mark the second instrument station R on the ground. Measure the distance RP accurately.

   Repeat steps (2) and (3) for both face observations. The mean values should be adopted.
4. With the vertical vernier set to zero reading, and the altitude bubble in the centre of its run, take the reading on the staff kept at the nearby B.M.
5. Shift the instrument to R and set up the theodolite there. Measure the vertical angle α_2 to Q with both face observations.
6. With the vertical vernier set to zero reading, and the altitude bubble in the centre of its run, take the reading on the staff kept at the nearby B.M.

In order to calculate the R.L. of Q, we will consider three cases : (*a*) when the instrument axes at A and B are at the same level, (*b*) when they are at different levels but the difference is small, and (*c*) when they are at very different levels.

(*a*) **Instrument axes at the same level** (Fig. 5.5)

Let $h = QQ'$

α_1 = angle of elevation from A at Q

α_2 = angle of elevation from B to Q

S = Staff reading on B.M. taken from both A and B, the reading being the same in both the cases

b = horizontal distances between the instrument stations.

D = horizontal distance between P and Q.

From triangle AQQ', $h = D\tan\alpha_1$ ...(1)

From triangle BQQ', $h = (b + D)\tan\alpha_2$ ...(2)

Equating (1) and (2), we get

$$D\tan\alpha_1 = (b + D)\tan\alpha_2$$

or $D(\tan\alpha_1 - \tan\alpha_2) = b\tan\alpha_2$

or
$$D = \frac{b\tan\alpha_2}{\tan\alpha_1 - \tan\alpha_2} \quad ...(5.2)$$

$$h = D\tan\alpha_1 = \frac{b\tan\alpha_1\tan\alpha_2}{\tan\alpha_1 - \tan\alpha_2} = \frac{b\sin\alpha_1\sin\alpha_2}{\sin(\alpha_1 - \alpha_2)} \quad ...(5.3)$$

R.L. of Q = R.L. of B.M. + $S + h$.

(*b*) **Instrument axes at different levels** [Fig. 5.6. and Fig. 5.7]

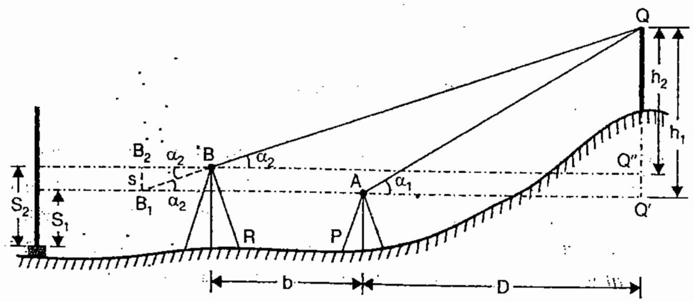


FIG. 5.6. INSTRUMENT AXES AT DIFFERENT LEVELS.

Figs. 5.6 and 5.7 illustrate the cases when the instrument axes are at different levels. If S_1 and S_2 are the corresponding staff readings on staff kept at B.M., the difference in levels of the instrument axes will be either $(S_2 - S_1)$ (if the axis at B is higher) or $(S_1 - S_2)$ (if the axis at A is higher). Let Q' be the projection of Q on horizontal line through A and Q'' be the projection on horizontal line through B. Let us derive the expression for Fig. 5.6. when S_2 is greater than S_1.

From triangle QAQ', $h_1 = D\tan\alpha_1$ ...(1)

From triangle BQQ'', $h_2 = (b + D)\tan\alpha_2$ ...(2)

Subtracting (2) from (1), we get

$$(h_1 - h_2) = D\tan\alpha_1 - (b + D)\tan\alpha_2$$

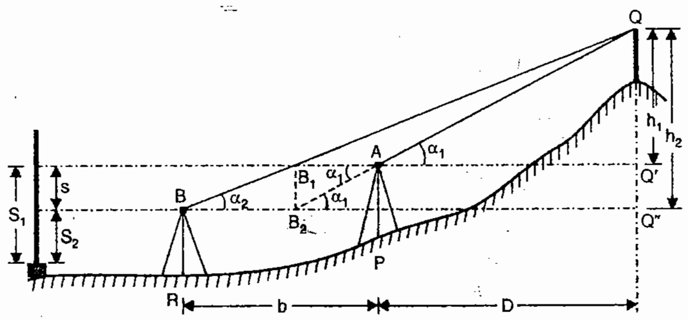


FIG. 5.7. INSTRUMENT AXES AT DIFFERENT LEVELS.

But $\quad h_1 - h_2$ = difference in levels of instrument axes $= S_2 - S_1 = s$ (say)

$$s = D \tan \alpha_1 - b \tan \alpha_2 - D \tan \alpha_2$$

or $\quad D(\tan \alpha_1 - \tan \alpha_2) = s + b \tan \alpha_2$

or $$D = \frac{s + b \tan \alpha_2}{\tan \alpha_1 - \tan \alpha_2}$$

or $$D = \frac{(b + s \cot \alpha_2) \tan \alpha_2}{\tan \alpha_1 - \tan \alpha_2} \quad ...[5.4\ (a)]$$

Now, $\quad h_1 = D \tan \alpha_1$

or $$h_1 = \frac{(b + s \cot \alpha_2) \tan \alpha_1 \tan \alpha_2}{\tan \alpha_1 - \tan \alpha_2}$$

or $$h_1 = \frac{(b + s \cot \alpha_2) \sin \alpha_1 \sin \alpha_2}{\sin (\alpha_1 - \alpha_2)} \quad ...[5.5\ (a)]$$

Expression 5.4 (*a*) could also be obtained producing the line of sight BQ backwards to meet the line $Q'A$ in B_1. Drawing B_1B_2 as vertical to meet the horizontal line $Q''B$ in B_2, it is clear that with the same angle of elevation if the instrument axis were at B_1, the instrument axes in both the cases would have been at the same elevation. *Hence the distance at which the axes are at the same level is* $AB_1 = b + BB_2 = b + s \cot \alpha_2$. Substituting this value of the distance between the instrument stations in equation 5.2, we get

$$D = \frac{(b + s \cot \alpha_2) \tan \alpha_2}{\tan \alpha_1 - \tan \alpha_2}$$ which is the same as equation 5.4 (*a*)

Proceeding on the same lines for the case of Fig. 5.7 where the instrument axes at A is higher, it can be proved that

$$D = \frac{(b - s \cot \alpha_2) \tan \alpha_2}{\tan \alpha_1 - \tan \alpha_2} \quad ...[5.4\ (b)]$$

and $$h_1 = \frac{(b - s \cot \alpha_2) \sin \alpha_1 \sin \alpha_2}{\sin (\alpha_1 - \alpha_2)} \quad ...[5.5\ (b)]$$

Thus, the general expressions for D and h_1 can be written as

$$D = \frac{(b \pm s \cot \alpha_2) \tan \alpha_2}{\tan \alpha_1 - \tan \alpha_2} \quad ...(5.4)$$

and $$h_1 = \frac{(b \pm s \cot \alpha_2) \sin \alpha_1 \sin \alpha_2}{\sin (\alpha_1 - \alpha_2)} \quad ...(5.5)$$

Use + sign with $s \cot \alpha_2$ when the instrument axis at A is lower and − sign when it is higher than at B.

R.L. of Q = R.L. of B.M. + $S_1 + h_1$.

(*c*) **Instrument axes at very different levels**

If $S_2 - S_1$ or s is too great to be measured on a staff kept at the B.M., the following procedure is adopted (Fig. 5.8 and 5.9):

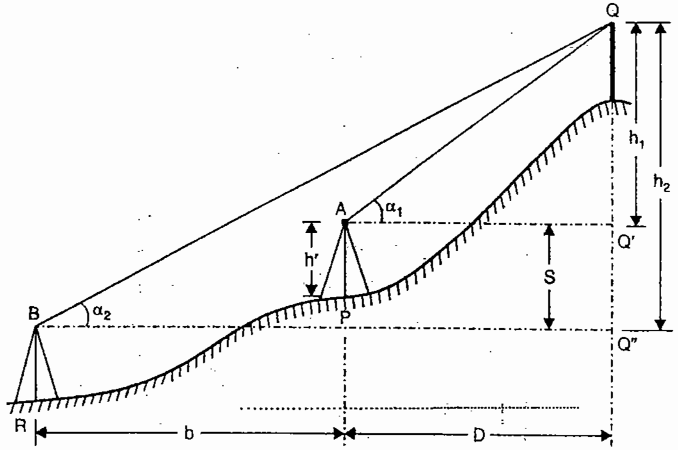


FIG. 5.8. INSTRUMENT AXES AT VERY DIFFERENT LEVELS.

(1) Set the instrument at P (Fig. 5.8), level it accurately with respect to the altitude bubble and measure the angle α_1 to the points Q.

(2) Transit the telescope and establish a points R at a distance b from P.

(3) Shift the instrument to R. Set the instrument and level it with respect to the altitude bubble and measure the angle α_2 to Q.

(4) Keep a vane of height r at P (or a staff) and measure the angle α to the top of the vane [or to the reading r if a staff is used (Fig. 5.9)].

Let s = Difference in level between the two axes at A and B. With the same symbols as earlier, we have

$$h_1 = D \tan \alpha_1 \quad ...(1)$$

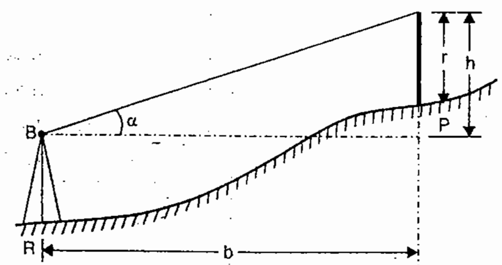


FIG. 5.9.

and $$h_2 = (b + D) \tan \alpha_2 \qquad ...(2)$$

Subtracting (1) from (2), we get

$$(h_2 - h_1) = s = (b + D) \tan \alpha_2 - D \tan \alpha_1$$

or $$D (\tan \alpha_1 - \tan \alpha_2) = b \tan \alpha_2 - s$$

$\therefore$ $$D = \frac{b \tan \alpha_2 - s}{\tan \alpha_1 - \tan \alpha_2} \qquad ...(3)$$

and $$h_1 = D \tan \alpha_1 = \frac{(b \tan \alpha_2 - s) \tan \alpha_1}{\tan \alpha_1 - \tan \alpha_2} = \frac{(b - s \cot \alpha_2) \sin \alpha_1 \sin \alpha_2}{\sin (\alpha_1 - \alpha_2)} \qquad ...[5.5\ (b)]$$

From Fig. 5.9, we have

Height of station P above the axis at $B = h - r = b \tan \alpha - r$

Height of axis at A above the axis at $B = s = b \tan \alpha - r + h'$

where h' the height of the instrument at P.

Substituting this value of s in (3) and equation 5.5 (b), we can get D and h_1.

Now R.L. of Q = R.L. of $A + h_1$ = R.L. of $B + s + h_1$

= (R.L. of B.M. + back sight taken from B) + $s + h_1$

where $$s = b \tan \alpha - r + h'.$$

5.4. BASE OF THE OBJECT INACCESSIBLE : INSTRUMENT STATIONS NOT IN THE SAME VERTICAL PLANE AS THE ELEVATED OBJECT

Let P and R be the two instrument stations *not* in the same vertical plane as that of Q. The procedure is as follows :

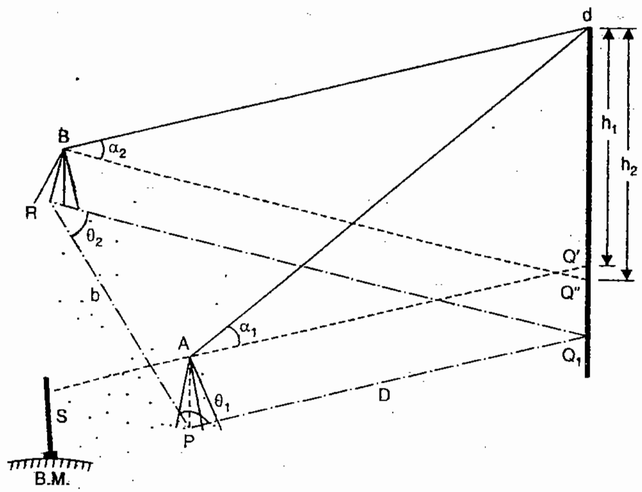


FIG. 5.10. INSTRUMENT AND THE OBJECT NOT IN THE SAME VERTICAL PLANE.

(1) Set the instrument at P and level it accurately with respect to the altitude bubble. Measure the angle of elevation α_1 to Q.

(2) Sight the point R with reading on horizontal circle as zero, and measure the angle RPQ_1 *i.e.*, the horizontal angle θ_1 at P.

(3) Take a back sight S on the staff kept at B.M.

(4) Shift the instrument to R and measure α_2 and θ_2 there.

In Fig. 5.10, AQ' is the horizontal line through A, Q' being the vertical projection of Q. Thus, AQQ' is a vertical plane. Similarly, BQQ'' is a vertical plane, Q'' being the vertical projection of Q on a horizontal line through B. PRQ_1 is a horizontal plane, Q_1 being the vertical projection of Q and R vertical projection of B on a horizontal plane passing through P. θ_1 and θ_2 are the horizontal angles, and α_1 and α_2 are the vertical angles measured at A and B respectively.

From triangle AQQ', $\quad QQ' = h_1 = D \tan \alpha_1 \qquad ...(1)$

From triangle PRQ_1, $\quad \angle PQ_1R = 180° - (\theta_1 + \theta_2) = \pi - (\theta_1 + \theta_2)$

From the sine rule,

$$\frac{PQ_1}{\sin \theta_2} = \frac{RQ_1}{\sin \theta_1} = \frac{RP}{\sin[\pi - (\theta_1 + \theta_2)]} = \frac{b}{\sin (\theta_1 + \theta_2)}$$

$\therefore$ $$PQ_1 = D = \frac{b \sin \theta_2}{\sin (\theta_1 + \theta_2)} \qquad ...(2)$$

and $$RQ_1 = \frac{b \sin \theta_1}{\sin (\theta_1 + \theta_2)} \qquad ...(3)$$

Substituting the value of D in (1), we get

$$h_1 = D \tan \alpha_1 = \frac{b \sin \theta_2 \tan \alpha_1}{\sin (\theta_1 + \theta_2)} \qquad ...(5.6)$$

$\therefore$ R.L. of Q = R.L. of B.M. + $S + h_1$

As a check, $$h_1 = RQ_1 \tan \alpha_2 = \frac{b \sin \theta_1 \tan \alpha_2}{\sin (\theta_1 + \theta_2)}$$

If a reading on a B.M. is taken from B, the R.L. of Q can be known by adding h_2 to R.L. of B.

Example 5.1. *An instrument was set up at P and the angle of elevation to a vane 4 m above the foot of the staff held at Q was 9° 30′. The horizontal distance between P and Q was known to be 2000 metres. Determine the R.L. of the staff station Q, given that the R.L. of the instrument axis was 2650.38.*

Solution.

Height of vane above the instrument axis

$$= D \tan \alpha = 2000 \tan 9° 30' = 334.68 \text{ m}$$

Correction for curvature and refraction $= \frac{6}{7} \frac{D^2}{2R}$

or $$C = 0.06728\, D^2 \text{ m, when } D \text{ is in km}$$

$$= 0.06728 \left(\frac{2000}{1000}\right)^2 = 0.269 \approx 0.27 \text{ m } (+\text{ve})$$

Height of vane above the instrument axis = 334.68 + 0.27 = 334.95

R.L. fo vane = 334.95 + 2650.38 = 2985.33 m

R.L. of Q = 2985.33 – 4 = **2981.33 m.**

Example 5.2. *An instrument was set up at P and the angle of depression to a vane 2 m above the foot of the staff held at Q was 5° 36′. The horizontal distance between P and Q was known to be 3000 metres. Determine the R.L. of the staff station Q, given that staff reading on a B.M. of elevation 436.050 was 2.865 metres.*

Solution.

The difference in elevation between the vane and the instrument axis

$= D \tan \alpha = 3000 \tan 5° 36' = 294.153$

Combined correction due to curvature and refraction $= \frac{6}{7}\frac{D^2}{2R}$

or $C = 0.06728\ D^2$ metres, when D is in km $= 0.06728 \left(\frac{3000}{1000}\right)^2 = 0.606$ m.

Since the observed angle is negative, the combined correction due to curvature and refraction is subtractive.

Difference in elevation between the vane and the instrument axis = 294.153 – 0.606

= 293.547 = h.

R.L. of instrument axis = 436.050 + 2.865 = 438.915

∴ R.L. of the vane = R.L. of instrument aixs. – h

= 438.915 – 293.547 = 145.368

R.L. of Q = 145.368 – 2

= **143.368 m.**

Example 5.3. *In order to ascertain the elevation of the top (Q) of the signal on a hill, observations were made from two instrument stations P and R at a horizontal distance 100 metres apart, the station P and R being in the line with Q. The angles of elevation of Q at P and R were 28° 42′ and 18° 6′ respectively. The staff reading upon the bench mark of elevation 287.28 were respectively 2.870 and 3.750 when the instrument was at P and at R, the telescope being horizontal. Determine the elevation of the foot of the signal if the height of the signal above its base is 3 metres.*

Solution. (Fig. 5.6)

Elevation of instrument axis at P = R.L of B.M. + Staff reading

= 287.28 + 2.870 = 290.15 m

Elevation of instrument axis at R = R.L. of B.M. + staff reading

= 287.28 + 3.750 = 291.03 m

Difference in level of the instrument axes at the two stations = s = 291.03 – 290.15 = 0.88 m

$\alpha_1 = 28° 42'$ and $\alpha_2 = 18° 6'$

$s \cot \alpha_2 = 0.88 \cot 18° 6' = 2.69$ m

From equation 5.4 (*a*), we have

$$D = \frac{(b + s \cot \alpha_2) \tan \alpha_2}{\tan \alpha_1 - \tan \alpha_2} = \frac{(100 + 2.69) \tan 18° 6'}{\tan 28° 42' - \tan 18° 6'} = \mathbf{152.1 \text{ m.}}$$

∴ $h_1 = D \tan \alpha_1 = 152.1 \tan 28° 42' = 83.272$ m

∴ R.L. of foot of signal = R.L. of inst. axis at $P + h_1$ – ht. of signal

= 290.15 + 83.272 – 3 = **370.422 m.**

Check : $(b + D)$ = 100 + 152.1 = 252.1 m

$h_2 = (b + D) \tan \alpha_2 = 252.1 \times \tan 18° 6' = 82.399$ m

R.L. of foot of signal = R.L. of inst. axis at $R + h_2$ + ht. of signal

= 291.03 + 82.399 – 3 = **370.429 m.**

Example 5.4. *The top (Q) of a chimney was sighted from two stations P and R at very different levels, the stations P and R being in the line with the top of the chimney. The angle of elevation from P to the top of the chimney was 38° 21′ and that from R to the top of the chimney was 21° 18′. The angle of elevation from R to a vane 2 m above the foot of the staff held at P was 15° 11′. The heights of the instrument at P and R were 1.87 m and 1.64 m respectively. The horizontal distance between P and R was 127 m and the reduced level of R was 112.78 m. Find the R.L. of the top of the chimney and the horizontal distance from P to the chimney.*

Solution. (Figs. 5.8 and 5.9)

(*i*) When the observations were taken from R to P

$h = b \tan \alpha = 127 \tan 15° 11' = 34.47$ m

R.L. of P = R.L. of R + height of instrument at $R + h - r$

= 112.78 + 1.64 + 34.47 – 2 = 146.89 m

R.L. of instrument axis at P = R.L. of P + ht. of instrument at P

= 146.89 + 1.87 = 148.76 m ...(*i*)

Difference in elevation between the instrument axes = s

= 148.76 – (112.78 + 1.64) = 34.34 m

$$\therefore \quad D = \frac{(b \tan \alpha_2 - s)}{\tan \alpha_1 - \tan \alpha_2} = \frac{127 \tan 21° 18' - 34.34}{\tan 38° 21' - \tan 21° 18'} = \frac{49.25 - 34.34}{0.79117 - 0.38988} = 37.8 \text{ m}$$

$h_1 = D \tan \alpha_1 = 37.8 \tan 38° 21' = 29.91$

∴ R.L. of Q = R.L. of instrument axis at $P + h_1$

= 148.76 + 29.91 = **178.67 m.**

Check : R.L. of Q = R.L. of instruments axis at $R + h_2$

$= (112.78 + 1.64) + (b + D) \tan \alpha_2 = 114.42 + (127 + 37.8) \tan 21° 18'$

= 114.42 + 64.25 = **178.67 m.**

Example 5.5. *To find the elevation of the top (Q) of a hill, a flag staff of 2 m height was erected and observations were made from two stations P and R, 60 metres* *apart. The horizontal angle measured at P between R and the top of the flat staff was 60° 30′ and that measured at R between the top of the flag staff and P was 68° 18′. The angle of elevation to the top of the flag staff was measured to be 10° 12′ at P. The angle of elevation to the top of the flag staff was measured to be 10° 48′ at R. Staff readings on B.M. when the instrument was at P = 1.965 m and that with the instrument at R = 2.055 m. Calculate the elevation of the top of the hill if that of B.M. was 435.065 m.*

Solution. (Fig. 5.10)

Given : $b = 60$ m

$\theta_1 = 60° 30'$; $\theta_2 = 68° 18'$

$\alpha_1 = 10° 12'$; $\alpha_2 = 10° 18'$

$$\therefore \quad PQ_1 = D = \frac{b \sin \theta_2}{\sin (\theta_1 + \theta_2)}$$

and $$h_1 = D \tan \alpha_1 = \frac{b \sin \theta_2 \tan \alpha_1}{\sin(\theta_1 + \theta_2)} = \frac{60 \sin 68° 18' \tan 10° 12'}{\sin (60° 30' + 68° 18')} = 12.87 \text{ m}$$

$\therefore$ R.L. of Q = (R.L. of instrument axis at P) + h_1

= (435.065 + 1.965) + 12.87 = **449.900 m.**

Check $$h_2 = \frac{b \sin \theta_1 \tan \alpha_2}{\sin(\theta_1 + \theta_2)} = \frac{60 \sin 60° 30' \tan 10° 48'}{\sin (60° 30' + 68° 18')} = 12.78 \text{ m}$$

$\therefore$ R.L. of Q = R.L. of instrument axis at $R + h_2$ = (435.065 + 2.055) + 12.78

= **449.9 m.**

5.5. DETERMINATION OF HEIGHT OF AN ELEVATED OBJECT ABOVE THE GROUND WHEN ITS BASE AND TOP ARE VISIBLE BUT NOT ACCESSIBLE

(*a*) Base line horizontal and in line with the object

Let A and B be the two instrument stations, b apart. The vertical angles measured at A are α_1 and α_2, and those at B are β_1 and β_2, corresponding to the top (E) and bottom (D) of the elevated object. Let us take a general case of instruments at different heights, the difference being equal to s.

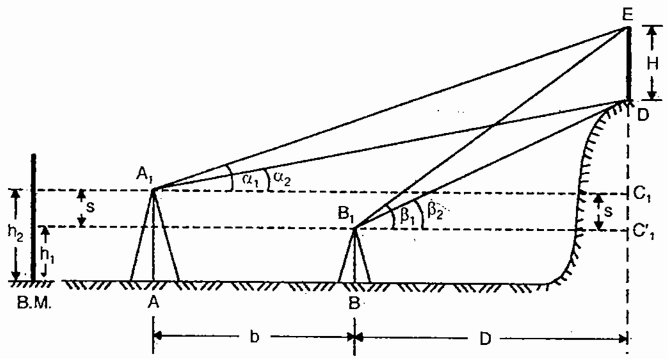


FIG. 5.11

Now $AB = b = C_1E \cot \alpha_1 - C_1'E \cot \beta_1 = C_1E \cot \alpha_1 - (C_1E + s) \cot \beta_1$

$\therefore \quad b = C_1E (\cot \alpha_1 - \cot \beta_1) - s \cot \beta_1$

or $$C_1E = \frac{b + s \cot \beta_1}{\cot \alpha_1 - \cot \beta_1} \qquad ...(1)$$

Also, $AB = b = C_1D \cot \alpha_2 - C_1'D \cot \beta_2 = C_1 D \cot \alpha_2 - (C_1D + s) \cot \beta_2$

or $b = C_1D (\cot \alpha_2 - \cot \beta_2) - s \cot \beta_2$

or $$C_1D = \frac{b + s \cot \beta_2}{\cot \alpha_2 - \cot \beta_2}$$

$$\therefore \quad H = C_1E - C_1D = \frac{b + s \cot \beta_1}{\cot \alpha_1 - \cot \beta_1} - \frac{b + s \cot \beta_2}{\cot \alpha_2 - \cot \beta_2} \qquad ...(5.7)$$

If heights of the instruments at A and B are equal, $s = 0$

$$\therefore \quad H = b\left[\frac{1}{\cot \alpha_1 - \cot \beta_1} - \frac{1}{\cot \alpha_2 - \cot \beta_2}\right] \qquad ...(5.7\ a)$$

Horizontal distance of the object from B

$EC_1' = D \tan \beta_1$ and $DC_1' = D \tan \beta_2$

$\therefore \quad EC_1' - DC_1' = H = D (\tan \beta_1 - \tan \beta_2)$

or $$D = \frac{H}{\tan \beta_1 - \tan \beta_2} \qquad ...(5.7\ b)$$

where H is given by Eq. 5.7.

(*b*) Base line horizontal but not in line with the object

Let A and B be two instrument stations, distant b. Let α_1 and α_2 be the vertical angles measured at A, and β_1 and β_2 be the vertical angle measured at B , to the top (E) and bottom (D) of the elevated object. Let θ and φ be the horizontal angles measured at A and B respectively.

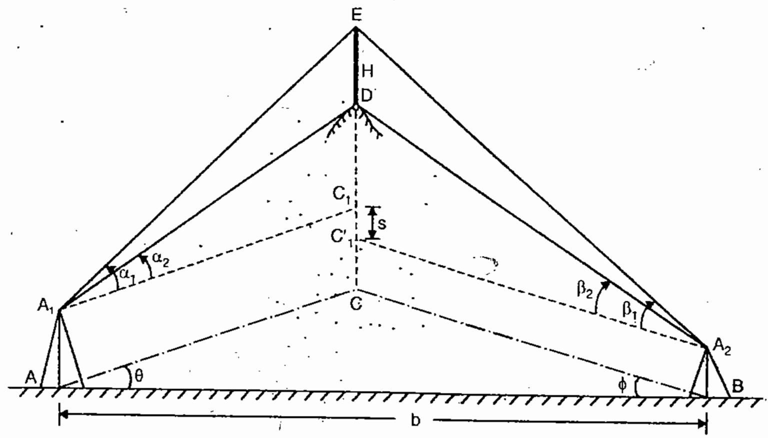


FIG. 5.12

From triangle ACB, $\frac{AC}{\sin\varphi} = \frac{BC}{\sin\theta} = \frac{AB}{\sin(180° - \theta - \varphi)}$

$\therefore$ $AC = b \sin\varphi \operatorname{cosec}(\theta + \varphi)$

and $BC = b \sin\theta \operatorname{cosec}(\theta + \varphi)$

Now $H = ED = A_1C_1 (\tan\alpha_1 - \tan\alpha_2) = AC (\tan\alpha_1 - \tan\alpha_2)$

or $H = b \sin\varphi \operatorname{cosec}(\theta + \varphi)(\tan\alpha_1 - \tan\alpha_2)$ ...(5.8 *a*)

Similarly $H = ED = BC_1' (\tan\beta_1 - \tan\beta_2) = BC (\tan\beta_1 - \tan\beta_2)$

or $H = b \sin\theta \operatorname{cosec}(\theta + \varphi)(\tan\beta_1 - \tan\beta_2)$ ...(5.8 *b*)

5.6. DETERMINATION OF ELEVATION OF AN OBJECT FROM ANGLES OF ELEVATION FROM THREE INSTRUMENT STATIONS IN ONE LINE

Let A, B, C be three instrument stations in one horizontal line, with instrument axes at the same height. Let E' be the projection of E on the horizontal plane through ABC, and let $EE' = h$. Let α, β and γ be the angles of elevation of the object E, measured from instruments at A, B and C respectively. Also let $AB = b_1$ and $BC = b_2$, be the measured horizontal distances.

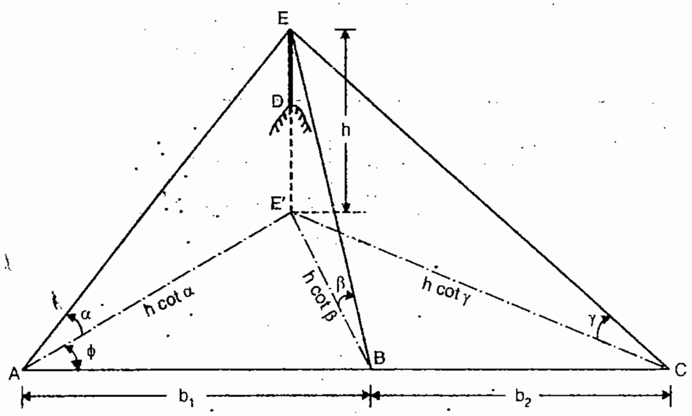


FIG. 5.13

From triangle $AE'B$, we have from *cosine rule*

$$\cos\varphi = \frac{h^2\cot^2\alpha + b_1^2 - h^2\cot^2\beta}{2b_1 h\cot\alpha} \quad ...(1) \quad ...(5.9)$$

Also, from triangle $AE'C$, $\cos\varphi = \frac{h^2\cot^2\alpha + (b_1+b_2)^2 - h^2\cot^2\gamma}{2(b_1+b_2)h\cot\alpha}$ ...(2)

Equating (1) and (2), $\frac{h^2\cot^2\alpha + b_1^2 - h^2\cot^2\beta}{2b_1 h\cot\alpha} = \frac{h^2\cot^2\alpha + (b_1+b_2)^2 - h^2\cot^2\gamma}{2(b_1+b_2)h\cot\alpha}$

or $(b_1+b_2)[h^2(\cot^2\alpha - \cot^2\beta) + b_1^2] = b_1[h^2(\cot^2\alpha - \cot^2\gamma) + (b_1+b_2)^2]$

or $h^2[(b_1+b_2)(\cot^2\alpha - \cot^2\beta) - b_1(\cot^2\alpha - \cot^2\gamma)] = b_1(b_1+b_2)^2 - b_1^2(b_1+b_2)$

or $$h^2 = \frac{(b_1+b)[b_1(b_1+b_2) - b_1^2]}{(b_1+b_2)(\cot^2\alpha - \cot^2\beta) - b_1(\cot^2\alpha - \cot^2\gamma)}$$

$$= \frac{(b_1+b_2)\,b_1 b_2}{(b_1+b_2)(\cot^2\alpha - \cot^2\beta) - b_1(\cot^2\alpha - \cot^2\gamma)}$$

or $$h = \left[\frac{b_1 b_2(b_1+b_2)}{b_1(\cot^2\gamma - \cot^2\beta) + b_2(\cot^2\alpha - \cot^2\beta)}\right]^{1/2} \quad ...(5.10)$$

If $b_1 = b_2 = b$

$$h = \frac{\sqrt{2}\,b}{(\cot^2\gamma - 2\cot^2\beta + \cot^2\alpha)^{1/2}} \quad ...(5.10\ a)$$

Example 5.6. *Determine the height of a pole above the ground on the basis of following angles of elevation from two instrument stations A and B, in line with the pole.*

Angles of elevation from A to the top and bottom of pole : 30° and 25°

Angles of elevation from B to the top and bottom of pole : 35° and 29°

Horizontal distance AB = 30 m.

The readings obtained on the staff at the **B.M.** *with the two instrument settings are 1.48 and 1.32 m respectively.*

What is the horizontal distance of the pole from A ?

Solution (Refer Fig. 5.11)

$s = 1.48 - 1.32 = 0.16$ m

$b = 30$ m ; $\alpha_1 = 30°$; $\alpha_2 = 25°$; $\beta_1 = 35°$; $\beta_2 = 29°$

Substituting the values in Eq. 5.7.

$$H = \frac{b + s\cot\beta_1}{\cot\alpha_1 - \cot\beta_1} - \frac{b + s\cot\beta_2}{\cot\alpha_2 - \cot\beta_2}$$

$$= \frac{30 + 0.16\cot 35°}{\cot 30° - \cot 35°} - \frac{30 + 0.16\cot 29°}{\cot 25° - \cot 29°}$$

$= 99.47 - 88.96 = \mathbf{10.51}$ **m**

Also, $D = \frac{H}{\tan\beta_1 - \tan\beta_2} = \frac{10.51}{\tan 35° - \tan 29°} = 72.04$ m

$\therefore$ Distance of pole from $A = b + D = 30 + 72.04 =$ **102.04 m**

Example 5.7. *A, B and C are stations on a straight level line of bearing 110° 16′ 48″. The distance AB is 314.12 m and BC is 252.58 m. With instrument of constant height of 1.40 m, vertical angles were successively measured to an inaccessible up station E as follows:*

At A : 7° 13′ 40″

At B : 10° 15′ 00″

At C : 13° 12′ 10″

Calculate (a) the height of station E above the line ABC

(b) the bearing of the line AE

(c) the horizontal distance between A and E :

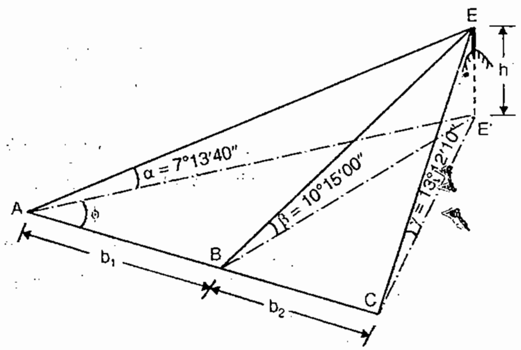


FIG. 5.14

Solution : Refer Fig. 5.14.

Given : $\alpha = 7°\ 13'\ 40''$;

$\beta = 10°\ 15'\ 00''$;

$\gamma = 13°\ 12'\ 10''$;

$b_1 = 314.12$ m;

and $b_2 = 252.58$ m

Substituting the values in Eq. 5.10, we get

$$EE' = h = \left[\frac{b_1\, b_2\,(b_1 + b_2)}{b_1\,(\cot^2 \gamma - \cos^2 \beta) + b_2\,(\cot^2 \alpha - \cot^2 \beta)}\right]^{1/2}$$

$$= \left[\frac{314.12 \times 252.58\ (314.12 + 252.58)}{314.12(\cot^2 13°\ 12'\ 10'' - \cot^2 10°\ 15'\ 00'') + 252.58\ (\cot^2 7°\ 13'\ 40'' - \cot^2 10°\ 15'\ 00'')}\right]^{1/2}$$

$= 104.97$ m

$\therefore$ Height of E above $ABC = 104.97 + 1.4 =$ **106.37 m**.

Also, From Eq. 5.9.

$$\cos \varphi = \frac{h^2\,(\cot^2 \alpha - \cot^2 \beta) + b_1^2}{2\, b_1\, h \cot \alpha}$$

$$= \frac{(104.97)^2\ (\cot^2 7°\ 13'\ 40'' - \cot^2 10°\ 15'\ 00'') + (314.12)^2}{2 \times 314.12 \times 104.97 \cot 7°\ 13'\ 40''}$$

$= 0.859205$

or $\varphi = 30°\ 46'\ 21''$

Hence bearing of $AE = 110°\ 16'\ 48'' - 30°\ 46'\ 21''$

$=$ **79° 30′ 27″**

Length $AE' = h \cot \alpha = 104.97 \cot 7°\ 13'\ 40''$

$=$ **827.70 m**

GEODETICAL OBSERVATIONS

5.7. TERRESTRIAL REFRACTION

The effect of refraction is to make the objects appear *higher* than they really are. In plane surveying where a graduated staff is observed either with horizontal line of sight or inclined line of sight, the effect of refraction is to decrease the staff reading and the correction is applied linearly to the observed staff reading. In trigonometrical levelling employed

for determining the elevations of widely distributed points, the correction is applied to the observed angles.

In Fig. 5.15, P and Q are the two points the difference in elevation between these being required.

Let O = centre of the earth

PO' = tangent to the level line through P = horizontal line at P

QO' = horizontal line at Q

$\angle P'PO' = \alpha_1$ = *observed* angle of elevation from P to Q (corrected for the difference in the heights of the signal and the instrument)

$\angle Q'QQ_2 = \beta_1$ = observed angle of depression from Q to P (corrected for the difference in the heights of the signal and the instrument).

r = angle of refraction or angular correction for refraction $= \angle P'PQ = \angle Q'QP$

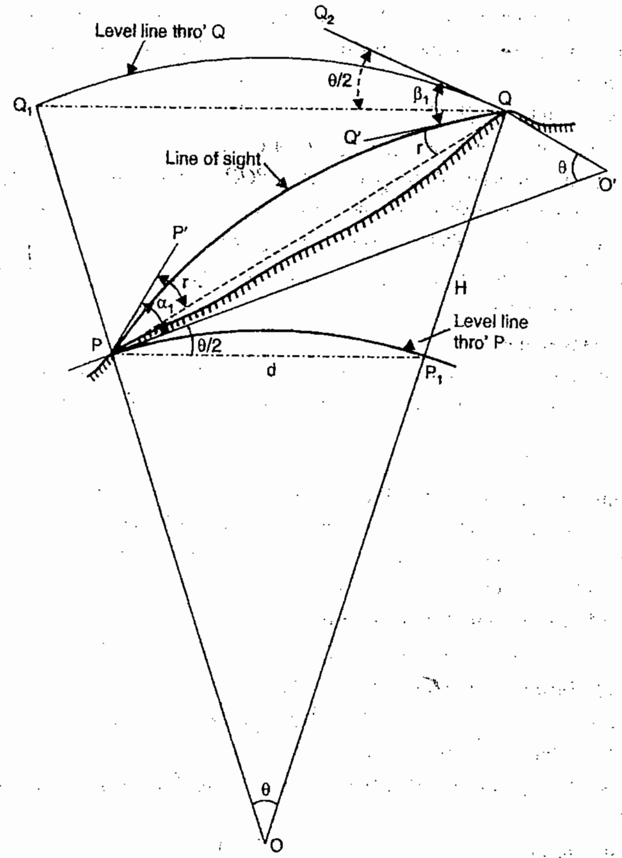


FIG. 5.15. TERRESTRIAL REFRACTION.

PP' = tangent at P to the curved line of sight PQ = apparent sight
QQ' = tangent at Q to the curved line of sight QP = apparent sight
d = horizontal distance between P and Q
R = mean radius of the earth = 6370 km
m = co-efficient of refraction
θ = angle subtended at the centre by the distance PP_1 over which the observations are made.

The actual line of sight between P and Q should have been along straight line PQ but due to the effect of terrestrial refraction, the actual line of sight is curved concave towards the ground surface. PP' is, therefore, the apparent sight from P to Q, and QQ' is the apparent sight from Q to P. Since the angles are measured on the circle of a theodolite, they are measured in the horizontal plane. The angle measured at P towards Q is, therefore, the angle between the apparent sight $P'P$ and the horizontal line PQ'. Hence $\angle P'PO'$ = observed angle α_1. The true angle of elevation, in the absence of refraction is $\angle QPO'$. Hence the correction for refraction is the $\angle P'PQ$. Calling this as r, the correction is evidently *subtractive*.

Thus, correct angle $= \angle QPO' = \angle P'PO' - \angle P'PQ = \alpha_1 - r$

Similarly, the angle measured at Q towards P is $\angle Q'QQ_2 = \beta_1$. The true angle of depression, in the absence of refraction is $\angle PQQ_2$. Hence the correction for refraction is $\angle PQQ'$ and should be *added* to the observed angle to get the correct angle.

Thus, correct angle $= \angle PQQ_2 = \angle Q'QQ_2 + \angle Q'QP = \beta_1 + r$

Thus, the correction for refraction is subtractive to the angle of elevation and additive to the angle of depression.

Co-efficient of refraction

The co-efficient of refraction (m) is the ratio of angle of refraction and the angle subtended at the centre of the earth by the distance over which observations are taken.

Thus, $$m = \frac{r}{\theta} \quad \text{or} \quad r = m\theta \qquad \ldots(5.11)$$

Thus, value of m varies roughly between 0.06 to 0.08. An average value of 0.07 may be taken if accurate data is not available. At a given place, its greatest value occurs in the early morning; it diminishes until 9 to 10 a.m. after which it remains fairly constant until about 4 o'clock after which it commences to increase.

Determination of correction for refraction (r)

In order to determine the angle of refraction r, we will take two cases :

Case (a) Distance d small and H large :

In this case, d is small and H is large so that one angle (α_1) is the angle of elevation and the other (β_1) is the angle of depression.

In Fig. 5.15, the angle $PO'Q$ between the two horizontal lines through P and Q is evidently θ.

In triangle PQO',

$$\angle PQQ_2 = \angle QPO' + \angle QO'P$$

Now $\angle PQQ_2 = \beta_1 \pm r$

$\angle QPO' = \alpha_1 - r$

and $\angle QO'P = \theta$

$\therefore \quad \beta_1 + r = \theta + \alpha_1 - r$

or $\quad 2r = \theta + \alpha_1 - \beta_1 = \theta - (\beta_1 - \alpha_1)$

or $$r = \frac{\theta}{2} - \left(\frac{\beta_1 - \alpha_1}{2}\right) \qquad \ldots(5.12)$$

It is assumed that the refraction error r is the same at both the stations.

Writing $r = m\theta$ and rearranging, we get

$2m\theta = \theta - (\beta_1 - \alpha_1)$

or $\quad \beta_1 = \alpha_1 + \theta(1 - 2m)$.

Thus, the observed angle of depression always exceeds the angle of elevation by the amount $\theta(1 - 2m)$

Case (b) Distance d large and H small :

In this case, both α_1 and β_1 are the angles of depression.

Changing the sign of α_1 in Eq. 5.12, we get

$$r = \frac{\theta}{2} - \left(\frac{\beta_1 + \alpha_1}{2}\right) \qquad \ldots[5.12\ (a)]$$

Which reduces to : $(\beta_1 + \alpha_1) = \theta(1 - 2m)$.

Correction for curvature

The effect of curvature is to make the objects appear *lower* than they really are. In spirit levelling, the effect is to increase the staff reading and the correction is, therefore, subtracted from the staff reading. The effect of refraction is in the opposite direction to that of curvature. In trigonometrical levelling employed for determining the elevation of widely distributed points the correction for curvature is applied directly to the observed angles.

Thus, in Fig. 5.15, the angle α_1 was measured with reference to the horizontal line PO' while it should be measured with the chord PP_1, where P_1 is the vertical projection of Q on a level line passing through P. Hence the correction $= \angle O'PP_1 = \frac{\theta}{2}$ and is *additive*.

Similarly, the angle β_1 was measured with reference to the horizontal line QO' while it should be measured with the chord QQ_1. Hence the correction $= \angle Q_2QQ_1 = \theta/2$ and is subtractive.

Thus, the correction for curvature is $+\theta/2$ for angles of elevation and $-\theta/2$ for angles of depression.

Combined correction

Now, $\quad \angle O'PP_1 = \frac{\theta}{2} = \frac{d}{2R}$ radians $= \frac{d}{2R \sin 1''}$ seconds

Angular correction of refraction $= m\theta = \dfrac{md}{R \sin 1''}$ seconds

Hence, combined angular correction $= \left[\dfrac{d}{2R \sin 1''} - \dfrac{md}{R \sin 1''}\right] = \dfrac{(1-2m)d}{2R \sin 1''}$ seconds ...(5.13)

The combined correction is positive for angles of elevation and negative for angles of depression.

5.8. AXIS SIGNAL CORRECTION (EYE AND OBJECT CORRECTION)

In order to observe the points from the theodolite station, signals of appropriate heights are erected at the points to be observed. The signals may or may not be of the same height as that of the instrument. If the height of the signal is not the same as that of the height of the instrument axis above the station, a correction known as the *axis signal correction* or *eye and object correction* is to be applied.

Let

h_1 = height of instrument at P, for observation to Q

h_2 = height of instrument at Q, for observation to P

s_1 = height of the signal at P, instrument being at Q

s_2 = height of the signal at Q, instrument being at P

d = horizontal distance between P and Q

α = observed angle of elevation uncorrected for the axis signal

β = observed angle of depression, uncorrected for axis signal

α_1 = angle of elevation corrected for axis signal

β_1 = angle of depression corrected for axis signal.

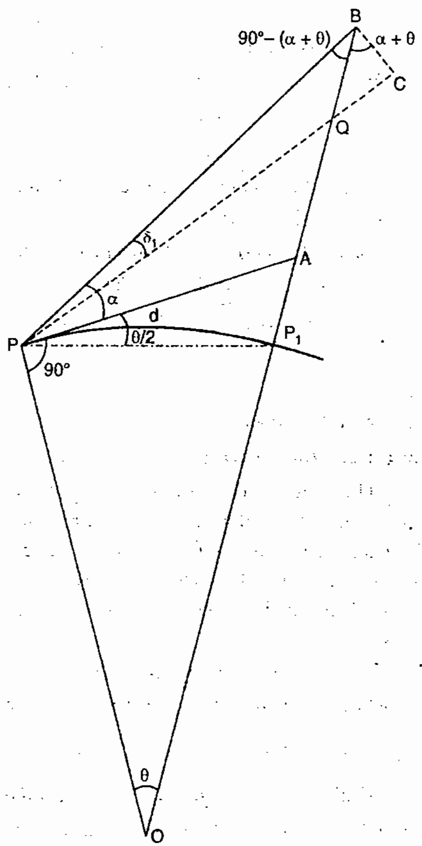


FIG. 5.16. AXIS SIGNAL CORRECTION.

In Fig. 5.16,

PA = horizontal line at P

Q = point observed

BQ = difference in the height of signal at Q and the height of instrument at P $= (s_2 - h_1)$

$\therefore$ $\angle BPA = \alpha$ = angle observed from P to Q

$\angle BPQ = \delta_1$ = axis signal correction (angular) at P.

At B, draw BC perpendicular to BP, to meet PQ produced in C.

For triangle PBO,

$$\angle BPO = \angle BPA + \angle APO = \alpha + 90° = 90° + \alpha$$

$$\angle POB = \theta$$

$$\therefore \quad \angle PBO = 180° - (90° + \alpha) - \theta = 90° - (\alpha + \theta)$$

$$\therefore \quad \angle QBC = 90° - [90° - (\alpha + \theta)] = (\alpha + \theta)$$

The angle δ_1 is usually very small and hence $\angle BCQ$ can be approximately taken equal to 90°.

$$\therefore \quad BC = BQ \cos(\alpha + \theta) \text{ very nearly} = (s_2 - h_1)\cos(\alpha + \theta) \quad ...(1)$$

For triangle PP_1B,

$$\angle BPP_1 = \alpha + \theta/2$$

$$\angle PBP_1 = 90° - (\alpha + \theta)$$

$$\angle PP_1B = 180° - [90° - (\alpha + \theta)] - (\alpha + \theta/2) = (90° + \theta/2)$$

Now $$\frac{PB}{\sin PP_1B} = \frac{PP_1}{\sin PBP_1}$$

or $$PB = PP_1 \cdot \frac{\sin PP_1B}{\sin PBP_1} = \frac{d \sin(90° + \theta/2)}{\sin[90° - (\alpha + \theta)]} = d\,\frac{\cos \theta/2}{\cos(\alpha + \theta)} \quad ...(2)$$

From triangle PBC,

$$\tan \delta_1 = \frac{BC}{PB}$$

Substituting the value of BC from (1), and of PB from (2),

we get $$\tan \delta_1 = \frac{(s_2 - h_1)\cos(\alpha + \theta)}{d\,\dfrac{\cos \theta/2}{\cos(\alpha + \theta)}}$$

or $$\tan \delta_1 = \frac{(s_2 - h_1)\cos^2(\alpha + \theta)}{d \cos \theta/2} \quad ...\text{(exact)} \quad ...(5.14)$$

Usually, θ is small in comparison to α and may be ignored

$$\tan \delta_1 = \frac{(s_2 - h_1)\cos^2 \alpha}{d} \quad ...[5.14\ (a)]$$

The correction is evidently *subtractive* for this case.

Similarly, if observations are taken from Q towards P, it can be proved that

$$\tan \delta_2 = \frac{(s_1 - h_2)\cos^2 \beta}{d} \quad \text{(additive)} \quad ...(5.15)$$

The correction for axis signal is negative for angles of elevation and positive for angles of depression.

If, however, the vertical angle α (or β) is very small, we can take, with sufficient accuracy,

$$\tan \delta_1 = \delta_1 = \frac{s_2 - h_1}{d \sin 1''} \text{ seconds} \quad ...[5.16\ (a)]$$

and $$\tan \delta_2 = \delta_2 = \frac{s_1 - h_2}{d \sin 1''} \text{ seconds} \qquad ...[5.16\ (b)]$$

Equation 5.16 can also be derived by considering $PB = PQ = PP_1 = d$, and taking BQ as the arc with radius equal to d. Then

$$\delta_1 = \frac{BQ}{d} \text{ radians}$$
$$= \frac{s_2 - h_1}{d} \text{ radians}$$
$$= \frac{s_2 - h_1}{d \sin 1''} \text{ seconds.}$$

This expression gives sufficiently accurate results when the vertical angle is small, the distance is large and the difference in height of the signal and that of the instrument is small. After having calculated δ_1 and δ_2, the angles corrected for axis signal are given by

$$\alpha_1 \text{ (elevation)} = \alpha - \delta_1$$
$$\beta_1 \text{ (depression)} = \beta + \delta_2.$$

5.9. DETERMINATION OF DIFFERENCE IN ELEVATION

The difference in elevation between the two points P and Q can be found out by two methods :

(*a*) *By single observation*

(*b*) *By reciprocal observation.*

(*a*) Difference in elevation by single observation

In this case, the observations are made from only one station (say P). The following corrections will have to be applied :

(1) Correction for curvature,

(2) Correction for refraction, and

(3) Correction for axis signal.

Since the sign of these corrections will depend upon the sign of the angle observed, we shall consider the following cases :

(*i*) When the observed angle is the angle of elevation

(*ii*) When the observed angle is the angle of depression.

(*i*) For angle of elevation

In Fig. 5.17, let

α = observed angle of elevation to Q

α_1 = observed angle corrected for axis signal = $(\alpha - \delta_1)$

$$= \left(\alpha - \frac{s_2 - h_1}{d \sin 1''}\right) \text{ second or } = \left(\alpha - \frac{\tan^{-1}(s_2 - h_1)\cos^2\alpha}{d}\right)$$

d = horizontal distance = arc $PP_1 \cong$ chord $PP_1 \approx PA$

$\angle P'PQ = r = m\theta$;

$QP_1 = H$ = difference in elevation between Q and P.

In triangle PQP_1,

$$\angle QPP_1 = \angle P'PA + \angle P'PQ + \angle APP_1$$
$$= \alpha_1 - m\theta + \frac{\theta}{2}$$
$$\angle PP_1Q = \left(90° - \frac{\theta}{2}\right) + \theta = 90° + \frac{\theta}{2}$$
$$\angle PQP_1 = 180° - \left(\alpha_1 - m\theta + \frac{\theta}{2}\right) - \left(90° + \frac{\theta}{2}\right)$$
$$= 90° - (\alpha_1 - m\theta + \theta)$$
$$\frac{QP_1}{\sin QPP_1} = \frac{PP_1}{\sin PQP_1}$$

or
$$QP_1 = H = PP_1 \frac{\sin QPP_1}{\sin PQP_1}$$
$$= \frac{d \sin\left(\alpha_1 - m\theta + \frac{\theta}{2}\right)}{\sin[90° - (\alpha_1 - m\theta + \theta)]}$$
$$= \frac{d \sin\left(\alpha_1 - m\theta + \frac{\theta}{2}\right)}{\cos(\alpha_1 - m\theta + \theta)} \qquad \text{....(exact)} \qquad ...(5.17)$$
$$= \frac{d \sin\left(\alpha_1 - \frac{md}{R \sin 1''} + \frac{d}{2R \sin 1''}\right)}{\cos\left(\alpha_1 - \frac{md}{R \sin 1''} + \frac{d}{R \sin 1''}\right)}$$
$$= \frac{d \sin\left\{\alpha_1 + (1 - 2m)\frac{d}{2R \sin 1''}\right\}}{\cos\left\{\alpha_1 + (1 - m)\frac{d}{R \sin 1''}\right\}} \qquad ...[5.17\ (a)]$$

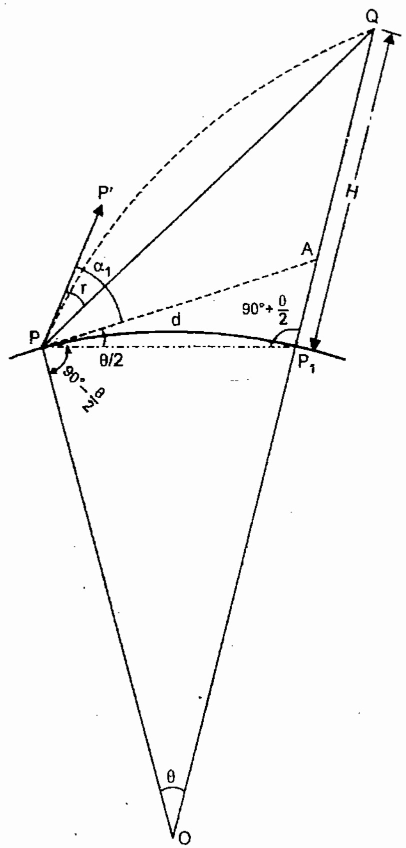


FIG. 5.17 DIFFERENCE IN ELEVATION BY SINGLE OBSERVATION : ANGLE OF ELEVATION.

where the quantities $(1 - 2m)\frac{d}{2R \sin 1''}$ and $(1 - m)\frac{d}{R \sin 1''}$ are in seconds.

Approximate Expressions

Equation 5.17 is the exact expression for the difference in elevation H. An approximate expression, however, can be had by considering $\angle PP_1Q$ to be equal to 90°, specially when θ is very small. Then

$$QP_1 = H = PP_1 \tan QPP_1 = d \tan\left(\alpha_1 - m\theta + \frac{\theta}{2}\right) \qquad ...[5.17\ (b)]$$
$$= d \tan\left\{\alpha_1 + (1 - 2m)\frac{d}{2R \sin 1''}\right\} \qquad ...[5.17\ (c)]$$

Equation 5.17 (*b*) or 5.17 (*c*) should be used only when θ is small. Otherwise equation 5.17 or 5.17 (*a*) should be used.

(*ii*) **For angle of depression**

In Fig. 5.18, let

β = observed angle of depression to P

β_1 = observed angle corrected for axis signal = $\beta + \delta_2$

or $$= \beta + \frac{s_1 - h_2}{d \sin 1''} = \beta + \frac{\tan^{-1}(s_1 - h_2)\cos^2\beta}{d}$$

d = horizontal distance = arc QQ_1 = chord $QQ_1 \approx QB$

$\angle Q'QP = r = m\theta$

$Q_1P = H$ = difference in elevation between P and Q.

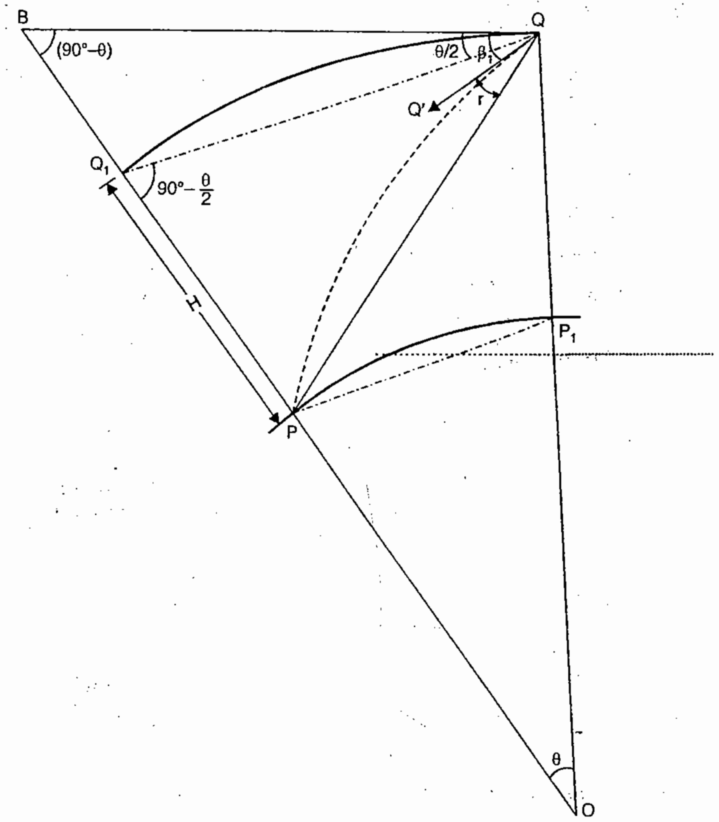


FIG. 5.18. DIFFERENCE IN ELEVATION BY SINGLE OBSERVATION : ANGLE OF DEPRESSION.

In triangle QPQ_1, $\angle PQQ_1 = \angle Q'QB + \angle Q'QP - \angle Q_1QB = \beta_1 + m\theta - \frac{\theta}{2}$ ...(1)

$$\angle PBQ = 90° - \theta$$

$\therefore$ $$\angle QQ_1P = (90° - \theta) - \frac{\theta}{2} = 90° - \frac{\theta}{2} \quad ...(2)$$

and $$\angle Q_1PQ = 180° - \left(90° - \frac{\theta}{2}\right) - \left(\beta_1 + m\theta - \frac{\theta}{2}\right) = 90° - (\beta_1 + m\theta - \theta) \quad ...(3)$$

Now, $$\frac{PQ_1}{\sin PQQ_1} = \frac{QQ_1}{\sin Q_1PQ}$$

or $$PQ_1 = H = QQ_1 \frac{\sin PQQ_1}{\sin Q_1PQ}$$

or $$PQ_1 = \frac{d \sin\left(\beta_1 + m\theta - \frac{\theta}{2}\right)}{\sin[90° - (\beta_1 + m\theta - \theta)]} = \frac{d \sin\left(\beta_1 + m\theta - \frac{\theta}{2}\right)}{\cos(\beta_1 + m\theta - \theta)} \text{ ...(exact)} \quad ...(5.18)$$

$$= \frac{d \sin\left\{\beta_1 - (1 - 2m)\frac{d}{2R \sin 1''}\right\}}{\cos\left\{\beta_1 - (1 - m) - \frac{d}{R \sin 1''}\right\}} \quad ...[5.18\ (a)]$$

where the quantities $(1 - 2m)\frac{d}{2R \sin 1''}$ and $(1 - m)\frac{d}{R \sin 1''}$ are in seconds.

Approximate Expressions

Equation 5.18 is the exact expression for the difference in elevation H. An approximate expression, however, can be had by considering $\angle PQ_1Q$ to be equal to 90° specially when θ is very small. Then

$$Q_1P = H = QQ_1 \tan PQQ_1 = d \tan\left(\beta_1 + m\theta - \frac{\theta}{2}\right) \quad ...[5.18\ (b)]$$

$$= d \tan\left\{\beta_1 - (1 - 2m)\frac{d}{2R \sin 1''}\right\} \quad ...[5.18\ (c)]$$

Application of corrections in linear measure

The difference in elevation between P and Q can also be obtained by applying the three corrections (*i.e.*, curvature, refraction and axis-signal) in linear measure.

Thus, axis-signal correction in linear measure = $s_2 - h_1$

$$\text{Curvature correction} = \frac{d^2}{2R}$$

$$\text{Refraction correction} = rd = m\theta \cdot d = m\frac{d}{R} \cdot d = \frac{md^2}{R}$$

$$\text{Combined correction for curvature and refraction} = \frac{d^2}{2R} - \frac{md^2}{R} = (1 - 2m)\frac{d^2}{2R}$$

If α is the observed angle, uncorrected for curvature, refraction and axis-signal, we have

$H = d \tan \alpha$ – (Ht. of signal – Ht. of instrument) + curvature correction – refraction correction

$$= d \tan \alpha - (s_2 - h_1) + \frac{d^2}{2R} - \frac{md^2}{R} = d \tan \alpha - (s_2 - h_1) + (1 - 2m)\frac{d^2}{2R} \quad ...[5.19\ (a)]$$

Similarly, for angle of depression β , we have

$H = d \tan \beta +$ (Ht of signal – Ht. of instrument) – curvature + refraction

$$= d \tan \beta + (s_1 - h_2) - \frac{d^2}{2R} + \frac{md^2}{R} = d \tan \beta + (s_1 - h_2) - (1 - 2m)\frac{d^2}{2R} \quad ...[5.19\ (b)]$$

(*b*) **Difference in elevation by reciprocal observations**

Reciprocal observations are generally made to eliminate the effect of refraction. In this method, observations are made simultaneously from both the stations (*i.e.*, *P* and *Q*) so that refraction effect is the same. However, if it is not possible to take the observations simultaneously, observations at one station may be taken on the first day and at the second on the next day, during the time during which refraction is almost constant (*i.e.*, between 10 a.m. to 4 p.m.). This method is more accurate than the single observation method, specially when the exact value of *m* is not known.

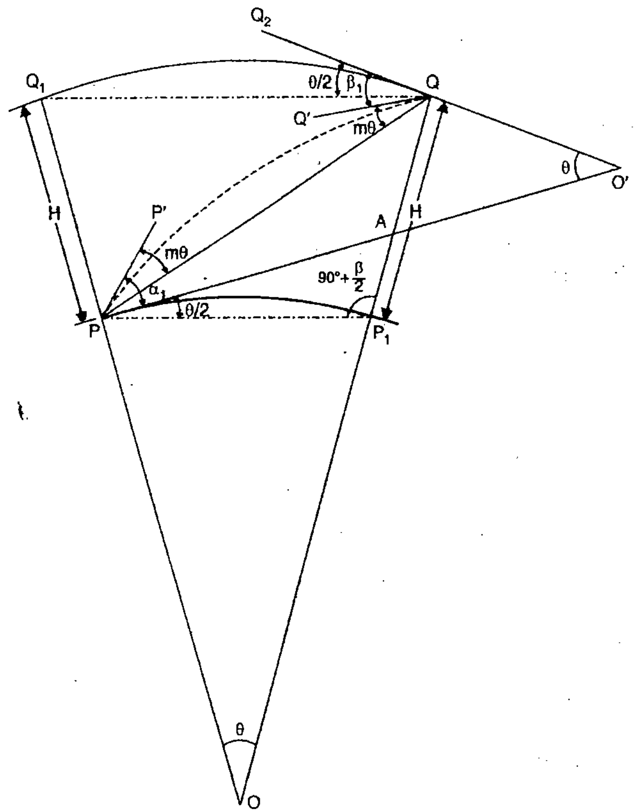


FIG. 5.19. RECIPROCAL OBSERVATIONS

In Fig. 5.19, let

$\angle P'PO' = \alpha_1$ = observed angle of elevation at *P* corrected for axis signal $= \alpha - \dfrac{s_2 - h_1}{d \sin 1''}$

$\angle P'PQ = r = m\,\theta$ = refraction error at *P*

$\angle O'PP_1 = \dfrac{\theta}{2}$ = curvature effect

$QP_1 = H$ = difference in elevation between *P* and *Q*

$\angle Q_2QQ' = \beta_1$ = observed angle of depression at *Q* corrected for axis signal $= \beta + \dfrac{s_1 - h_2}{d \sin 1''}$

$\angle Q'QP = r = m\theta$ = refraction error at *Q*

$\angle Q_1QQ_2 = \dfrac{\theta}{2}$ = curvature effect

Arc PP_1 = chord PP_1 = arc QQ_1 = chord $QQ_1 = d$ = horizontal distance

$\angle QPP_1$ = angle of elevation corrected for axis signal, curvature and refraction

$$= \alpha_1 + \frac{\theta}{2} - m\theta$$

Similarly

$\angle PQQ_1$ = angle of depression corrected for axis signal, curvature and refraction.

$$= \beta_1 - \frac{\theta}{2} + m\theta$$

Since PP_1 and Q_1Q are parallel to each other.

$$\angle QPP_1 = \angle PQQ_1$$

$$\therefore \quad \alpha_1 + \frac{\theta}{2} - m\theta = \beta_1 - \frac{\theta}{2} + m\theta = \frac{1}{2}\left\{\left(\alpha_1 + \frac{\theta}{2} - m\theta\right) + \left(\beta_1 - \frac{\theta}{2} + m\theta\right)\right\} = \frac{\alpha_1 + \beta_1}{2} \quad ...(1)$$

Thus, each corrected angle $= \dfrac{\alpha_1 + \beta_1}{2}$

In triangle QPP_1,

$$\angle QPP_1 = \alpha_1 + \frac{\theta}{2} - m\theta \quad ; \qquad \angle PQP_1 = 90° - (\beta_1 + m\theta)$$

$$\frac{QP_1}{\sin QPP_1} = \frac{PP_1}{\sin PQP_1}$$

$$QP_1 = H = PP_1 \frac{\sin QPP_1}{\sin PQP_1} = d\,\frac{\sin\left(\alpha_1 + \frac{\theta}{2} - m\theta\right)}{\sin\,[90° - (\beta_1 + m\theta)]} = d\,\frac{\sin\left(\alpha_1 + \frac{\theta}{2} - m\theta\right)}{\cos\,(\beta_1 + m\theta)}$$

But $\left(\alpha_1 + \dfrac{\theta}{2} - m\theta\right) = \dfrac{\alpha_1 + \beta_1}{2}$ from (1)

and $\beta_1 - \dfrac{\theta}{2} + m\theta = \dfrac{\alpha_1 + \beta_1}{2}$ or $\beta_1 + m\theta = \dfrac{\alpha_1 + \beta_1}{2} + \dfrac{\theta}{2}$.

Substituting these values, we get

$$H = \frac{d \sin\left(\frac{\alpha_1 + \beta_1}{2}\right)}{\cos\left\{\left(\frac{\alpha_1 + \beta_1}{2}\right) + \frac{\theta}{2}\right\}} \quad ...(5.20)$$

If however, $\frac{\theta}{2}$ is small in comparison to $\frac{\alpha_1 + \beta_1}{2}$, it can be neglected. Then

$$H = d \frac{\sin\left(\frac{\alpha_1 + \beta_1}{2}\right)}{\cos\left(\frac{\alpha_1 + \beta_1}{2}\right)} = d \tan \frac{\alpha_1 + \beta_1}{2} \quad ...[5.20\ (a)]$$

If, however, both α_1 and β_1 are the angles of depression, the expression for H can be obtained by changing the sign of α_1 in equation 5.20 :

$$H = \frac{d \sin\left(\frac{\beta_1 - \alpha_1}{2}\right)}{\cos\left(\frac{\beta_1 - \alpha_1}{2} + \frac{\theta}{2}\right)} \quad ...(5.21)$$

If the value of H obtained from the above expression is positive, Q is higher than P. If H is negative, Q will be lower than P.

Thus, in general, the expression for H is

$$H = \frac{d \sin\left(\frac{\beta_1 \pm \alpha_1}{2}\right)}{\cos\left\{\frac{\beta_1 \pm \alpha_1}{2} + \frac{\theta}{2}\right\}}$$

Use plus sign when α_1 is the angle of elevation and minus sign when it is the angle of depression.

Example 5.8. *Correct the observed altitude for the height of signal, refraction and curvature from the following data :*

Observed altitude	$= +2°\ 48'\ 39''$
Height of instrument	$= 1.12$ m
Height of signal	$= 4.87$ m
Horizontal distance	$= 5112$ m
Co-efficient of refraction	$= 0.07$ m
$R \sin 1''$	$= 30.88$ m.

Solution.

Given : $\alpha = +2°\ 48'\ 39''$; $h = 1.12$ m ; $s = 4.87$ m

$d = 5112$ m ; $m = 0.07$

The axis signal correction $= \delta = \frac{s - h}{d \sin 1''}$ seconds

$$= \frac{4.87 - 1.12}{5112 \sin 1''} = \frac{3.75 \times 206265}{5112} \quad \left(\text{since } \sin 1'' = \frac{1}{206265}\right)$$

$= 151''.31 = 2'\ 31''.31$ (subtractive)

The central angle $= \theta = \frac{d}{R \sin 1''} = \frac{5122}{30.88} = 165''.54$

Curvature correction $= \frac{\theta}{2} = 82''.77$ (additive)

Refraction correction $= r = m\theta = 0.07 \times 165.54$

$= 11''.59$ (subtractive)

Total correction $= \frac{\theta}{2} - \delta - r = 82''.77 - 151''.31 - 11''.59$

$= -80''.13 = 1'20''.13$ (subtractive)

Correct altitude $= 2°\ 48'\ 39'' - 1'\ 20''.13 = 2°\ 47'\ 18''.87$

Example 5.9. *Find the R.L. of Q from the following observations:*

Horizontal distance between P and Q	$= 9290$ m
Angle of elevation from P to Q	$= 2°\ 06'\ 18''$
Height of signal at Q	$= 3.96$ m
Height of instrument at P	$= 1.25$ m
Co-efficient of refraction	$= 0.07$
$R \sin 1''$	$= 30.88$ m
R.L. of P	$= 396.58$ m.

Solution.

Given : $d = 9290$ m ; $\alpha = +2°\ 06'\ 18'$; $s = 3.96$ m

$h = 1.25$ m ; $R \sin 1'' = 30.88$ m ; $m = 0.07$

Axis signal correction $= \delta = \frac{s - h}{d \sin 1''} = \frac{(3.96 - 1.25)}{9290 \sin 1''}$ seconds

$= \frac{2.71 \times 206265}{9290} = 60''.17$ (subtractive)

$\alpha_1 = \alpha - \delta = 2°\ 06'\ 18'' - 60''.17 = 2°\ 05'\ 17''.83$

$\theta = \frac{d}{R \sin 1''}$ seconds $= \frac{9290}{30.88} = 300''.84 = 5'0''.84$

$\frac{\theta}{2} = 150''.42 = 2'30''.42$

$r = m\theta = 0.07 \times 300.84 = 21''.06$

Now, from equation 5.17, we have

$$H = \frac{d \sin\left(\alpha_1 - m\theta + \frac{\theta}{2}\right)}{\cos(\alpha_1 - m\theta + \theta)} = \frac{9290 \sin(2°\ 5'\ 17''.83 - 21''.06 + 2'\ 30''.42)}{\cos(2°\ 5'\ 17''.83 - 21''.06 + 5'0''.84)}$$

$$= \frac{9290 \sin 2°\ 7'27''.19}{\cos 2°\ 9'\ 57''.61} = 344.59 \text{ m}$$

R.L. of Q = R.L. of $P + H = 396.58 + 344.59 =$ **741.17 m.**

Example 5.10. *Find the difference of levels of the points P and Q and the R.L. of P from the following data :*

Horizontal distance between P and Q = 7118 *m*
Angle of depression to P at Q = 1° 32′ 12″
Height of signal at P = 3.87 *m*
Height of instrument at Q = 1.27 *m*
Co-efficient of refraction = 0.07

R sin 1″ = 30.88 m ; m = 0.07

R.L. of Q = 417.860 m

Solution.

Given : $d = 7118$ m ; $\beta = 1° 32' 12''$; $s = 3.87$ m

$h = 1.27$ m ; $R \sin 1'' = 30.88$ m

$$\text{Axis signal correction} = \delta = \frac{s-h}{d \sin 1''} = \frac{(3.87 - 1.27)}{7118 \sin 1''} \text{ seconds}$$

$$= \frac{2.60 \times 206265}{7118} = 1' \, 15''.34 \quad \textbf{(additive)}$$

$\therefore$ $\beta_1 = \beta + \delta = 1° 32' 12'' + 1' 15''.34 = 1° 33' 27''.34$

$$\theta = \frac{d}{R \sin 1''} = \frac{7118}{30.88} = 230''.50 = 3' \, 50''.5$$

$$\frac{\theta}{2} = 115''.25 = 1' \, 55''.25$$

$$r = m\theta = 16''.14$$

Now from equation 5.18,

$$H = \frac{d \sin\left(\beta_1 + m\theta - \frac{\theta}{2}\right)}{\cos(\beta_1 + m\theta - \theta)} = \frac{7118 \sin (1° 33' 27''.34 + 16''.14 - 1' 55''.25)}{\cos (1° 33' 27''.34 + 16''.14 - 3' 50''.5)}$$

$$= \frac{7118 \sin 1° 31' 48''.23}{\cos 1° 29' 52''.98} = 190.13$$

$\therefore$ R.L. of $P = 417.86 - 190.13 = \mathbf{227.73}$ **m.**

Example 5.11. *The following reciprocal observations were made from two points P and Q* :

Horizontal distance between P and Q = 6996 *m*
Angle of elevation of Q at P = 1° 56′ 10″
Angle of depression of P at Q = 1° 56′ 52″
Height of signal at P = 4.07 *m*
Height of signal at Q = 3.87 *m*
Height of instrument at P = 1.27 *m*
Height of instrument at Q = 1.48 *m*

Find the difference in level between P and Q and the refraction correction. Take R sin 1″ = 30.88 m.

Solution.

Given : $d = 6996$ m ; $\alpha = +1° 56' 10''$; $\beta = -1° 56' 52''$

$h_1 = 1.27$ m ; $h_2 = 1.48$ m ; $s_1 = 4.07$ m ; $s_2 = 3.87$ m

$$\text{Axis signal correction at } P = \delta_1 = \frac{s_2 - h_1}{d \sin 1''} = \frac{(3.87 - 1.27)}{6996 \sin 1''} = \frac{2.60 \times 206265}{6996}$$

$$= 76''.66 = 1' \, 16''.66 \quad \text{(subtractive)}$$

$$\text{Axis signal correction at } Q = \delta_2 = \frac{s_1 - h_2}{d \sin 1''} = \frac{4.07 - 1.48}{6996 \sin 1''} = \frac{2.59 \times 206265}{6996}$$

$$= 76''.36 = 1' \, 16''.36 \quad \text{(additive)}$$

$$\text{Central angle} \quad \theta = \frac{d}{R \sin 1''} = \frac{6996}{30.88} = 3' \, 46''.55 = 226''.55$$

$$\frac{\theta}{2} = 1' \, 53''.28$$

Now $\alpha_1 = \alpha - \delta_1 = 1° 56' 10'' - 1' 16''.66 = 1° 54' 53''.44$

$\beta_1 = \beta + \delta_2 = 1° 56' 52'' + 1' 16''.36 = 1° 58' 08''.36$

$$\frac{\beta_1 + \alpha_1}{2} = \tfrac{1}{2}(1° 58' 08''.36 + 1° 54' 53''.34) = 1° 56' 30''.85$$

$$\therefore \quad \frac{\beta_1 + \alpha_1}{2} + \frac{\theta}{2} = 1° 56' 30''.85 + 1' 53''.28 = 1° 58' 24''.13$$

$$\frac{\beta_1 - \alpha_1}{2} = \tfrac{1}{2}(1° 58' 08''.36 - 1° 54' 53''.34) = 1' 37''.51$$

From equation 5.20,

$$H = \frac{d \sin\left(\frac{\alpha_1 + \beta_1}{2}\right)}{\cos\left(\frac{\alpha_1 + \beta_1}{2} + \frac{\theta}{2}\right)} = \frac{6996 \sin 1° 56' 30''.85}{\cos 1° 58' 24''.13} = 237.21 \text{ m}$$

Also from equation 5.12,

$$r = \frac{\theta}{2} - \frac{\beta_1 - \alpha_1}{2} = 1' \, 53''.28 - 1' \, 37''.51 = 15''.77$$

$$\text{Co-efficient of refraction} = m = \frac{r}{\theta} = \frac{15''.77}{226''.55} = \mathbf{0.0696.}$$

Example 5.12. *The following reciprocal observations were made from two points P and Q* :

Horizontal distance between P and Q = 33128 *m*
Angle of depression of Q at P = 6′ 20″
Angle of depression of P at Q = 8′ 10″
Height of signal at P = 4.87 *m*

Height of signal at Q $= 4.07\ m$

Height of the instrument at P $= 1.27\ m$

Height of the instrument at Q $= 1.34\ m$

Calculate (a) the R.L. of Q, if that of P is 1248.65 m and (b) the average co-efficient of refraction at the time of observations.

Take R sin 1″ = 30.88 m.

Solution.

Given : $d = 33128$ m ; $\alpha = -6' 20''$; $\beta = -8' 10''$

$s_1 = 4.87$ m ; $s_2 = 4.07$ m ; $h_1 = 1.27$ m ; $h_2 = 1.34$ m

$$\text{Axis signal correction at } P = \delta_1 = \frac{s_2 - h_1}{d \sin 1''} = \frac{4.07 - 1.27}{33128 \sin 1''} = \frac{2.80 \times 206265}{33128}$$

$$= 17''.43 \text{ (additive to } \alpha \text{)}$$

$$\text{Axis signal correction at } Q = \delta_2 = \frac{s_1 - h_2}{d \sin 1''} = \frac{4.87 - 1.34}{33128 \sin 1''} = \frac{3.53 \times 206265}{33128}$$

$$= 21''.98 \text{ (additive to } \beta \text{)}$$

$\therefore$ $\alpha_1 = \alpha + \delta_1 = 6' 20'' + 17''.43 = 6' 37''.43$ (depression)

$\beta_1 = \beta + \delta_2 = 8' 10'' + 21''.98 = 8' 31''.98$ (depression)

$$\frac{\beta_1 - \alpha_1}{2} = \frac{1}{2}(8' 31''.98 - 6' 37''.43) = 57''.27$$

$$\frac{\beta_1 + \alpha_1}{2} = \frac{1}{2}(8' 31''.98 + 6' 37''.43) = 7' 34''.71$$

$$\theta = \frac{d}{R \sin 1''} = \frac{33128}{30.88} = 1072''.8 = 17' 52''.8$$

$$\frac{\theta}{2} = 8' 56''.4$$

$$r = \frac{\theta}{2} - \left(\frac{\beta_1 + \alpha_1}{2}\right) = 8' 56''.4 - 7' 34''.71 = 1' 21''.69 = 81''.69$$

$$m = \frac{r}{\theta} = \frac{81.69}{1072.8} = \mathbf{0.0762}$$

The difference in elevation (H) is given by

$$H = \frac{d \sin\left(\frac{\beta_1 - \alpha_1}{2}\right)}{\cos\left(\frac{\beta_1 - \alpha_1}{2} + \frac{\theta}{2}\right)} = \frac{33128 \sin 57''.27}{\cos (57''.27 + 8' 56''.4)}$$

$$= \frac{33128 \sin 57''.27}{\cos 9' 53''.67} = 9.20 \text{ m}$$

$\therefore$ R.L. of θ = R.L. of $P + H = 1248.65 + 9.20 = \mathbf{1257.85}$ **m.**

Example 5.13. *In the trigonometrical measurement of the difference in level of two stations P and Q, 10480 m apart, the following data were obtained :*

Instrument at P, angle of elevation of Q $= 0' 15''$

Instrument at Q, angle of depression of P $= 3' 33''$

Height of instrument at P $= 1.42\ m$

Height of instrument at Q $= 1.45\ m$

Height of signal at P $= 3.95\ m$

Height of signal at Q $= 3.92\ m$

Find the difference in level between P and Q, and the curvature and refraction correction. Take R sin 1″ = 30.38 metres.

Solution.

Given : $d = 10480$ m ; $\alpha = +0' 15''$; $\beta = -3' 33''$

$h_1 = 1.42$ m ; $h_2 = 1.45$ m ; $s_1 = 3.95$ m ; $s_2 = 3.92$ m

$$\text{Axis signal correction at } P = \delta_1 = \frac{s_2 - h_1}{d \sin 1''} = \frac{3.92 - 1.42}{10480 \times \sin 1''}$$

$$= \frac{2.50 \times 206265}{10480} = 49''.30$$

This is subtractive since α is the angle of elevation.

$$\text{Axis signal correction at } Q = \delta_2 = \frac{s_1 - h_2}{d \sin 1''} = \frac{3.95 - 1.45}{10480 \sin 1''}$$

$$= \frac{2.5 \times 206265}{10480} = 49''.30$$

This is additive since β is the angle of depression.

$\therefore$ $\alpha_1 = \alpha - \delta_1 = 0' 15'' - 49''.30 = -34''.30$ *i.e.*, $34''.30$ (dep.)

$\beta_1 = -(\beta + \delta_2) = -(3' 33'' + 49''.30) = -4' 22''.30$

$$\theta = \frac{d}{R \sin 1''} = \frac{10480}{30.80} = 339''.38 = 5' 39''.38$$

$$\frac{\theta}{2} = 2' 49''.69$$

$\therefore$ Curvature correction $= \frac{\theta}{2} = \mathbf{2' 49''.69}$.

Also, from Eq. 5.12 (a),

$$r = \frac{\theta}{2} - \left(\frac{\beta_1 + \alpha_1}{2}\right)$$

$$= 2' 49''.69 - \left(\frac{4' 22''.30 + 34''.30}{2}\right) = 21''.39$$

Refraction correction $= r = \mathbf{21''.39}$.

From Eq. 5.21, we have

$$H = \frac{d \sin\left(\frac{\beta_1 - \alpha_1}{2}\right)}{\cos\left(\frac{\beta_1 - \alpha_1}{2} + \frac{\theta}{2}\right)}$$

Here $\beta_1 = 4' \, 22''.30 \;;\; \alpha_1 = 0'34''.30$

$\therefore$ $\beta_1 - \alpha_1 = 3' \, 48'' \;;\; \frac{\beta_1 - \alpha_1}{2} = 1'54''$

$$\frac{\theta}{2} = 2' \, 49''.69$$

$$\frac{\beta_1 - \alpha_1}{2} + \frac{\theta}{2} = 4' \, 43''.69$$

$$H = \frac{10480 \sin 1' \, 54''}{\cos 4' \, 43''.69} = \mathbf{5.792 \ m.}$$

PROBLEMS

1. A theodolite was set up at a distance of 200 m from a tower. The angle of elevation to the top of the parapet was 8° 18′ while the angle of depression to the foot of the wall was 2° 24′. The staff reading on the B.M. having R.L. 248.362 with the telescope horizontal was 1.286 m. Find the height of the tower and the R.L. of the top of the parapet.

2. To determine the elevation of the top of a flagstaff, the following observations were made:

Inst. station	Reading on B.M.	Angle of elevation	Remarks
A	1.266	10° 48′	R.L. of B.M.= 248.362
B	1.086	7° 12′	

Stations *A* and *B* and the top of the aerial pole are in the same vertical plane.

Find the elevation of the top of the flagstaff, if the distance between *A* and *B* is 50 m.

3. Find the elevation of the top of the chimney from the following data :

Inst. station	Reading on B.M.	Angle of elevation	Remarks
A	0.862	18° 36′	R.L. of B.M.= 421.380 m
B	1.222	10° 12′	Distance AB =50 m

Stations *A* and *B* and the top of chimney are in the same vertical plane.

4. The top (*Q*) of a chimney was sighted from two stations *P* and *R* at very different levels, the stations *P* and *R* being in line with the top of the chimney. The angle of elevation from *P* to the top of chimney was 36° 12′ and that from *R* to the top of the chimney was 16° 48′. The angle of elevation from *R* to a vane 1 m above the foot of the staff held at *P* was 8° 24′. The heights of instrument at *P* and *R* were 1.85 m and 1.65 m respectively. The horizontal distance between *P* and *R* was 100 m and the R.L. of *R* was 248.260 m. Find the R.L. of the top of the chimney and the horizontal distance from *P* to the chimney.

5. Obtain an expression for the difference of level between two points *A* and *B*, a considerable distance apart, *B* being higher, by vertical angle readings from the point *A*. Take into account the height of the instrument at *A* and the height of the target at *B*. What is the assumption made in obtaining your equation for the difference of level ?

6. Obtain an expression for the difference in level between two points by reciprocal vertical angle readings from two stations. Heights of instruments and targets should not be ignored.

7. Two stations, *A* and *B* are at a horizontal distance from one another of 11439 metres. At *B*, a depression angle of 3° 4′ 2″ is recorded to *A*. Assuming that 30.88 metres subtend 1″ at the earth's centre and that the correction for refraction is $\frac{1}{7}$ of that for curvature, calculate the difference in heights of *A* and *B*, given that the height of theodolite was 1.433 m and that of the signal 3.871 m.

Why should this measured depression angle from *B* to *A* be greater than the elevation angle from *A* and *B* ? In what circumstances might it be possible for both angles to be measured as depressions?

8. The mean vertical angle from *A*, reading on to a target at *B*, is 24′ 30″. The height of instrument at *A* is 5.00 ft and the height of the target at *B* is 4.00 ft. If the distance between the two points is 20 miles, assume the usual value for the co-efficient of refraction and obtain the difference of ground level between the two points. The radius of the earth may be taken as 3956 miles.

9. The following reciprocal observations were made from two points *P* and *Q* :

Horizontal distance between *P* and *Q*	= 4860 m
Angle of elevation of *Q* at *P*	= 1° 5′ 21″
Angle of depression of *P* at *Q*	= 1° 0′ 50″
Height of instrument at *P*	= 1.35 m
Height of signal at *P*	= 6.10 m
Height of instrument at *Q*	= 1.38 m
Height of signal at *Q*	= 6.21 m

Find the difference in level between *P* and *Q* and the co-efficient of refraction. Take $R \sin 1'' = 30.88$ m.

10. The following reciprocal observations were made from two points *P* and *Q* :

Horizontal distance	= 16440 m
Angle of depression of *Q* at *P*	= 0° 3′ 42″
Angle of depression of *P* at *Q*	= 0° 2′ 4″
Heights of instrument at *P* and *Q*	= 1.42 m
Height of signal at *P* and *Q*	= 5.53 m

Calculate (*a*) the R.L. of *Q*, if that of *P* is 346.39 m ; and

(*b*) the average co-efficient of refraction at the time of observation.

Take $R \sin 1'' = 30.88$ m.

11. In trigonometrical measurement of the difference in level of two stations *P* and *Q*, 61760 m apart, the following data were obtained :

Instrument at *P*, angle of elevation of *Q*	= 0′ 32″
Instrument at *Q*, angle of depression of *P*	= 3′ 33″
Height of instrument at *P*	= 1.44 m
Height of instrument at *Q*	= 1.50 m
Height of signal at *P*	= 13.84 m
Height of signal at *Q*	= 13.80 m

Find the difference in level between P and Q, and the curvature and refraction corrections. Take $R \sin 1'' = 30.88$ m.

ANSWERS

1. 37.558 m ; 278.824 m.
2. 267.796.
3. 442.347.
4. 290.335 ; 33.9 m.
7. 606.55 m.
8. 44142 ft.
9. m=0.0693 ; H=89.13 m.
10. m= 0.0784 ; 342.48 m.

6

Hydrographic Surveying

6.1. INTRODUCTION

Hydrographic survey is that branch of surveying which deals with the measurement of bodies of water. It is the art of delineating the submarine levels, contours and features of seas, gulfs, rivers and lakes. It is used for :

(1) making nautical charts for navigation and determination of rocks, sand bars, lights and buoys ;

(2) making subaqueous investigations to secure information needed for the construction, development and improvement of port facilities ;

(3) measurement of areas subject to scour or silting and to ascertain the quantities of dredged material ;

(4) controlling and planning of engineering projects like bridges, tunnels, dams, reservoirs, docks and harbours ;

(5) establishing mean sea level and observation of tides ;

(6) determination of shore lines ; and

(7) measurement of discharge of rivers.

Horizontal and Vertical Control

The main operation in hydrographic surveying is to determine the depth of water at a certain point. The measurement of depth below the water surface is called *sounding*. Thus, to take the sounding, a vertical control is necessary and to locate the sounding (*i.e.*, the point where the sounding is taken), a horizontal control is necessary. The *horizontal control* may consist of either a triangulation or a traverse. For surveys of large extent, a second or third order triangulation may be used as the main control. For surveys of small extent, a transit-tape-traverse may be used. For small detached surveys, a control system may be developed by a combination of stadia and graphical triangulation procedures with plane table. In the case of a long narrow river, the horizontal control is established by running a single traverse line on one shore. If the width of body of water is more than 1 kilometre, traverse may be run on both the shores and may be connected at intervals.

When the soundings are recorded, it is essential to know the gauge reading. *i.e.*, the level of water which continously goes on changing. Tide or water-stage gauges are kept in operation to establish the common datum and to give the height of water for which each sounding is taken. Before sounding operations are begun, a *vertical control* must be established to connect these gauges with shore elevations and with each other.

6.2. SHORE LINE SURVEY

The shore line surveys consist of :

(*i*) determination or delineation of shore lines,

(*ii*) location of shore details and prominent features to which soundings may be connected,

(*iii*) determination of low and high water lines for average spring tides.

The determination or delineation of shore lines is done by traversing along the shore and taking offsets to the water edge by tape, or stadia or plane table. If the river is narrow, both the banks may be located by running a sigle line of traverse on one bank. For wide rivers, however, transverse may be run along both the banks. The traverse should be connected at convenient intervals to check the work. Thus, in Fig. 6.1, the two traverses XY and X_1Y_1 along the two opposite shores may be checked by taking observations from A and B to the points C and D. When the instrument is at B, angles ABC and ABD can be measured. From the measured length of AB and the four angles, the length CD can be calculated. If this agrees with the measured length of CD, the work is checked. Sometimes, a triangulation net is run along a wide river. In sea shore survey, buoys anchored off the shore and light houses are used as reference points and are located by triangulation.

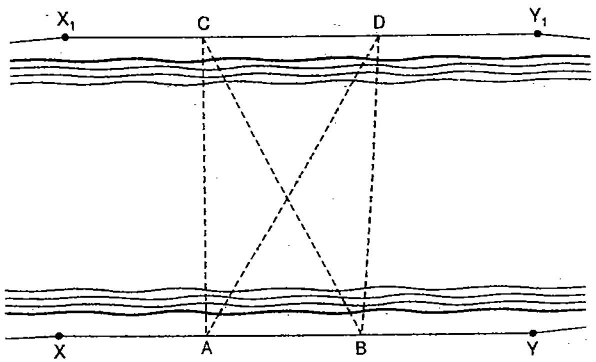


FIG. 6.1

In the case of tidal water, it is necessary to locate the high and low water lines. The position of high water line may be determined roughly from shore deposits and marks on rocks. To determine the high water line accurately, the elevation of *mean high water of ordinary spring tide* is determined and the points are located on the shore at *that* elevation as in direct method of contouring. The low water line can also be determined similarly. However, since the limited time is available for the survey of low water line, it is usually located by interpolation from soundings.

6.3. SOUNDINGS

The measurement of depth below the water surface is called *sounding*. This corresponds to the ordinary spirit levelling in land surveying where depths are measured below a horizontal line established by a level. Here, the horizontal line or the datum is the surface of water, the level of which continously goes on changing with time. The object of making soundings is thus to determine the configuration of the subaqueous source. As stated earlier, soundings are required for :

(*i*) making nautical charts for navigation ;

(*ii*) measurement of areas subject to scour or silting and to ascertain the quantities of dredged material ;

(*iii*) making sub-aqueous investigations to secure information needed for the construction, development and improvement of port facilities.

For most of the engineering works, soundings are taken from a small boat. The equipment needed for soundings are :

(*i*) Sounding boat
(*ii*) Sounding rods or poles
(*iii*) Lead lines
(*iv*) Sounding machine
(*v*) Fathometer.

(*i*) Sounding boat

A row-boat for sounding should be sufficiently roomy and stable. For quiet water, a flat bottom boat is more suitable, but for rough water round-bottomed boat is more suitable. For regular soundings, a row boat may be provided with a well through which soundings are taken. A sounding platform should be built for use in smaller boat. It should be extended far enough over the side to prevent the line from striking the boat. If the currents are strong, a motor or stream launch may be used with advantage.

(*ii*) Sounding rods or poles

A sounding rod is a pole of a sound straight-grained well seasoned tough timber usually 5 to 8 cm in diameter and 5 to 8 metres long. They are suitable for shallow and quiet waters. An arrow or lead shoe of sufficient weight is fitted at the end. This helps in holding them upright in water. The lead or weight should be of sufficient area so that it may not sink in mud or sand. Between soundings it is turned end for end without removing it from the water. A pole of 6 m can be used to depths upto 4 metres.

(*iii*) Lead lines

A *lead line* or *a sounding line* is usually a length of a such cord, or tiller rope of Indian hemp or braided flax or a brass chain with a sounding lead attached to the end. Due to prolonged use, a line of hemp or cotton is liable to get stretched. To graduate such a line, it is necessary to stretch it thoroughly when wet before it is graduated. The line should be kept dry when not in use. It should be soaked in water for about one hour before it is used for taking soundings. The length of the line should be tested frequently with a tape. For regular sounding, a chain of brass, steel or iron is preferred. Lead lines are usually used for depths over about 6 metres.

Sounding lead is a weight (made of lead) attached to the line. The weight is conical in shape and varies from 4 to 12 kg depending upon the depth of water and the strength of the current. The weight should be somewhat streamlined and should have an eye at the top for attaching the cord. It often has cup-shaped cavity at the bottom so that it may be armed with lard or tallow to pick up samples from the bottom. Where the bottom surface is soft, lead-filled pipe with a board at the top is used with the lead weight. The weight penetrates in the mud and stops where the board strikes the mud surface.

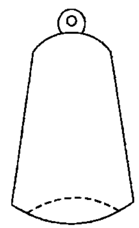


FIG. 6.2

Suggested system of marking poles and lead lines

The U.S. Coast and Geodetic survey recommends the following system of marking the poles and the lead lines :

Poles : Make a small permanent notch at each half foot. Paint the entire pole white and the spaces between the 2- and 3-, the 7- and 8- and the 12- and 13-ft marks black. Paint $\frac{1}{2}''$ red bands at the 5- and 10-ft marks, a $\frac{1}{2}''$ in black band at each of the other foot marks and $\frac{1}{4}''$ bands at the half foot marks. These bands are black where the pole is white and *vice-versa*.

Lead Lines : A lead line is marked in feet as follow :

Feet	*Marks*
2, 12, 22 etc	Red bunting
4, 14, 24 etc.	White bunting
6, 16, 26 etc.	Blue bunting
8, 18, 28 etc.	Yellow bunting
10, 60, 110 etc.	One strip of leather
20, 70, 120 etc.	Two strips of leather
30, 80, 130 etc.	Leather with two holes
40, 90, 140 etc.	Leather with one hole
50	Star-shaped leather
100	Star-shaped leather with one hole.

The intermediate odd feet (1, 3, 5, 7, 9 etc.) are marked by white seizings.

(*iv*) **Sounding Machine**

Where much of sounding is to be done, a sounding machine as very useful. The sounding machine may either be hand driven or automatic. Fig. 6.3 shows a typical hand driven Weddele's sounding machine.

The lead weight is carried at the end of a flexible wire cord attached to the barrel and can be lowered at any desired rate, the speed of the drum being controlled by means of a brake. The readings are indicated in two dials — the outer dial showing the depth in feet and the inner showing tenths of a foot. A handle is used to raise the level which can be suspended at any height by means of a paul and ratchet. The sounding

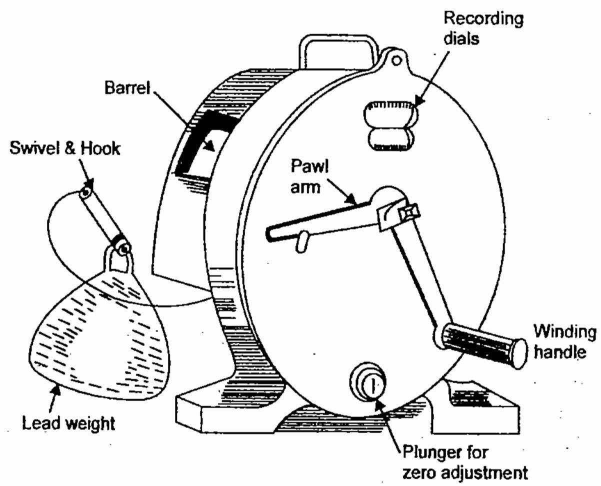


FIG. 6.3. WEDDELE'S SOUNDING MACHINE.

machine is mounted in a sounding boat and can be used up to a maximum depth of 100 ft.

(*v*) **Fathometer : Echo-sounding**

A fathometer is used for ocean sounding where the depth of water is too much, and to make a continuous and accurate record of the depth of water below the boat or ship at which it is installed. It is an *echo-sounding* instrument in which water depths are obtained by determining the time required for the sound waves to travel from a point near the surface of the water to the bottom and back. It is adjusted to read depth in accordance with the velocity of sound in the type of water in which it is being used. A fathometer may indicate the depth visually or indicate graphically on a roll which continuously goes on revolving and provide a virtual profile of the lake or sea.

The main parts of an echo-sounding apparatus are :

1. Transmitting and receiving oscillators.
2. Recorder unit.
3. Transmitter/Power unit

Fig. 6.4. illustrates the principle of echo-sounding. It consists in recording the interval of time between the emission of a sound impulse direct to the bottom of the sea and the reception of the wave or echo, reflected from the bottom. If the speed of sound in that water is v and the time interval between the transmitter and receiver is t, the depth h is given by

$$h = \frac{1}{2} vt \qquad \ldots(6.1)$$

Due to the small distance between the receiver and the transmitter, a slight correction is necessary in shallow waters. The error between the true depth and the recorded depth can be calculated very easily by simple geometry. If the error is plotted against the recorded depth, the true depth can be easily known. The recording of the sounding is produced by the action of a small current passing through chemically impregnated paper from a rotating stylus to an anode plate. The stylus is fixed at one end of a radial arm which revolves at constant speed. The stylus makes a record on the paper at the instants when the sound impulse is transmitted and when the echo returns to the receiver.

The record of depth is made by a stylus on a moving band of dry paper as shown in Fig.6.5

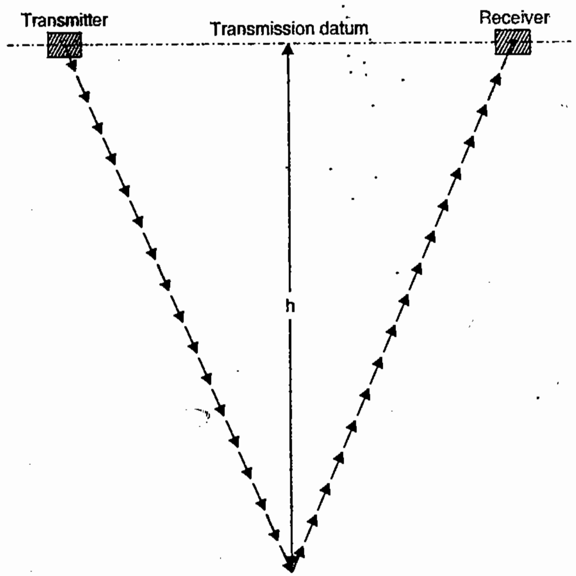


FIG. 6.4. PRINCIPLE OF ECHO-SOUNDING.

for the Kelvin Huges MS48 Echo-sounder. The draught of the vessel can be compensated for, so that transmission is effective from water level.

Accuracy of Measurement

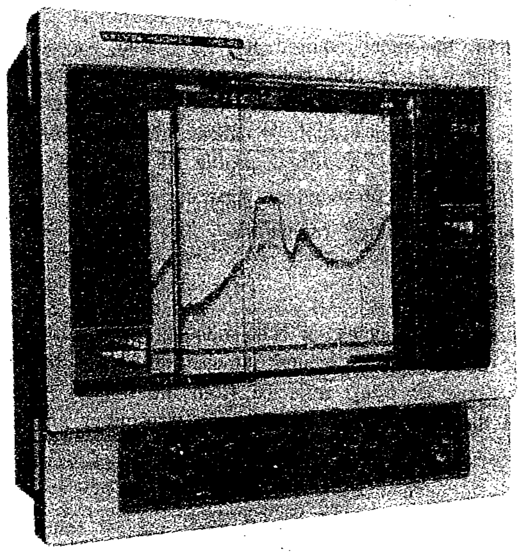


FIG. 6.5. KELVIN HUGES MS 48 ECHO SOUNDER

The accuracy of measurement depends upon matching the readers time scale with the velocity of accoustic pulse in sea water, the value of which is approximately 1500 m/s. This velocity varies with salinity and temperature of sea water, which in turn, vary with the depth, weather and time. Following equation is one of the several expressions used to calculate accoustic velocity in sea water

$$V = 1410 + 4.21\,T - 0.037\,T^2 + 1.14\,S \quad \ldots(6.2)$$

where V = Velocity of sound in sea water (m/s)

T = Surface temperature in °C

S = Salinity in parts of sodium chloride per 1000.

Advantage of echo-sounding

Echo-sounding has the following advantages over the older method of lead line and rod :

1. It is more accurate as a truly vertical sounding is obtained. The speed of the vessel does deviate it appreciably from the vertical. Under normal water conditions, in ports and harbours an accuracy of 7.5 cm may be obtained.

2. It can be used when a strong current is running and when the weather is unsuitable for the soundings to be taken with the lead line.

3. It is more sensitive than the lead line.

4. A record of the depth is plotted immediately and provides a continuous record of the bottom as the vessel moves forward.

5. The speed of sounding and plotting is increased.

6. The error due to estimation of water level in a choppy sea is reduced owing to the instability of the boat.

7. Rock underlying softer material is recorded and this valuable information is obtained more cheaply than would be the case where sub-marine borings are taken.

6.4. MAKING THE SOUNDINGS

If the depth is less than 25 m, the soundings can be taken when the boat is in motion. In the case of soundings with rod the leadsman stands in the bow and plunges the rod at a forward angle, depending on the speed of the boat, such that the rod is vertical when the boat reaches the point at which soundings is being recorded. The rod should be read very quickly. The nature of the bottom should also be recorded at intervals in the note-book.

If the sounding is taken with a lead, the leadsman stands in the bow of the boat and casts the lead forward at such a distance that the line will become vertical and will reach the bottom at a point where sounding is required. The lead is withdrawn from the water after the reading is taken. If the depth is great, the lead is not withdrawn from the water, but is lifted between the soundings.

The water surface, which is also the reference datum, changes continuously. It is, therefore, essential to take the readings of the tide gauges at regular interval so that the soundings can be reduced to a fixed datum. To co-relate each sounding with the gauge reading, it is essential to record the time at which each sounding is made.

6.5. METHODS OF LOCATING SOUNDINGS

The soundings are located with reference to the shore traverse by observations made (*i*) entirely from the boat, (*ii*) entirely from the shore or (*iii*) from both.

The following are the methods of location :

(a) By conning the survey vessel

1. By cross rope
2. By range and time intervals

(b) By observations with sextant or theodolite

3. By range and one angle from the shore
4. By range and one angle from the boat
5. By two angles from the shore
6. By two angles from the boat
7. By one angle from shore and one from boat
8. By intersecting ranges
9. By tacheometry.

(c) By theodolite angles and EDM distances from the shore

(d) By microwave systems

(a) By conning the survey vessel

The process of keeping the survey vessel or boat on a known course is known as *conning* the vessel. The task of conning is mainly one of seamanship. One of the most common method of conning is to fix markers (poles, beacons etc.) on the shore, thus providing the 'ranges' along which the vessel is run. The method is suitable for work in rivers and open seas upto 5 km off shore.

Range. A range or range line is the line on which soundings are taken. They are, in general, laid perpendicular to the shore line and parallel to each other if the shore is straight or are arranged radiating from a prominent object when the shore line is very irregular.

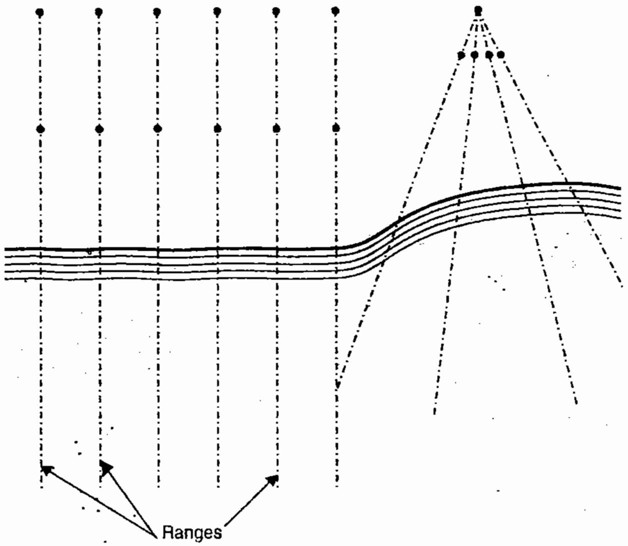


FIG. 6.6. RANGES.

Shore signals. Each range line is marked by means of signals erected at two points on it at a considerable distance apart. Signals can be constructed in a variety of ways. They should be readily seen and easily distinguished from each other. The most satisfactory and economic type of signal is a wooden tripod structure dressed with white and coloured signal of cloth. The position of the signals should be located very accurately since all the soundings are to be located with reference to these signals.

1. Location by Cross-Rope

This is the most accurate method of locating the soundings and may be used for rivers, narrow lakes and for harbours. It is also used to determine the quantity of materials removed by dredging, the soundings being taken before and after the dredging work is done. A single wire or rope is stretched across the channel etc. as shown in Fig. 6.7 and is marked by metal tags at appropriate known distance along the wire from a reference point or zero station on shore. The soundings are then taken by a weighted pole. The position of the pole during a sounding is given by the graduated rope or line.

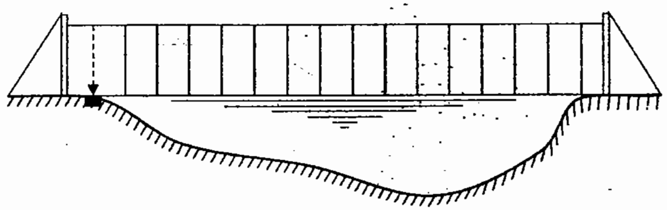


FIG. 6.7. SOUNDING FROM A GRADUATED LINE.

In another method, specially used for harbours etc., *a reel boat* is used to stretch the rope. The zero end of the rope is attached to a spike or any other attachment on one shore. The rope is wound on a drum on the reel boat. The reel boat is then rowed across the line of sounding, thus unwinding the rope as it proceeds. When the reel boat reaches the other shore, its anchor is taken ashore and the rope is wound as tightly as possible. If anchoring is not possible, the reel is taken ashore and spiked down. Another boat, known as the sounding boat, then starts from the previous shore and soundings are taken against each tag of the rope. At the end of the soundings along that line, the reel boat is rowed back along the line thus winding in the rope. The work thus proceeds.

2. Location by Range and Time Intervals

In this method, the boat is kept in range with the two signals on the shore and is rowed along it at constant speed. Soundings are taken at different time intervals. Knowing the constant speed and the total time elapsed at the instant of sounding, the distance of the total point can be known along the range. The method is used when the width of channel is small and when great degree of accuracy is not required. However, the method is used in conjunction with other methods, in which case the first and the last soundings along a range are located by angles from the shore and the intermediate soundings are located by interpolation according to time intervals.

(b) By observations with sextant or theodolite

3. Location by Range and One Angle from the Shore

In this method, the boat is ranged in line with the two shore signals and rowed along the ranges. The point where sounding is taken is fixed on the range by observation of the angle from the shore. As the boat proceeds along the shore, other soundings are also fixed by the observation of angles from the shore. Thus, in Fig. 6.8 (*a*), *B* is the instrument station, A_1A_2 is the range along which the boat is rowed and α_1, α_2, α_3 etc., are the angles measured at *B* to points 1, 2, 3 etc. The method is very accurate and very convenient for plotting. However, if the angle at the sounding point (say angle β) is less than 30°, the fix becomes poor. The nearer the intersection angle (β) is to a right angle, the better. If the angle diminishes to about

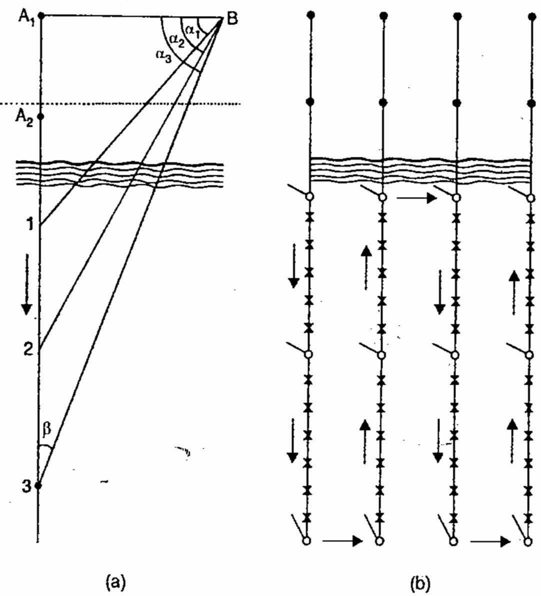


FIG. 6.8. LOCATION BY RANGE AND ONE ANGLE FROM THE SHORE

30° a new instrument station must be chosen. The only defect of the method is that the surveyor does not have an immediate control in all the observations. If all the points are to be fixed by angular observations from the shore, a note-keeper will also be required along with the instrument man at shore since the observations and the recordings are to be done rapidly. Generally, the first and last soundings and every tenth sounding are fixed by angular observations and the intermediate points are fixed by time intervals. Thus, in Fig. 6.8 (*b*), the points with round mark are fixed by angular observations from the shore and the points with cross marks are fixed by time intervals. The arrows show the course of the boat, seaward and shoreward on alternate sections.

To fix a point by observations from the shore, the instrument man at *B* orients his line of sight towards a shore signal or any other prominent point (known on the plan) when the reading is zero. He then directs the telescope towards the leadsman or the bow of the boat, and is kept continually pointing towards the boat as it moves. The surveyor on the boat holds a flag for a few seconds and on the fall of the flag, the sounding and the angle are observed simultaneously.

The angles are generally observed to the nearest 5 minutes. The time at which the flag falls is also recorded both by the instrument man as well as on the boat. In order to avoid acute intersections, the lines of soundings are previously drawn on the plan and suitable instrument stations are selected.

4. Location by Range and One Angle from the Boat

The method is exactly similar to the previous one except that the angular fix is made by angular observation from the boat. The boat is kept in range with the two shore signals and is rowed along it. At the instant the sounding is taken, the angle, subtended at the point between the range and some prominent point *B* on the shore is measured with the help of a sextant. The telescope is directed on the range signals, and the side object is brought into coincidence at the instant the sounding is taken. The accuracy and ease of plotting is the same as obtained in the previous method. Generally, the first and the last soundings, and some of the intermediate soundings are located by angular observations and the rest of the soundings are located by time intervals.

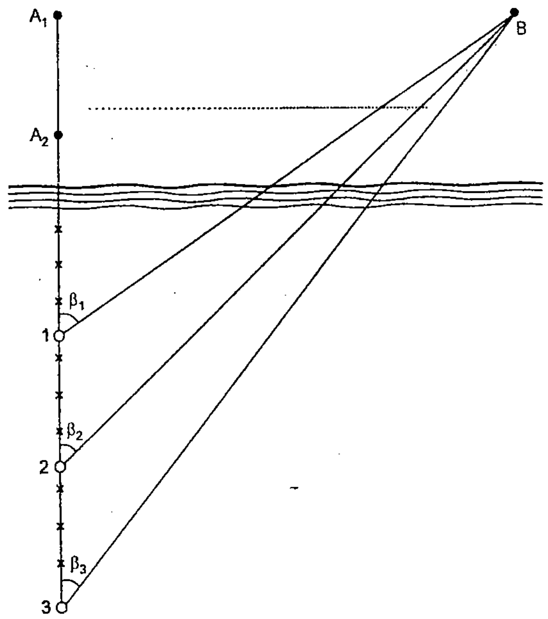


FIG. 6.9. LOCATION BY RANGE AND ONE ANGLE FROM THE BOAT.

As compared to the previous methods, this method has the following advantages :

1. Since all the observations are taken from the boat, the surveyor has better control over the operations.

2. The mistakes in booking are reduced since the recorder books the readings directly as they are measured.

3. On important fixes, check may be obtained by measuring a second angle towards some other signal on the shore.

4. To obtain good intersections throughout, different shore objects may be used for reference to measure the angles.

5. Location by Two Angles from the Shore

In this method, a point is fixed independent of the range by angular observations from two points on the shore. The method is generally used to locate some isolated points. If this method is used on an extensive survey, the boat should be run on a series of approximate ranges. Two instruments and two instrument men are required. The position of instrument is selected in such a way that a strong fix is obtained. New instrument stations should be chosen when the intersection angle (θ) falls below 30°. Thus, in Fig. 6.10, *A* and *B* are the two instrument stations. The distance *d* between them is very accurately measured. The instrument stations *A* and *B* are precisely connected to the ground traverse or triangulation, and their positions on plan are known. With both the plates clamped to zero, the instrument man at *A* bisects *B*; similarly with both the plates clamped to zero, the instrument man at *B* bisects *A*.

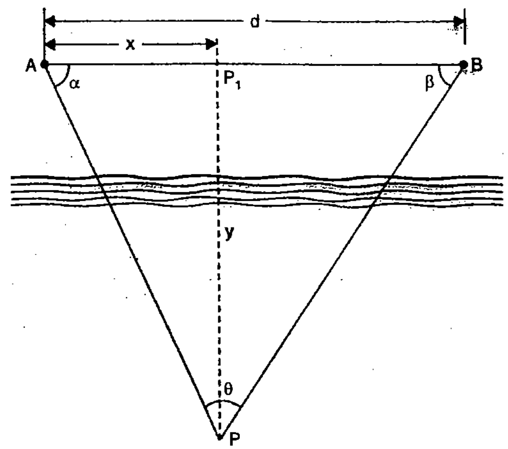


FIG. 6.10. LOCATION BY TWO ANGLES FROM THE SHORE.

Both the instrument men then direct the line of sight of the telescope towards the leadsman and continuously follow it as the boat moves. The surveyor on the boat holds a flag for a few seconds, and on the fall of the flag the sounding and the angles are observed simulutaneously. The co-ordinates of the position *P* of the sounding may be computed from the relations :

$$x = \frac{d \tan \beta}{\tan \alpha + \tan \beta} \qquad ...(6.3)$$

and

$$y = \frac{d \tan \alpha \tan \beta}{\tan \alpha + \tan \beta} \qquad ...(6.4)$$

The method has got the following advantages :

1. The preliminary work of setting out and erecting range signals is eleminated.

2. It is useful when there are strong currents due to which it is difficult to row the boat along the range line.

The method is, however, laborious and requires two instruments and two instrument-men.

6. Location by Two Angles from the Boat

In this method, the position of the boat can be located by the solution of the three-point problem by observing the two angles subtended at the boat by three suitable shore objects of known position. The three-shore points should be well-defined and clearly visible. Prominent natural objects such as church spire, lighthouse, flagstaff, buoys etc, are selected for this purpose. If such points are not available, range poles or shore singals may be taken. Thus, in Fig. 6.11, A, B and C are the shore objects and P is the position of the boat from which the angles α and β are measured. Both the angles should be observed simultaneously with the help of two sextants, at the instant the sounding is taken. If both the angles are observed by surveyour alone, very little time should be lost in taking the observation. The angles on the circle are read afterwards. The method is used to take the soundings at isolated points. The surveyor has better control on the operations since the survey party is concentrated in one boat. If sufficient number of prominent points are available on the shore, preliminary work of setting out and erecting range signals is eliminated. The position of the boat is located by the solution of the three point problem either analytically or graphically.

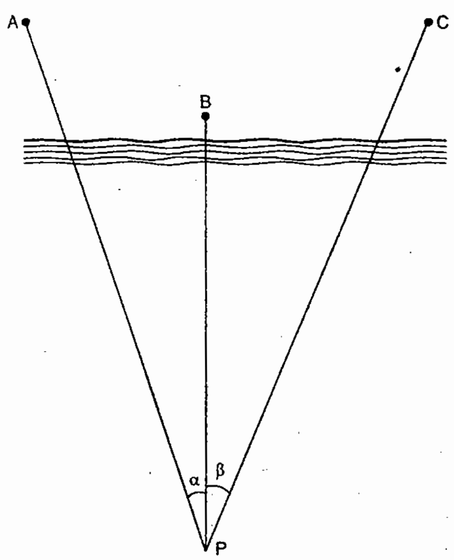


FIG. 6.11. LOCATION BY TWO ANGLES FROM THE BOAT.

7. Location by One Angle from the Shore and the other from the Boat.

This method is the combination of methods 5 and 6 described above and is used to locate the isolated points where soundings are taken. Two points A and B (Fig. 6.12) are chosen on the shore, one of the points (say A) is the instrument station where a theodolite is set up, and the other (say B) is a shore signal or any other prominent object. At the instant the sounding is taken at P, the angle α at A is measured with a theodolite while the angle β at the boat is measured with the help of a sextant. Knowing the distance d between the two points A and B by ground survey, the position of P can be located by calculating the two co-ordinates x and y.

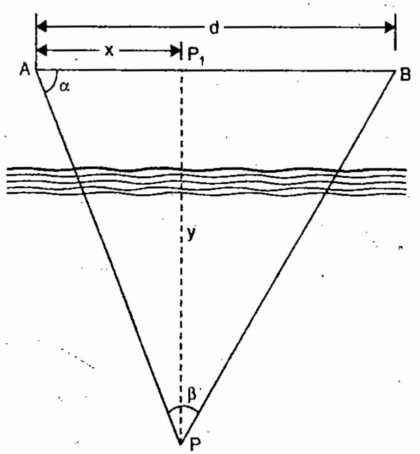


FIG. 6.12. LOCATION BY ONE ANGLE FROM THE SHORE AND THE OTHER FROM THE BOAT.

8. Location by Intersecting Ranges

This method is used when it is required to determine by periodical sounding at the *same* points, the rate at which silting or scouring is taking place. This is very essential on the harbours and reservoirs. The position of sounding is located by the intersection

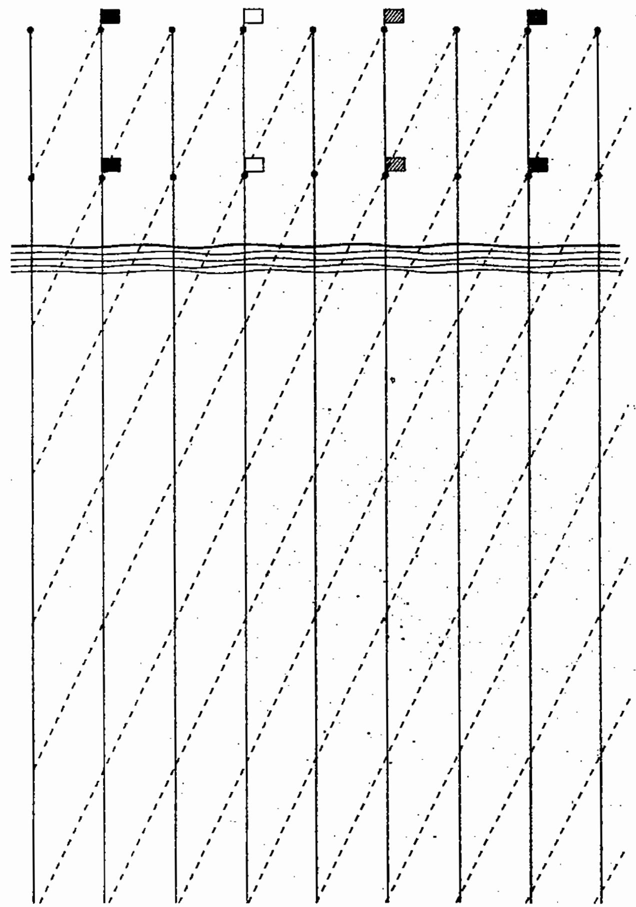


FIG. 6.13. LOCATION BY INTERSECTING RANGES.

of two ranges, thus completely avoiding the angular observations. Suitable signals are erected at the shore. The boat is rowed along a range perpendicular to the shore and soundings are taken at the points in which inclined ranges intersect the range, as illustrated in Fig. 6.13. However, in order to avoid the confusion, a definite system of flagging the range poles is necessary. The position of the range poles is determined very accurately by ground survey.

9. Location by Tacheometric Observations

The method is very much useful in smooth waters. The position of the boat is located by tacheometric observations from the shore on a staff kept vertically on the boat. Observing the staff intercept s at the instant the sounding is taken, the horizontal distance between the instrument stations and the boat is calculated by

$$d = \frac{f}{i}\,s + (f + d) \qquad \ldots(6.4)$$

The direction of the boat (P) is established by observing the angle (α) at the instrument station B with reference to any prominent object A [Fig. 6.14 (a) and (b)]. The transit station should be near the water level so that there will be no need to read vertical angles. The method is unsuitable when soundings are taken far from shore.

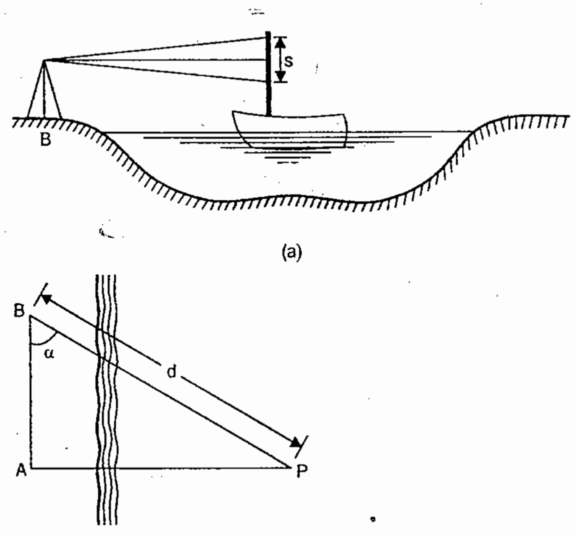


FIG. 6.14. LOCATION BY STADIA METHOD.

(c) **Fixing by theodolite angles and EDM distances from the shore**

This is a modern method of fixing the position of the vessel, wherein a theodolite and infra-red EDM instrument, set up at a *shore station* is used to fix the boat by the polar method of range and bearing. However, the main problem in this method lies in maintaining the orientation of the EDM *onshore* and reflector *off-shore* (i.e. on the boat) so that a return signal is constantly received and obviously the calmer the water the easier the work. For working and details of EDM instruments, reader may refer to chapter 15 on EDM.

(d) **Fixing by microwave systems**

In the method of fixing the position of the moving boat by microwave system, the Tellurometer MRD$_1$ (see chapter 15) is used, for positions upto 100 km from the shore, using the technique of 'two range' or 'range-range' technique. Distances are measured from the master unit on the vessel to two remote shore stations. Fig. 6.15 shows two shore stations A and B, at a fixed distance d, where the *remote unit* of the tellurometer is placed. P is the position of the boat on which the *master* unit is placed. By determining distances D_A and

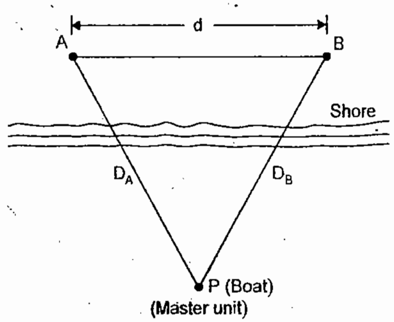


FIG. 6.15

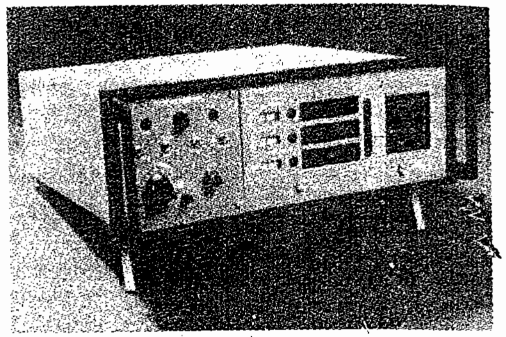


FIG. 6.16 (a) MASTER UNIT OF TELLUROMETER MRD1

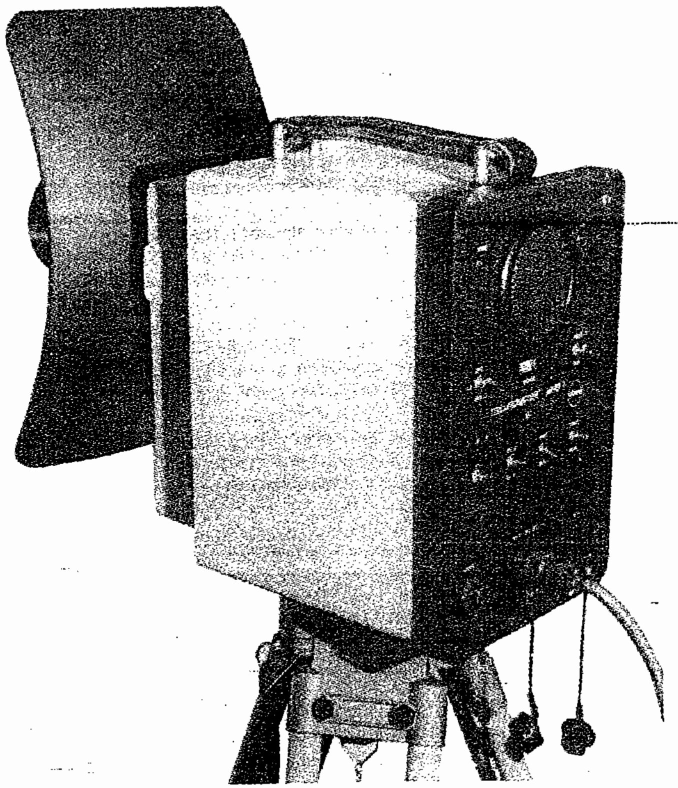


FIG. 6.16 (b) REMOTE UNIT OF TELLUROMETER MRD2

D_B by the microwave system (see chapter 15), a fix is obtainable since the three sides of the triangle ABP are now known. It must be borne in mind that sloping distances are measured, and that '*line of sight*' conditions must be satisfied when selecting the shore stations. The transmissions should clear the sea surface by at least 3 m, and well conditioned triangles should be sought for accuracy of fix. With the best conditions, an accuracy of ± 0.1 m is claimed.

Fig. 6.16 (*a*) shows the master unit of Tellurometer MRD1 while Fig. 6.16 (*b*) shows the remote unit of Tellurometer MRD1, placed at the shore station To measure a single range, two instruments, one the *master*, being on the vessel, and second the *remote*, at the shore station, whilst a *third unit* the *master antenna* completes the basic system. The master unit contains all the required circuitory to produce two sets of range information. The *master antenna* unit is connected by cable to the master unit and the two can be upto 30 m apart.

6.6. REDUCTION OF SOUNDINGS

The *reduced soundings* are the reduced levels of the sub-marine surface in terms of the adopted datum. When the soundings are taken, the depth of water is measured with reference to the existing water level at that time. If the gauge readings are also taken at the same time, the soundings can be reduced to a common unvarying datum. The datum most commonly adopted is the '*mean level of low water of spring tides*' and is written either as L.W.O.S.T. (low water, ordinary spring tides) or M.L.W.S. (mean low water springs). For reducing the soundings, a correction equal to the difference of level between the actual water level (read by gauges) and the datum is applied to the observed soundings, as illustrated in the table given below :

Gauge Reading at L.W.O.S.T. = 3.0 m.

	Gauge *(m)*	*Distance* *(m)*	*Sounding* *(m)*	*Correction* *(m)*	*Reduced sounding* *(m)*	*Remarks*
8.00 A.M.	3.5	10	2.5	– 0.5	2.0	
		20	3.2		2.7	
		30	3.9		3.4	
		40	4.6		4.1	
8.10 A.M.	3.5	50	5.3	– 0.5	4.8	
		60	5.4		4.9	
		70	5.1		4.6	
		80	4.7		4.2	
		90	3.6		3.1	
8.20 A.M.	3.5	100	2.1	–0.5	1.6	

6.7. PLOTTING OF SOUNDINGS

The method of plotting the soundings depends upon the method used for locating the soundings. If the soundings have been taken along the range lines, the position of shore signals can be plotted and the sounding located on these in the plan. In the fixes

by angular methods also, the plotting is quite simple, and requires the simple knowledge of geometry. However, if the sounding has been located by two angles from the boat by observations to three known points on the shore, the plotting can be done either by the mechanical, graphical or the analytical solution of the three-point problem.

THE THREE POINT PROBLEM

***Statement** : Given the three shore signals A, B and C, and the angles α and β subtended by AP, BP and CP at the boat P, it is required to plot the position of P (Fig. 6.17).*

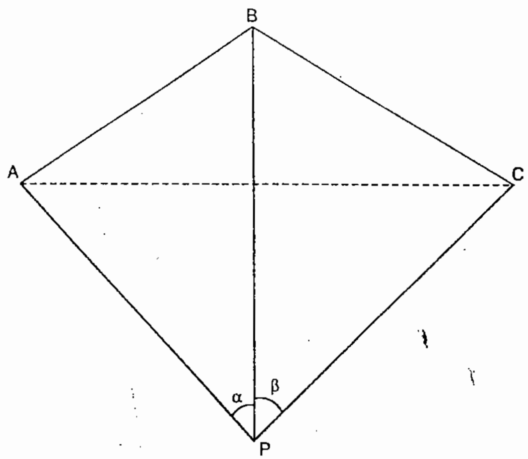


FIG. 6.17. THE THREE-POINT PROBLEM.

1. Mechanical Solution

(i) By Tracing Paper

Protract angles α and β between three radiating lines from any point on a piece of tracing paper. Plot the positions of signals *A*, *B*, *C* on the plan. Applying the tracing paper to the plan, move it about until all the three rays simultaneously pass through *A*, *B* and *C*. The apex of the angles is then the position of *P* which can be pricked through.

(ii) By Station Pointer : (Fig. 6.18)

The station pointer is a three-armed protractor and consists of a graduated circle with fixed arm and two movable arms to the either side of the fixed arm. All the three arms have bevelled or fiducial edges. The fiducial edge of the central fixed arm corresponds to the zero of the circle. The fiducial edges of the two moving arms can be set to any desired reading and can be clamped in position. They are also provided with verniers and slow motion screws to set the angle very precisely. To plot position of *P*, the movable arms are clamped to read the angles α and β very precisely. The station pointer is then moved on the plan till the three fiducial edges simultaneously touch *A*, *B* and *C*.

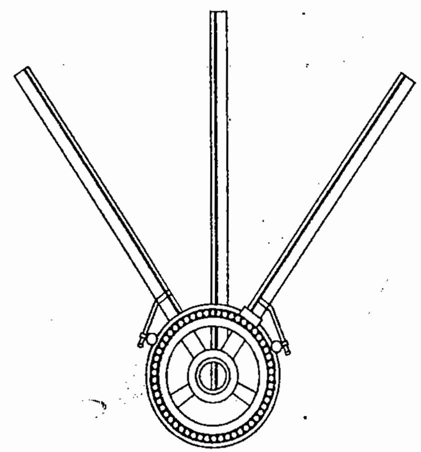


FIG. 6.18. STATION POINTER.

The centre of the pointer then represents the position of *P* which can be recorded by a prick mark.

2. **Graphical Solutions**

(*a*) **First Method** : (Fig. 6.19)

In Fig. 6.19, let *a*, *b* and *c* be the plotted positions of the shore signals *A*, *B* and *C* respectively and let α and β be the angles subtended at the boat. The point *p* of the boat position *P* can be obtained as under:

1. Join *a* and *c*.
2. At *a*, draw *ad* making an angle β with *ac*. At *c*, draw *cd* making an angle α with *ca*. Let both these lines meet at *d*.
3. Draw a circle passing through the points *a*, *d* and *c*.
4. Join *d* and *b*, and prolong it to meet the circle at the point *p* which is the required position of the boat.

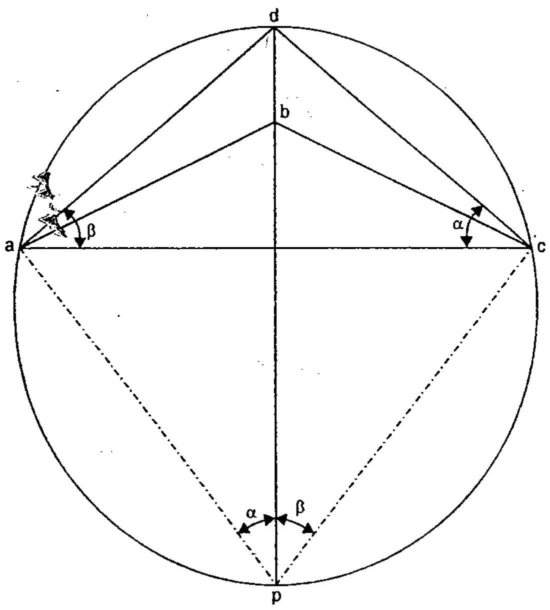


FIG. 6.19.

Proof. From the properties of a circle,

$$\angle apd = \angle acd = \alpha \quad \text{and} \quad \angle cpd = \angle cad = \beta$$

which is the required condition for the solution.

(*b*) **Second Method** (Fig. 6.20)

1. Join *ab* and *bc*.

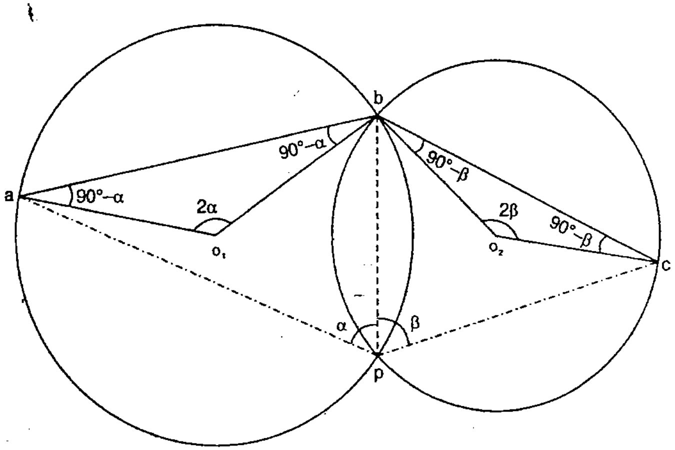


FIG. 6.20.

2. From *a* and *b*, draw lines ao_1 and bo_1 each making an angle (90° – α) with *ab* on the side towards *p*. Let them intersect at o_1.
3. Similarly, from *b* and *c*, draw lines bo_2 and co_2 each making an angle (90° – β) with *ab* on the side towards *p*. Let them intersect at o_2.
4. With o_1 as the centre, draw a circle to pass through *a* and *b*. Similarly, with o_2 as the centre draw a circle to pass through *b* and *c*. Let both the circles intersect each other at a point *p*. *p* is then the required position of the boat.

Proof. $\angle ao_1b = 180° - 2\,(90° - \alpha) = 2\alpha$

$$\angle apb = \frac{1}{2}\angle ao_1b = \alpha$$

Similarly, $\angle bo_2c = 180° - 2\,(90° - \beta) = 2\beta$

and $$\angle bpc = \frac{1}{2}\angle bo_2c = \beta\ .$$

The above method is sometimes known as the *method of two intersecting circles*.

(*c*) **Third Method** (Fig. 6.21)

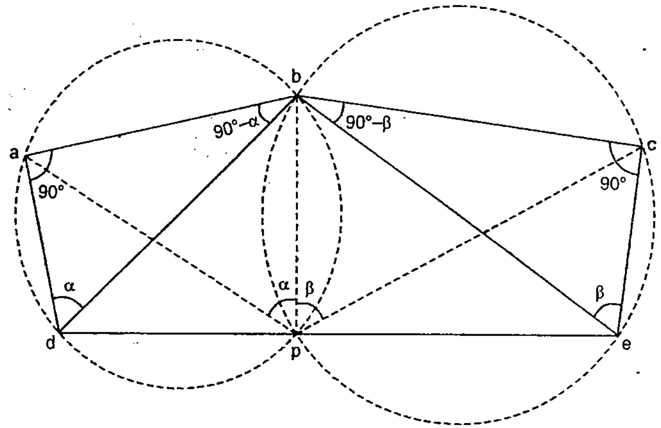


FIG. 6.21

1. Join *ab* and *bc*.
2. At *a* and *c*, erect perpendiculars *ad* and *ce*.
3. At *b*, draw a line *bd* subtending angle (90° – α) with *ba*, to meet the perpendicular through *a* in *d*.
4. Similarly, draw a line *be* subtending an angle (90° – β) with *bc*, to meet the perpendicular through *c* in *e*.
5. Join *d* and *e*.
6. Drop a perpendicular on *de* from *b*. The foot of the perpendicular (*i.e.* *p*) is then the required position of the boat.

Proof. Since ∠*bad* and ∠*bpd* are each equal to 90°, the quadrilateral *abpd* is concyclic. Similarly, the quadrilateral *bcep* is concyclic.

Hence $\angle adb = \angle apd = \alpha$

and $\angle bpc = \angle bec = \beta$

The problem is, however, indeterminate, if the points A, B, C and P are concyclic.

3. Analytical Solution

In Fig. 6.22, let A, B, and C be the shore signals whose position is known. Let α and β be the observed angles at P.

Let $\angle BAP = x$; $\angle BCP = y$; $\angle ABC = z$

a = distance BC

b = distance AC

and c = distance AB.

Now $x + y = 360° - (\alpha + \beta + z) = \theta$ (say) ...(1) ...(6.5)

Since α, β and z are known, θ can be calculated.

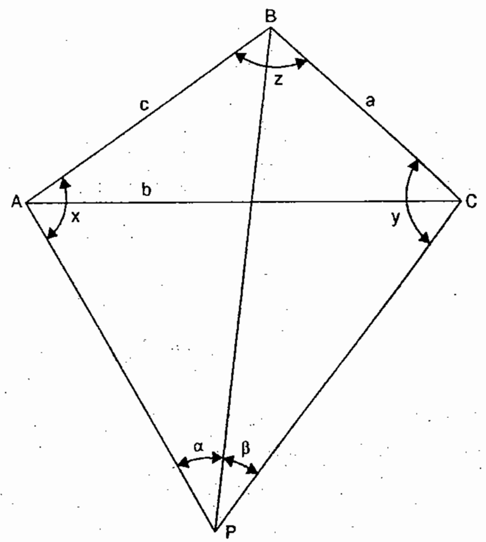


FIG. 6.22.

From the triangle PAB,

$$PB = \frac{c}{\sin\alpha}\sin x$$

From the triangle PCB,

$$PB = \frac{a}{\sin\beta}\sin y$$

Equating the two, we get

$$c \cdot \frac{\sin x}{\sin\alpha} = a \cdot \frac{\sin y}{\sin\beta}$$

$$\sin y = \frac{c \sin x \sin\beta}{a \sin\alpha}$$

But $y = \theta - x$, from (1)

Hence, $$\sin(\theta - x) = \frac{c \sin x \sin\beta}{a \sin\alpha}$$

or $$\sin\theta\cos x - \cos\theta\sin x = \frac{c \sin x \sin\beta}{a \sin\alpha}$$

Dividing both the sides by $\sin\theta \sin x$, we get

$$\cot x - \cot\theta = \frac{c \sin\beta}{a \sin\alpha \sin\theta}$$

or $$\cot x = \cot\theta + \frac{c\sin\beta}{a\sin\alpha\sin\theta}$$

or $$\cot x = \cot\theta\left\{1 + \frac{c\sin\beta\sec\theta}{a\sin\alpha}\right\} \qquad ...(2)$$

The value of x can, thus, be calculated from (2). Knowing the angle x, the angle y can be calculated from the relation $y = \theta - x$

Again, from ΔABP,

$$AP = \frac{c}{\sin\alpha}\sin\angle ABP = \frac{c}{\sin\alpha}\sin(180° - x - \alpha)$$

$$= \frac{c}{\sin\alpha}\cdot\sin(x + \alpha) \qquad ...(6.6)$$

and $$BP = \frac{c}{\sin\alpha}\sin x$$

Similarly, from ΔBPC,

$$BP = \frac{a}{\sin\beta}\sin y \qquad ...(6.7)$$

and $$CP = \frac{a}{\sin\beta}\sin CBP$$

$$= \frac{a}{\sin\beta}\sin(180° - y - \beta)$$

$$= \frac{a}{\sin\beta}\sin(y + \beta) \qquad ...(6.8)$$

Calculating AP, BP and CP, the position of P can be plotted.

Three cases may arise according to the position of the boat (P) with respect to the ground signals A, B and C :

Case 1. When B and P are to the opposite sides of the line AC (Fig. 6.22).

Case 2. When B and P are to the same side of the line AC [Fig. 6.23 (*a*)].

Case 3. When P is within the triangle ABC [Fig. 6.23 (*b*)].

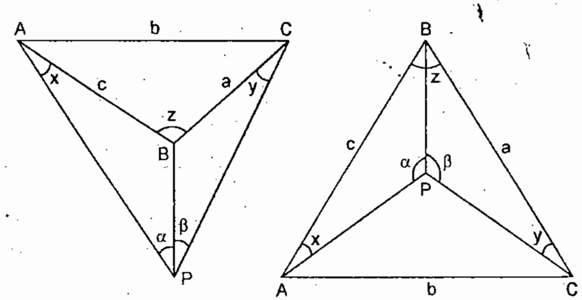


FIG. 6.23

In case (2), Fig. 6.23 (*a*), we have

$$x + y = z - (\alpha + \beta) = \theta \qquad ...(6.9)$$

In case (3), Fig. 6.20 (*b*), we have

$$x + y = 360° - (\alpha + \beta + z) = \theta \qquad ...(6.10)$$

Knowing the value of θ, the value of x can be calculated from Eq. (2) derived above.

Example 6.1. *A, B and C are three visible stations in a hydrographical survey. The computed sides of the triangle ABC are : AB, 1130 m ; BC, 1372 m ; and CA, 1889 m. Outside this triangle (and nearer to AC), a station P is established and its position is to be found by three point resection on A, B and C, the angles APB and BPC being respectively 42° 35′ and 54° 20′.*

Determine the distances PA and PC.

Solution. (Fig. 6.22)

Given : $c = AB = 1130$ m ; $a = BC = 1372$ m ; $b = CA = 1889$ m

Then $\angle ABC = z$ is given by

$$b^2 = c^2 + a^2 - 2ac \cos z$$

or
$$\cos z = \frac{c^2 + a^2 - b^2}{2ac} = \frac{(1130)^2 + (1372)^2 - (1889)^2}{2(1372)(1130)} = -0.1328980$$

$\therefore$ $\cos(180° - z) = 0.1328980$

$\therefore$ $180° - z = 82° 22' 14''$

or $z = 180° - 82° 22' 14'' = 97° 37' 46''$

Now $\theta = x + y = 360° - (\alpha + \beta + z)$

$$\theta = 360° - (42° 35' + 54° 20' + 97° 37' 46'') = 165° 27' 14'' \quad ...(ii)$$

Now
$$\cot x = \cot \theta + \frac{c \sin \beta}{a \sin \alpha \sin \theta}$$

$\therefore$
$$\cot x = \cot 165° 27' 14'' + \frac{1130 \sin 54° 20'}{1372 \sin 42° 35' \sin 165° 27' 24''}$$

$$= -3.98154 + 3.93594 = -0.04560$$

From which $x = 92° 36' 39''$

and $y = 165° 27' 14'' - 92° 36' 39'' = 72° 50' 35''$

$\therefore$ $\angle ABP = 180° - x - \alpha = 180° - 92° 36' 39'' - 42° 35' = 44° 48' 21''$

$\angle CBP = 180° - y - \beta = 180° - 72° 50' 35'' - 54° 20' = 52° 49' 25''$

Hence
$$AP = \frac{c}{\sin \alpha} \sin ABP = \frac{1130}{\sin 42° 35'} \sin 44° 48' 21'' = \mathbf{1176.83 \text{ m.}}$$

and
$$CP = \frac{a}{\sin \beta} \sin CBP = \frac{1372}{\sin 54° 20'} \sin 52° 49' 25'' = \mathbf{1346.00 \text{ m.}}$$

6.8. THE TIDES

All celestial bodies exert a gravitational force on each other. These forces of attraction between earth and other celestial bodies (mainly moon and sun) cause periodical variations in the level of a water surface, commonly known as *tides*. There are several theories about the tides, but none adequately explains all the phenomenon of tides. However, the commonly used theory is after Newton, and is known as the *equilibrium theory*. According to this theory, a force of attraction exists between two celestial bodies, acting in the straight line joining the centre of masses of the two bodies, and the magnitude of this force is proportional to the product of the masses of the bodies and is inversely proportional to the square of the distance between them. We shall apply this theory to the tides produced on earth due to the force of attraction between earth and moon. However, the following *assumptions* are made in the equilibrium theory :

1. The earth is covered all round by an ocean of uniform depth.

2. The ocean is capable of assuming instantaneously the *equilibrium figure* [Fig. 6.24 (*e*)], required by the tide producing forces. This is possible if we neglect (*i*) inertia of water, (*ii*) viscosity of water, and (*iii*) force of attraction between parts of itself.

1. The Lunar Tides

Fig. 6.24 (*a*) shows the earth and the moon, with their centres of masses O_1 and O_2 respectively. Since moon is very near to the earth, it is the major tide producing force. To start with, we will ignore the daily rotation of the earth on its axis. Both earth and moon attract each other, and the force of attraction would act along O_1O_2. Let O be the common centre of gravity of earth and moon. The earth and moon revolve monthly about O, and due to this revolution their separate positions are maintained. Fig. 6.24 (*b*) shows the distribution of force of attraction on earth, due to moon. The distribution of force is not uniform, but it is more for the points facing the moon and less for remote points. Due to the revolution of earth about the common centre of gravity O, centrifugal force of uniform intensity is exerted on all the particles of the earth. The direction of this centrifugal force is parallel to $O_1 O_2$ and acts outward, as shown in Fig. 6.24 (*c*). Thus, the total force of attraction due to moon is counter-balanced by the total centrifugal force, and the earth maintains its position relative to the moon. However, since the force of attraction is not uniform, the resultant force will vary all along, as shown in Fig. 6.24 (*d*). *The resultant forces are the tide producing forces*. Assuming that water has no inertia and viscosity, the ocean enveloping the earth's surface will adjust itself to the unbalanced resultant forces, giving rise to the *equilibrium figure* shown in Fig. 6.24 (*e*). Thus, there are two lunar tides at A and B, and two low water positions at C and

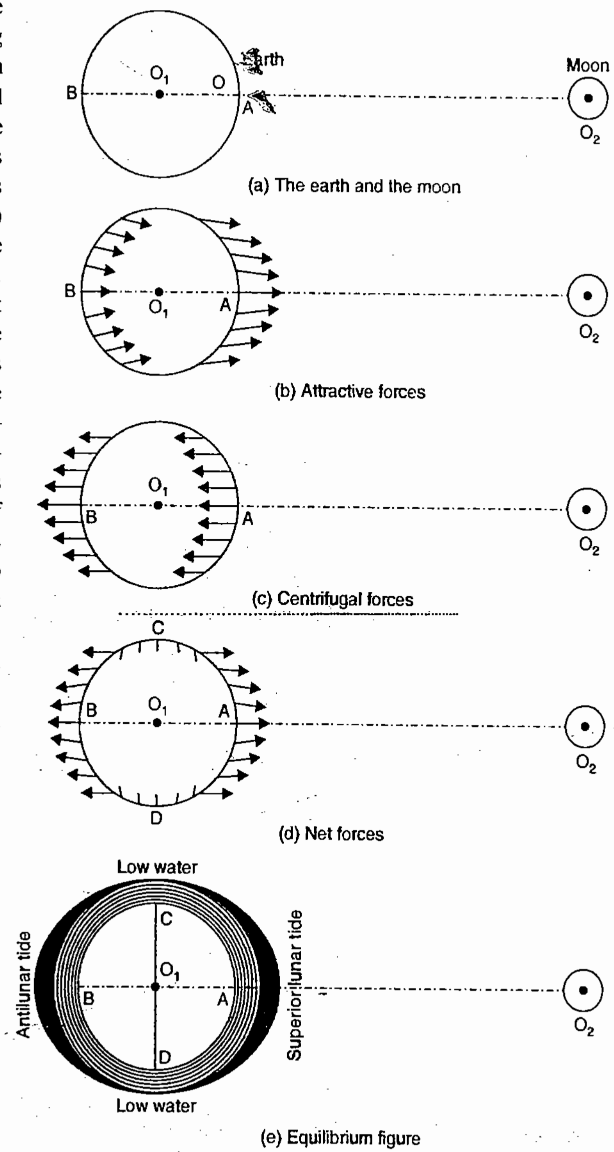


FIG. 6.24. PRODUCTION OF TIDES : EQUILIBRIUM THEORY.

D. The tide at *A* is called the *superior lunar tide* or tide of moon's upper transit, while tide at *B* is called *inferior* or *antilunar tide*.

Now let us consider the earth's rotation on its axis. Assuming the moon to remain stationary, the major axis of lunar tidal equilibrium figure would maintain a constant position. Due to rotation of earth about its axis from west to east, once in 24 hours, point *A* would occupy successive positions *C*, *B* and *D* at intervals of 6 h. Thus, point *A* would experience regular variation in the level of water. It will experience high water (tide) at intervals of 12 h and low water midway between. This interval of 6 h variation is true only if moon is assumed stationary. However, in a lunation of 29.53 days the moon makes one revolution relative to sun from the new moon to new moon. This revolution is in the same direction as the diurnal rotation of earth, and hence there are 29.53 transits of moon across a meridian in 29.53 mean solar days. This is on the assumption that the moon does this revolution in a plane passing through the equator. Thus, the interval between successive transits of moon or any meridian will be 24 h, 50.5 m. Thus, the average interval between successive high waters would be about 12 h 25 m. The interval of 24 h 50.5 m between two successive transits of moon over a meridian is called the *tidal day*.

2. The Solar Tides

The phenomenon of production of tides due to force of attraction between earth and sun is similar to the lunar tides. Thus, there will be *superior solar tide* and an *inferior* or *anti-solar tide*. However, sun is at a large distance from the earth and hence the tide producing force due to sun is much less.

Let M_E = mass of earth

M_M = Mass of moon

M_S = mass of sun

D_M = mean distance from the centre of earth to the centre of the moon

D_S = mean distance from the centre of earth to the centre of the sun

R = radius of earth

K = constant of gravitation

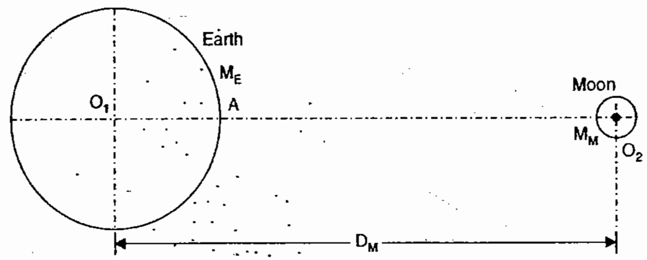


FIG. 6.25.

Consider point *A*, facing the moon. Tide producing force F_M of the moon on unit mass at *A* is given by

$$F_M = KM_M\left[\frac{1}{(D_M-R)^2}-\frac{1}{D_M^2}\right] = KM_M\left[\frac{(D_M^2-D_M^2-R^2+2D_MR)}{(D_M-R)^2\,D_M^2}\right]$$

$$= KM_M\left[\frac{R\,(2D_M-R)}{(D_M-R)^2\,D_M^2}\right]$$

Assuming radius of the earth *R* very small in comparsion to the distance between earth and moon, we have

$$F_M \approx KM_M\left(\frac{2R}{D_M^3}\right) \quad ...(6.11)$$

Similarly, tide producing force F_S of the sun on unit mass at *A* is given by

$$F_S \approx KM_S\left(\frac{2R}{D_S^3}\right) \quad ...(6.12)$$

Hence $$\frac{F_S}{F_M}=\frac{M_S}{M_M}\left(\frac{D_M}{D_S}\right)^3 \quad ...(6.13)$$

Now mass of sun, $M_S = 331{,}000\, M_E$

Mass of moon, $M_M = \frac{1}{18} M_E$

$D_S = 149{,}350{,}600$ km ; $D_M = 384{,}630$ km

Substituting the values in Eq. 6.13, we get

$$\frac{F_S}{F_M} = 0.458 \quad ...(6.14)$$

Hence solar tide = 0.458 lunar tide.

3. Combined effect : Spring and neap tides

Equation 6.14 shows that the solar tide force is less than half the lunar tide force. However, their combined effect is important, specially at the new moon when both the sun and moon have the same celestial longitude, they cross a meridian at the same instant. Assuming that both the sun and moon lie in the same horizontal plane passing through the equator, the effects of both the tides are added, giving rise to *maximum* or *spring tide of new moon*. The term 'spring' does not refer to the season, but to the springing or waxing of the moon. After the new moon, the moon falls behind the sun and crosses each meridian 50 minutes later each day. In after $7\frac{1}{2}$ days, the difference between longitude of the moon and that of sun becomes 90°, and the moon is in quadrature as shown in Fig. 6.26 (*b*). The crest of moon tide coincides with the trough of the solar tide, giving rise to the *neap tide of the first quarter*. During the neap tide, the high water level is below the average while the low water level is above the average. After about 15 days of the start of lunation, when full moon occurs, the difference between moon's longitude and of sun's longitude is 180°, and the moon is in opposition. However, the crests of both the tides coincide, giving rise to *spring tide of full moon*. In about 22 days after the start of lunation, the difference in longitudes of the moon and the sun becomes

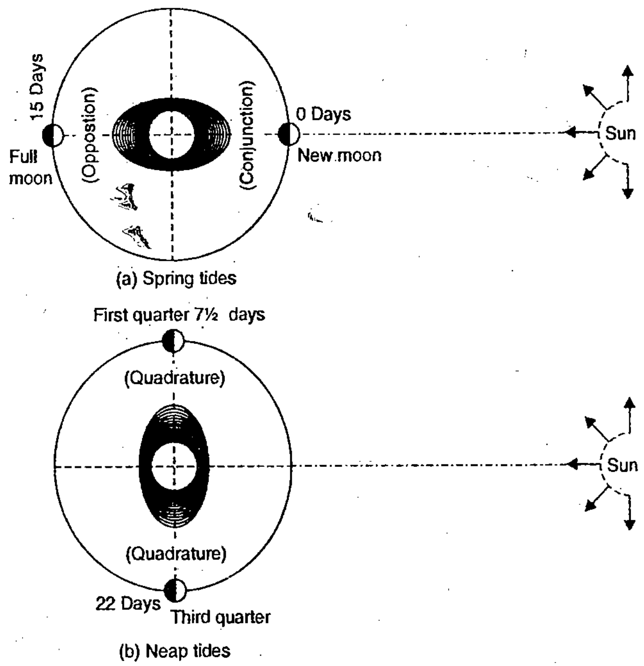


FIG. 6.26. SPRING AND NEAP TIDES.

270° and *neap tide of third quarter is formed*. Finally, when the moon reaches to its new moon position, after about $29\frac{1}{2}$ days of the previous new moon, both of them have the same celestial longitude and the spring tide of new moon is again formed making the beginning of another cycle of spring and neap tides.

4. Other Effects

The length of the tidal day, assumed to be 24 hours and 50.5 minutes is not constant because of (*i*) varying relative positions of the sun and moon, (*ii*) relative attraction of the sun and moon, (*iii*) ellipticity of the orbit of the moon (assumed circular earlier) and earth, (*v*) declination (or deviation from the plane of equator) of the sun and the moon, (*v*) effects of the land masses and (*vi*) deviation of the shape of the earth from the spheroid. Due to these, the high water at a place may not occur exactly at the moon's upper or lower transit. The effect of varying relative positions of the sun and moon gives rise to what are known as *priming of tide* and *lagging of tide*.

At the new moon position, the crest of the composite tide is under the moon and normal tide is formed. For the positions of the moon between new moon and first quarter, the high water at any place occurs *before* the moon's transit, the interval between successive high water is less than the average of 12 hours 25 minutes and the tide is said to prime. For positions of moon between the first quarter and the full moon [Fig. 6.27 (*b*)], the high water at any place occurs *after* the moon transits, the interval between successive

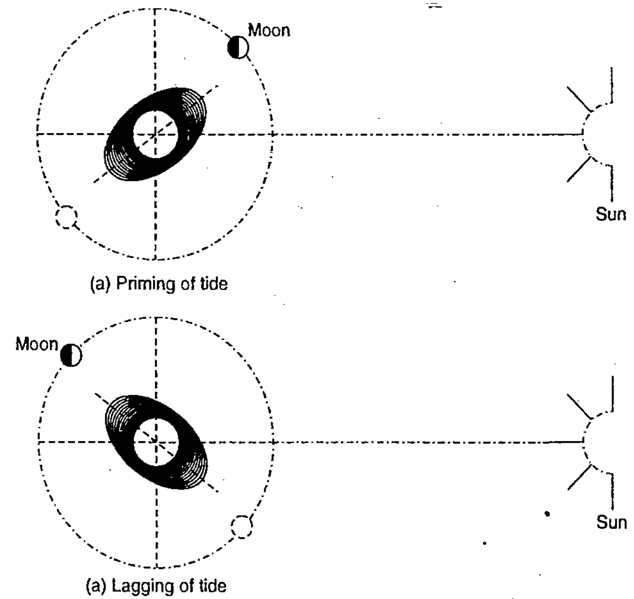


FIG. 6.27. PRIMING AND LAGGING.

high water is more than the average, and tide is said to lag. Similarly, between full moon and 3rd quarter position, the tide primes while between the 3rd. quarter and full moon position, the tide lags. At first quarter, full moon and third quarter position of moon, normal tide occurs.

Due to the several assumptions made in the equilibrium theory, and due to several other factors affecting the magnitude and period of tides, close agreement between the results of the theory, and the actual field observations is not available. Due to obstruction of land masses, tide may be heaped up at some places. Due to inertia and viscosity of sea water, equilibrium figure is not achieved instantaneously. Hence prediction of the tides at a place must be based largely on observations.

6.9. PREDICTION OF TIDES

The two elements required in the prediction of tide at a place are : (*i*) time of occurrence of tide and (*ii*) height of tide above datum. There are two principal methods of tide prediction :

1. Prediction by use of non-harmonic constants.
2. Prediction by use of harmonic constants.

1. PREDICTION BY USE OF NON-HARMONIC CONSTANTS

The various non-harmonic constants that are used for prediction of tide at a place are (*a*) age of tide, (*b*) lunitidal interval, (*c*) mean establishment, and (*d*) vulgar establishment.

(*a*) **AGE OF TIDE**

In the equilibrium theory, the earth is assumed to be enveloped with sea of uniform depth. This condition is fulfiled only in Southern Ocean extending southwards from about 40° S latitude. Therefore, it is only in this portion of ocean where equilibrium figure may be developed. *Primary tide waves* are, therefore, generated there and *derivative* or *secondary waves* are propagated into the Pacific, Atlantic and Indian Oceans. These derivative waves proceed in a general north and south direction, though their direction is influenced by the form of coast lines, and intervention of land masses. The velocity of wave travel may exceed 1000 km per hour, though it is less in shallow water. The amplitude, *i.e.*, the vertical range from crest to trough, is not more than 60 to 90 cm. *Due to the direction of propagation of tide wave, high or low water occurs at different times at various places on the same meridian.* Thus, the greatest spring tide arrives several tides after transits at new or full moon. *The time which elapse between the generation of spring tide and its arrival at the place is called the age of the tide at that place.* The age of tide varies for different places, upto a maximum of 3 days, and is reckoned to the nearest $\frac{1}{4}$ day. It is obtained as the mean of several observations. The *age of the tide* is one of the non-harmonic constants and its values for different ports are published in section I of part II of the Admiralty Tide Table.

(*b*) **LUNITIDAL INTERVAL**

Lunitidal interval is the time interval that elapses between the moon's transits and the occurrence of the next high water. The value of lunitidal interval is found to vary because of existence of priming and lagging. The values of lunitidal interval can be *observed* and if they are plotted for a fortnight against the times of moon's transits, a curve such as shown in Fig. 6.28 is obtained. Such a curve has approximately the same form for each fortnight and hence may be used for the rough prediction of time of tide at a place. The time of transit of moon at Greenwich is given in the *Nautical Almanac*. The time of transit at the given place can be derived by adding 2 m for every hour of west longitude

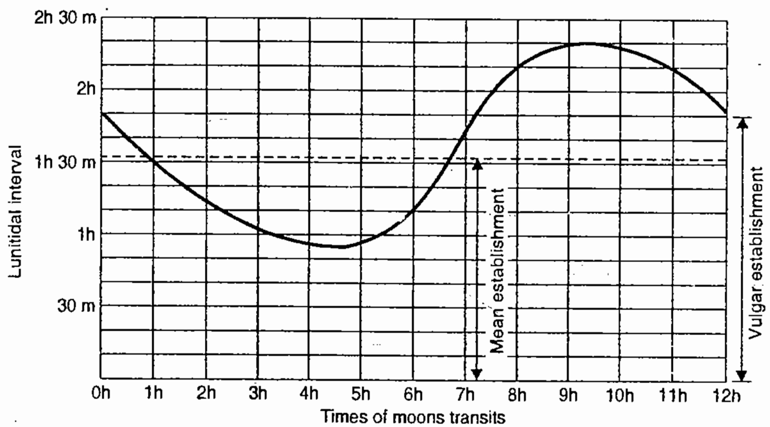


FIG. 6.28. LUNITIDAL INTERVAL.

and subtracting 2 m for every hour of east longitude of the place, to the time of transit at Greenwich. Knowing the time of moon's transit at the place, lunitidal interval is obtained from the curve (Fig. 6.28) and added to the time of preceding transit to know the approximate time of occurence of next high water at the place.

(*c*) **MEAN ESTABLISHMENT**

The average value of lunitidal interval at a place is known as its *mean establishment*, as shown by dotted line in Fig. 6.28. If the value of mean establishment is known, the lunitidal interval and hence the time of high water at a place can be estimated, provided the age of the tide at the place is also known. The procedure of determination is as follows :

1. Find from the charts, the age of tide and mean establishment for the place.

2. Knowing the hour of moon's transit at the place, on the day in question, determine the time of moon's transit on the day of generation of the tide (the day of generation of tide is equal to the day in question minus the age of the tide).

3. Corresponding to the time of transit of moon on the day of generation of tide (determined in step 2), find out the amount of priming or lagging correction from the table given below :

Hour of moon's transit	0	1	2	3	4	5	6	7	8	9	10	11	12
Correction in minutes	0	– 16	– 31	– 41	– 44	– 31	0	+31	+44	+41	+31	+16	0

4. Add algebraically the priming or lagging correction to the mean establishment to get the lunitidal interval for the day in question.

5. Add the lunitidal interval to the time of moon's transit on the day in question, to get the approximate time of high water.

Example 6.2. *Find the time of afternoon high water at a place with the following data :*

(*i*) *time of moon's transit on that day* = 4 h 40 m

(*ii*) *mean establishment* = 3 h 10 m

(*iii*) *age of tide* = 2 *days*.

Solution : We know that moon falls behind the sun at the rate of 50 m per day. Hence at the birth of tide, 2 days earlier, the time of moon's transit = 4 h 40 m – 2 × 50 m = 3 h 0 m.

From the table, corresponding to the time of transit of 3 h 0 m the correction for priming = – 41 m.

∴ Lunitidal interval = mean establishment – correction

= 3 h 10 m – 41 m = 2 h 29 m

∴ Time of high water

= Time of moon's transit + Lunitidal interval

= 4 h 40 m + 2 h 29 m = 7 h 09 m = **7.09 P.M.**

(*d*) VULGAR ESTABLISHMENT

Vulgar establishment is defined as the value of lunitidal interval on the day of full moon or change of moon. Its value is always more than establishment since the lagging correction in the second or fourth quadrant is positive. The difference between vulgar establishment and mean establishment depends upon the age of the tide. The value of vulgar establishment is approximately equal to the clock time at which high water occurs on the days of full moon or change of moon. Admiralty Tide Tables give the value of this non-harmonic constant for all principal ports. If the vulgar establishment is known, the mean establishment can be known for that place by the relation :

Mean establishment* = *vulgar establishment – lagging correction.

For finding the lagging correction from the Table, the age of tide must be known. Suppose the age of tide is 2 days. Then time of transit of moon on the day of generation of tide = 12 h – 2 × 50 m = 10 h 20 m. Corresponding to this time of transit of moon, lagging correction comes out to be 26 m. Hence for the place at which age of tide is 2 days, we have

***Mean establishment* = *vulgar establishment* – 26 m**

Thus, mean establishment is known. Once this is known, the time of tide on any other day can be determined by the procedure described in the previous para.

Height of tide

Another item in the prediction of tide is the estimation of the height of rise of tide, *i.e.*, vertical distance of the high water level above some suitable reference datum. Commonly adopted datum is *the low water of ordinary spring tides* for the place. The vertical distance from the low water level to the succeeding high water level is known as the *range of the tide*. The rate of variation of water level is small at high or low water and greatest at half tide. The approximate height of tide of known rise or range, at any time between high and low water can be ascertained from the following expression:

$$H = h + \frac{1}{2} r \cos \theta \qquad \ldots(6.15)$$

where H = required height of tide above datum

h = height of mean tide level above datum

r = range of the tide

$$\theta = \frac{\text{interval from high water}}{\text{interval between high and low water}} \times 180°$$

2. PREDICTION BY USE OF HARMONIC CONSTANTS

Prediction of tide with the help of non-harmonic constants is not very much used because the results obtained from these are often erroneous. Modern practice is to use harmonic constants. There are more than 36 tidal constituents of harmonic type. Out of these, the values of 10 important constituents are given in part II (Section II) of Admiralty Tide Tables, for different ports. These constituents, along with their symbols are given in the table given at next page.

For prediction of tide, the following expression is used :

$$V = f H \cos (E - g) \qquad \ldots(6.16)$$

Symbol for constiuent	*Description or name*	*Period*
M_2	Lunar semi-diurnal	$\frac{1}{2}$ Lunar day
S_2	Solar semi-diurnal	$\frac{1}{2}$ Solar day
N_2	Larger elliptic-semi-diurnal	—
K_2	Luni-solar semi-diurnal	$\frac{1}{2}$ Sideral day
K_1	Luni-solar diurnal	Sideral day
O_1	Larger diurnal (declinational)	—
P_1	Solar diurnal (declinational)	—
M_4	First overtake of semi-diurnal	$\frac{1}{4}$ Lunar day
MS_4	Compound luni-solar $\frac{1}{2}$ diurnal	—

where V = value of constituent at zero hour on the day in question.

H = mean amplitude (half range) of the constituent at the port in question.

f = factor, the value of which is very near to unity, and which varies slowly from year to year.

E = angle (same for all ports).

g = constant, special to the port and the constituent.

For a particular port, the harmonic constants are : mean sea level (A_0) and the values of H and g for the various constituents.

To determine the value of E at zero hour, we have

$$E \text{ (at zero hour)} = m + d \qquad \ldots(6.17)$$

where m = value of E at zero hour of the first day of each month

d = increment in E from zero hour of the first day of the month to the zero hour of the day in question.

The values of m, d, f, H etc. can be easily obtained for each constituent. Hence hourly heights can be easily obtained for each constituent. The height of the tide above the port datum at any hour will be equal to the mean sea level (A_0) plus the sum of height for the different constituents for that hour.

Alternatively, a *tide predicting machine* may be used. The various separate harmonic motions corresponding to the harmonic constituents are traced out by some suitable mechanism and their combined effect can be obtained in a graphical form. The tide curve so traced out gives heights and times of high and low waters.

6.10. TIDE GAUGES

The height of high and low waters, and its variations with time can be measured at site with the help of tide gauges. Following are some of the common types of tide gauges used :

1. Non-registering type of tide gauges

(*i*) Staff gauge (*ii*) Float gauge (*iii*) Weight gauge.

2. Self-registering type tide gauges.

Non-registering type tide gauges are those in which an attendant is required to take reading from time to time. In the self-registering type, no attendant is required.

1. Staff gauge. [Fig. 6.29 (*a*)]. This is the simplest type of gauge, which is firmly fixed in vertical position. The gauge consists of a board, about 15 cm to 25 cm broad, and of suitable height, having graduation to a least count of 5 to 10 cm. The zero of gauge is fixed at the predetermined level. Alternatively, the elevation of zero of the level may be determined by levelling. The staff is read directly, from some distance.

2. Float gauge. [Fig. 6.29 (*b*)]. On account of the wash of the sea, it may be difficult to read a staff gauge accurately. In that case a float gauge shown in Fig. 6.29 (*b*) may be used. It consists of a simple float with a graduated vertical rod, enclosed in a long wooden box of 30 cm ×30 cm square section. The box has few holes at the bottom through which water may enter and lift the float. The reading are taken through a slit window against some suitable index.

3. Weight gauge. The weight gauge, shown in Fig. 6.29 (*c*) consists of a weight attached to a wire or chain. The chain passes through a pulley, along the side of a graduated board. The weight is lowered to touch the water surface and the reading is taken against

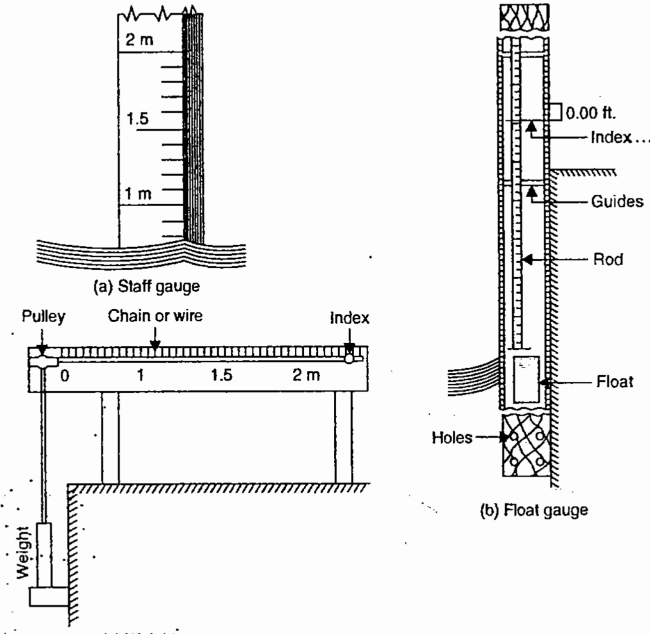


FIG. 6.29. NON-SELF REGISTERING TYPE TIDE GAUGES.

an *index* attached to the chain. The reduced, level of the water surface corresponding to the zero reading is determined earlier, by attaching the foot of the staff against the bottom of the suspended weight and taking its reading with a level, when the index of the chain is against zero reading.

Self-registering gauges

A self-registering gauge automatically registers the variation of water level with time. It essentially consists of a float protected from wind, waves etc. The float has attached to it a wire or cord which passes over a wheel (called the float wheel) and is maintained at constant tension by some suitable arrangement. The movement of the float is transferred to the wheel which reduces it through some gear system, and is finally communicated to a pencil attached to a lever. The movement of the pencil, corresponding to the movement of the float is recorded on a graph paper wound round a **drum** which is rotated at constant speed by some suitable clock-work. Thus graphical record of movement of the float with time is recorded automatically. Such a gauge is usually housed in a well constructed under a building so that effect of wind and other disturbances is reduced.

6.11. MEAN SEA LEVEL AS DATUM

For all important surveys, the datum selected is the mean sea level at a certain place. *The mean sea level may be defined as the mean level of the sea, obtained by taking the mean of all the height of the tide, as measured at hourly intervals over some stated period covering a whole number of complete tides,* The mean sea level, defined above shows appreciable variations from day to day, from month to month and from year to year. Hence the period for which observations should be taken depends upon the purpose for which levels are required. The daily changes in the level of sea may be more. The monthly changes are more or less periodic. The mean sea level in a particular month may be low while it may be high in some other months. Mean sea level may also show appreciable variations in its annual values. Due to variations in the annual values and due to greater accuracy needed in modern geodetic levelling, it is essential to base the mean sea level on observations extending over a period of about 19 years. During this period, the moon's nodes complete one entire revolution. The height of mean sea level so determined is referred to the datum of tide gauge at which the observations are taken. The point or place at which these observations are taken is known as a tidal station. If the observations are taken on two stations, situated say at a distance of 200 to 500 kms on an open coast, one of the station is called primary tidal station while the other is called secondary tidal station. Both the stations may then be connected by a line of levels.

PROBLEMS

1. In a harbour development scheme at the mouth of a tidal river, it has been found necessary to take soundings in order to buoy the navigation channel.

Explain clearly how you would determine the levels of points on the river bed and fix the positions of the soundings.

(a) by use of sextant in a boat :

(b) by use of the theodolite on the shore. (*U.L.*)

2. Describe briefly the location of sounding stations by means of (*a*) cross rope soundings, (*b*) intersecting ranges.

From a stationary boat, off-shore sextant readings are taken to three signals *A*, *B*, *C* on land and the measured angles subtended by *AB* and *BC* are 32° 30′ and 62° 30′ respectively. The positions of the three shore signals are such that $AB = 300$ m, $BC = 512.5$ m and the angle *ABC* on the landward side is 233° 30′. Determine graphically the distance of the boat from *B*.

The boat is now moved in-shore and sextant readings again taken, with boat stationary, to *A*, *B* and *C* and it is found that the angles now subtended by *AB* and *BC* are 90° 00′ and 113° 30′ respectively. Determine graphically the distance between the two stationary positions of the boat at which soundings are taken. Use scale of 1 cm = 500 m.

3. In a triangulation survey it becomes necessary to incorporate a station *S* not in the original net, and its position is determined by angular observations on three visible stations *P*, *Q* and *R*, the total co-ordinates of which are appended, with the two horizontal angles observed from *S*.

Station	*Latitude*	*Departure*	*Angle*
P	+ 18,400	+ 72,800	
			52° 12′ 20″
Q	+ 18,400	+ 94,600	
			68° 30′ 15″
R	+ 2,200	+ 107,400	

Determine analytically the co-ordinates of the station *S*. (*U.L.*)

4. In making a survey for a new town it was necessary to fix a new control point, *D*, by resection from three known points *A*, *B*, *C*. From the co-ordinates of the latter the following data were obtained :

Length *AB* = 701.5 m
Length *BC* = 741.5 m
Angle *ABC* = 125° 50′ 58″.

The following angles were measured by theodolite *D* : *ADB* 53° 31′ 54″ and *BDC* 61° 39′ 39″.

(*a*) How would you decide whether point *D* could be determined without ambiguity ?

(*b*) State briefly any method you known for determining the co-ordinates of *D*.

(*c*) Calculate angles *BAD* and *BCD*.

The angles at *C* and *D* are the internal angles of the quadrilateral.

ANSWERS

2. 532.5 m ; 432.5 m
3. Lat. − 1120 ; Dep. + 91934.
4. 61° 10′ 12″ ; 57° 47′ 17″ .

7

Mine Surveying

(UNDERGROUND SURVEYS)

7.1. GENERAL

The general and basic principles of underground surveys meant for mines and tunnel setting out are almost the same as for surface except for the fact that the conditions under which the work is to be done are entirely different. The special conditions confronted in the underground surveys make the following changes :

(*i*) Due to limitations of space, small instruments of special designs with extension tripod legs or suspension rods are used.

(*ii*) Due to very short sights, and sometimes very steep or vertical, special methods of observations are necessary with particular care to avoid the accumulation of excessive errors in measurements.

(*iii*) Usually the transit station is in the roof (*i.e.* above the transit), and hence the procedure of traversing is modified.

(*iv*) Due to darkness, special arrangements for illumination of both the instrument as well as the target are necessary.

(*v*) Usually, the distances are measured on slopes, the traverse measurements include vertical angles and hence it is necessary to determine the three rectangular co-ordinates of all instrument stations.

7.2. EQUIPMENT FOR MINE SURVEYS : THE TRANSIT

The mine transit (Fig. 7.1) is usually of a smaller size than the ordinary instrument. Special provisions are, however, made for steep or vertical sights. Due to very steep sights (say more than 50° or 60°) the horizontal circle of the ordinary transit will obstruct the pointings of the telescope of an ordinary transit. To overcome this difficulty, an *auxiliary telescope* is attached either at one end of the horizontal axis or above the main telescope and at a distance there-from somewhat more than one-half of the diameter of the horizontal plate. The two mountings are arranged in such a way that the auxiliary telescope is interchangeable between the top and side positions. In each position a counterpoise is attached to keep the telescopes in balance. In either position, the line of sight of the auxiliary telescope is parallel to that of the main telescope. For steep sights upward, a prismatic eye-piece is attached to the main telescope. The instrument is generally mounted on an extension leg tripod. For ease in reading the vertical angles by the transitman, the vertical circle is sometimes graduated on the edge instead of the side. The centre point of the transit is definitely marked on the top of the telescope.

In places where a tripod cannot be used, suspension type mine transit is employed. Fig. 7.2 shows a typical *suspension theodolite* by Funnel Kassel. The instrument is supported on a bracket being screwed horizontally into adjacent mine timbers. The horizontal circle along with its verniers are on the top of the telescope and the vertical circle, and hence the use of auxiliary telescope is obviated. The horizontal circle is 9 cm ($3\frac{1}{2}$ in.) in diameter, the vertical circle 7 cm ($2\frac{3}{4}$ in.) in diameter and the whole instrument weighs only 5.5 lb. If required, the instrument can also be supported on tripod, and for this purpose the vertical circle and the telescope are provided with sensitive *reversion spirit levels*.

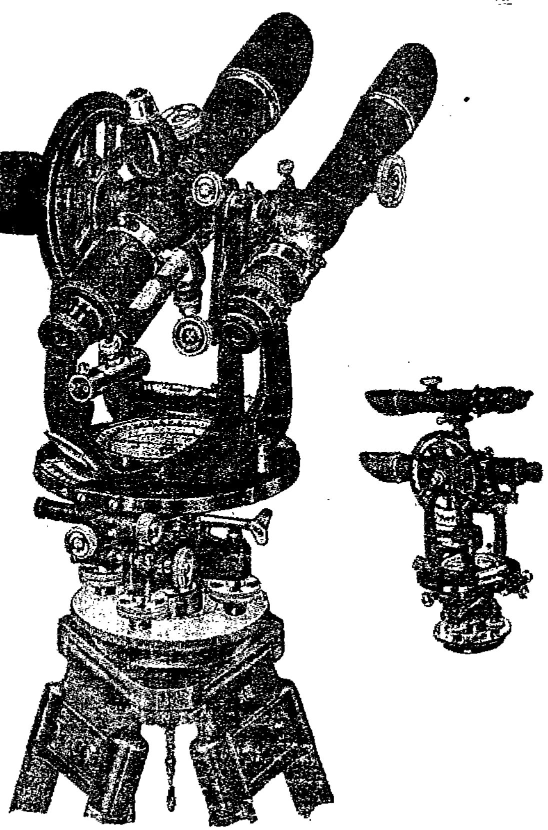


FIG. 7.1. MINING TRANSIT SHOWING THE AUXILIARY TELESCOPE IN BOTH POSITIONS.

The Correction for Side-Telescope Horizontal Angles

The side telescope is fitted at a slight distance away from the main telescope ; this eccentricity affects the horizontal angles measured with the auxiliary telescope.

Thus, in Fig. 7.3, *O* is the centre of the main telescope and *C* is the centre of the auxiliary telescope at distance *OC* from the main. The circle denotes the locus of the centre *C* of the auxiliary telescope when the instrument is revolved in azimuth for the measurement of the horizontal angle. *A* and *B* are the two points, and it is required to measure the true horizontal angle θ between these two subtended at the centre of the instrument. When the point *A* is sighted through the auxiliary telescope, the line of sight *CA* is tangential to the circle, reading on the horizontal circle being zero-zero. To sight the point *B*, the instrument is rotated in azimuth, so that the centre *C* of the auxiliary telescope comes to a position *C'*, the line *C'B* being the line of sight tangential to the

circle. The measured angle θ' is then the angle through which the line of sight has been rotated.

Evidently, the correct angle θ is given by the relation

$$\theta + \alpha = \theta' + \beta$$

or $$\theta = \theta' + (\beta - \alpha) \quad ...(i)$$

where $$\alpha = \sin^{-1}\frac{OC}{AO} = \tan^{-1}\frac{OC}{AC} \quad ...(ii)$$

and $$\beta = \sin^{-1}\frac{OC'}{OB} = \tan^{-1}\frac{OC'}{BC'} \quad ...(iii)$$

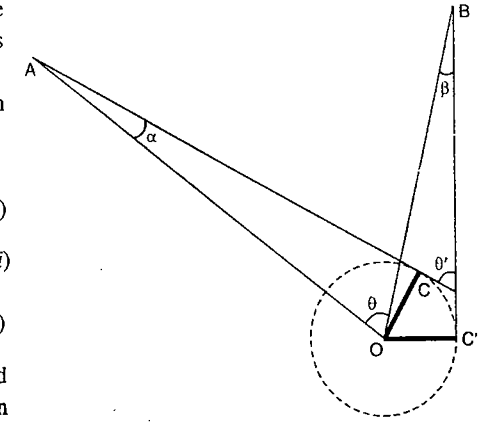


FIG. 7.3. CORRECTION FOR SIDE-TELESCOPE HORIZONTAL ANGLES.

Thus, the correct angle θ can be computed by applying, algebraically, the correction (β – α) to the observed angle θ'. However, the observed angle can be directly equal to the true angle, if :

(*i*) both the sights *OA* and *OB* are of equal length, thus making α and β equal;

(*ii*) by taking both face observations, one with telescope direct and the other with telescope reversed, the mean reading being used to give the true angle.

The above correction for horizontal angle is, however, not necessary if the auxiliary telescope is fitted on the top of the main telescope.

The Correction for Top-Telescope Vertical Angles

If the auxiliary telescope is fitted to the side of the main telescope, the horizontal angles need correction while the vertical angles do not need the correction. However, if the auxiliary telescope is fitted to the top of the main telescope, the vertical angle needs correction.

Thus, in Fig. 7.4, *A* is the point to be observed, *B* and *C* are the centres of the main and auxiliary telescopes, respectively. α' is the vertical angle measured with the aux-

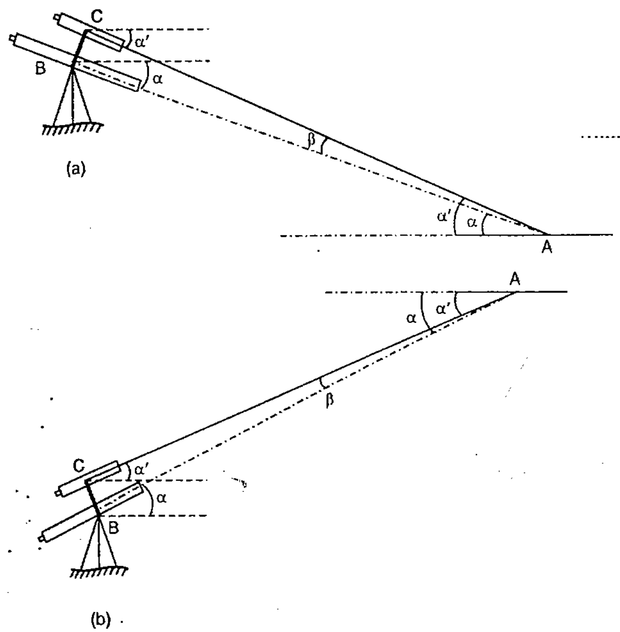


FIG. 7.4. CORRECTION FOR TOP TELESCOPE VERTICAL ANGLE.

iliary telescope and α is the true vertical angle to the point A.

Evidently, $\alpha = \alpha' - \beta$ (For angles of depression) ...(*iv*)

or $\alpha = \alpha' + \beta$ (For angles of elevation) ...(*v*)

where $\beta = \sin^{-1} \frac{BC}{AB}$ ...(*vi*)

The correction β varies inversely as the distance AB, since BC is the distance of the auxiliary telescope from the main and is constant. Tables are usually prepared or available before hand giving the value of β for different values of the inclined distance AB.

The process of computing the true horizontal and vertical angles are sometimes termed as 'Reduction to Centre'

Example 7.1. *The horizontal angle between two points A and B observed with the side telescope (auxiliary) of a mining transit is 54° 18′. The distance between the centres of the main and auxiliary telescopes is 6 cm. The distances of A and B from the auxiliary telescope are 30.25 and 18.32 metres respectively. Reduce the angle to the centre.*

Solution.

The correction $\alpha = \tan^{-1} \frac{OC}{AC} = \tan^{-1} \frac{6}{100 \times 30.25} = 0° \, 7'$

The correction $\beta = \tan^{-1} \frac{OC'}{BC'} = \tan^{-1} \frac{6}{100 \times 18.32} = 0° \, 11'$

Hence $\theta = \theta' + (\beta - \alpha) = 54° \, 18' + (0° \, 11' - 0° \, 7') = \mathbf{54° \, 22'}$.

Example 7.2. *The vertical angle observed with a top telescope (auxiliary) is 25° 18′. The distance between the centres of the main and auxiliary telescope is 6 cm. The inclined distance from the centre of the main telescope to the point observed is 60.32 metres. Compute the true vertical angle if the observed angle is (a) angle of depression, (b) angle of elevation.*

Solution.

The correction $\beta = \sin^{-1} \frac{BC}{AB} = \sin^{-1} \frac{6}{100 \times 60.32} = 0° \, 8'$

(*a*) For angle of depression,

the true vertical angle $= \alpha' - \beta = 25° \, 18' - 0° \, 8' = \mathbf{25° \, 10'}$

(*b*) For angle of elevation,

the true vertical angle $= \alpha' + \beta = 25° \, 18' + 0° \, 8' = \mathbf{25° \, 26'}$.

7.3. THE STATIONS AND STATION MARKERS

The stations used in mine surveying are chosen at suitable points and are located either in the roof or on the floor. The floor stations are more convenient, though there is always danger of their being displaced or lost. The roof stations, though incovenient, are therefore, preferred. In the case of a floor station, a spike or a nail in a tie or a wooden plug are driven into the holes drilled in the floor. The marks may also be made in brass nails set in the stout stakes driven in the floor. The stakes should be surrounded

by brickwork plastered over with cement flush with the top of the stake. The roof marker usually consists of a wooden plug from 2 to 5 cm diameter, driven to a tight fit into a hole from 10 to 15 cm long, drilled into the rock. The exact point is marked by a bent nail, a stable, a brass screw eye or a spad of some kind. The roof marker must have some provision for suspending plumb bobs or lamps from these nails. *These marks serve as instrument stations.* The markers should be of non-rusting material, and they should be referenced to nearby objects to detect any movement due to operations in the mine.

Illumination.

The following are various kinds of illuminated signals used for sighting underground:

(*i*) A plumb line seen against a white background of a sheet of oiled paper from behind by a suitable lamp. The device is most suitable for short sights.

(*ii*) Carriage candles

(*iii*) An Argand oil lamp of about 50 candle power. Both candles and lamps are supported in suitable metal frames, which are adjusted in position until the axis of the frame is vertically under the point of suspension.

(*vi*) A plummet lamp.

7.4. MEASUREMENT OF DISTANCE AND DIFFERENCE IN ELEVATION

In mine surveys, the distances are usually measured on the slope due to great difference in elevation between the instrument station and the point under observation. The horizontal and vertical distances are then computed from the known vertical angle and the inclined distance. The tape generally used is 100 to 200 ft long, a steel tape being preferred.

To measure the inclined distance between the instrument and the point, a plumb bob is hung from the instrument station to the centre of the instrument (set below it) and the distance is measured precisely along it. This distance is known as the *height of the instrument* and is positive if the instrument is above the floor station and is negative if the instrument is below the roof station.

The plumb bob is then transferred to the point and is hung from it. A suitable mark is made at some convenient distance along it, the distance being known as the *height of the point* (H.P.). The angle of elevation (or depression) is then measured to this point, and the distance is measured along the inclined line joining the centre

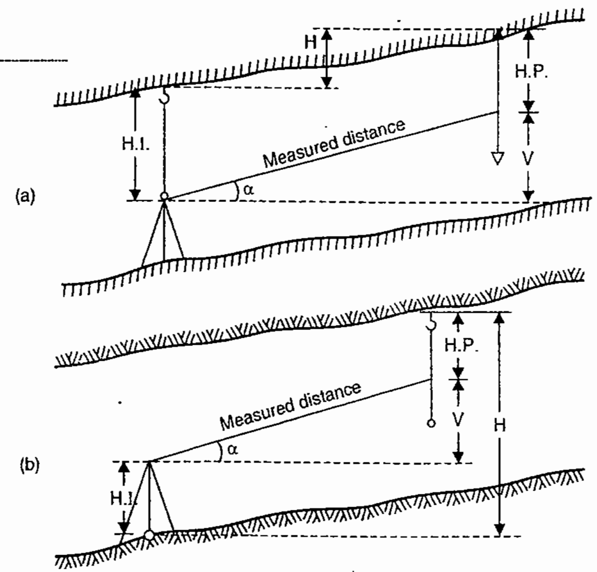


FIG. 7.5.

of the instrument to the mark made on the plumb line. The height of point (H.P.) is considered to be positive when it is measured above the floor and is negative when measured below the roof.

The difference in height between the instrument station and the point sighted can then be calculated.

Thus in Fig. 7.5 (*a*), the difference in height (H) is given by

$$\text{H.I.} + H = \text{H.P.} + V \quad \text{or} \quad H = (\text{H.P.} - \text{H.I.}) + V \qquad \ldots(i)$$

In Fig. 7.5 (*b*)

$$H = (\text{H.P.} + \text{H.I.}) + V \qquad \ldots(ii)$$

In both the expressions, $V = L \sin \alpha$ ...(*iii*)

where L is the measured inclined distance.

The horizontal distance D is given by

$$D = L \cos \alpha \qquad \ldots(iv)$$

7.5 TUNNEL ALIGNMENT AND SETTING OUT

Tunnels are usually constructed in the mountainous districts when a section of the road is subject to avalanches, and they not only protect the road, but serve as places of refuge for travellers. In cities, tunnels are sometimes employed for underground railways or roads to relieve the traffic congestion. Tunnels are also used in mining operations. They are sometimes constructed under rivers, where the construction of a bridge is considered undersirable or impracticable. In the case of mountainous railways, they are employed either to have the shortest route between two points or where the cost of cutting is expensive.

Tunnels are entered either on the level or by inclines. For the purpose of facilitating the construction of the operations, and for checking the accuracy of the alignment and levels, *vertical shafts* are often used. For proper drainage, a tunnel may be made slightly inclined to the horizontal, the gradient being in one direction if the tunnel is short, and in both the directions from the centre if it is long.

The setting out of a tunnel comprises four operations :

(*i*) Surface surveys or setting out

(*ii*) The connection of surface and underground surveys

(*iii*) Setting out underground

(*iv*) Level in tunnels.

Surface Alignment and Measurements

The centre line of the proposed tunnel should be accurately marked on the surface of the ground whenever it is possible. In the case of high snow-clad mountain ranges, this may not be possible. In such cases, the centre line must at least be set out over the contiguous shafts near the ends of the tunnel. The shaft at the extremities of the tunnel must be connected by triangulation or precise traverse with greatest possible care to ensure accuracy.

To set out the centre line on the surface, specially for a tunnel straight in plan, a suitable point is chosen on the centre line from which both the extremities can be commanded. An observatory is erected, and an instrument is erected in it, the instrument being centered exactly over the centre line. Two points in the preliminary setting out are taken as fixed, one at the observatory and the other being some conveniently situated point in the line. From the main station at the observatory, the line is set on suitable permanent objects at the ends of the tunnel and near to each shaft. In towns, the centre line is marked on the surface by driving spikes or wedges of iron, the centre line being marked on these with a steel punch. In order that these may not be replaced, measurements are taken from the corners of buildings or permanent marks. In the coverby, the centre line on the surface may be marked by stout pegs having brass nails driven into them, the exact line being marked on these with a steel punch.

The exact horizontal distance between two terminals of the tunnel is then measured. An accurate steel tape must be used, and all the corrections must be applied to the observed distance to get the correct distance. For very accurate results, the distance may be measured by the usual equipment used for base measurement in triangulation. The corrections for tension, temperature, grade, sag and absolute length are applied in the usual way to obtain the true horizontal length of the centre line. In case it is not possible to measure the distance directly due to obstacles etc. then length and direction of the centre line of the tunnel must be obtained by precise traversing or triangulation.

Transferring Surface Line Down Shafts and Setting Out Underground Line

After having fixed the centre line on the surface the setting out of underground line can be done by transferring surface line down the shafts wherever they are vertical. The points are selected in the centre line near the mouth of each shaft in a position clear of the works in connection with the sinking operations. A theodolite is then set over one of these points on the surface and the line of sight is directed towards the other point. The line is then set out accurately on two baulks of timber kept across the shaft perpendicular to the centre line and very near to the two edges of the shaft. From these marked points on the baulks, two plumb lines are suspended down the shaft [Fig. 7.6 (*a*)].

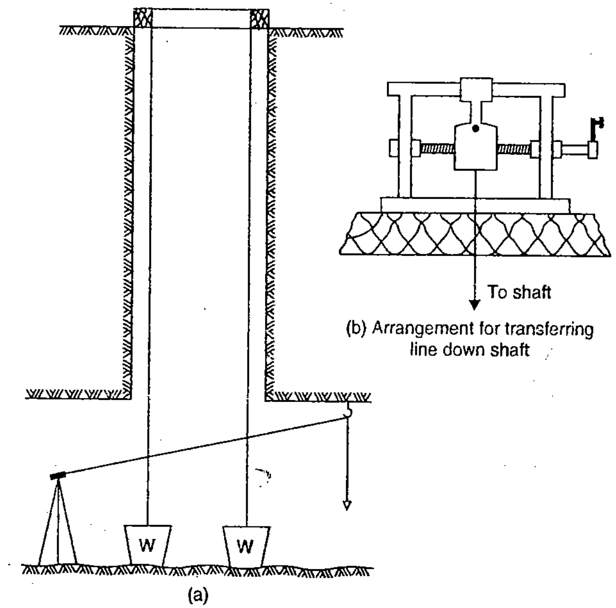


FIG. 7.6. TRANSFERRING SURFACE LINE UNDERGROUND.

The theodolite is then transferred underground and set exactly in line with the two suspended wires. The line joining these wires and hence the line of sight of that theodolite gives the direction of the centre line of the tunnel underground. The line is then set with the instrument on nails driven into convenient byates of timber from which plumb bobs or lamps may be suspended. The exact centre line is marked by steel punch or a file mark.

The plumb wires are fine wire stretched tight by attaching weight at their lower ends. In order to still their vibrations, the weights are suspended freely in a vessel of water. The wires must be so suspended that they do not touch the sides of the shaft. If the wires were permanently left suspended like this, there may be hindrances in mining operations underground. To avoid this, there should be some arrangement for removing them, and again placing them if required. Fig 7.6 (*b*) shows such an arrangement with the help of which the wire can be lifted up or lowered down.

Weisbach Triangle Method

Since line joining the two suspended wires gives the direction of the centre line, accurate setting can be achieved only if the theodolite can be set out exactly in the line with the wires. The operation is quite difficult and time consuming, and requires several trials. An alternative method, also known as *Weisbach triangle method*, may be used to connect surface and underground survey.

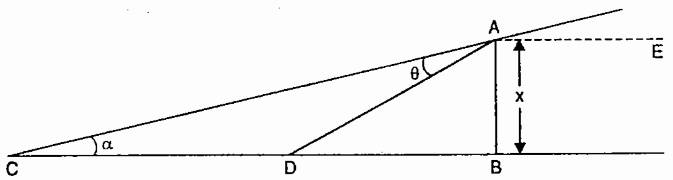


FIG. 7.7. THE WEISBACH TRIANGLE.

Thus in Fig. 7.7, *C* and *D* are the two wires (in plan) and *A* is the selected position of the instrument very near to the line *CD* and almost in line with it judged by eye. At *A*, the angle *CAD* (= θ) is measured very accurately by taking both face observations. The distances *CA* and *DA* are also measured precisely. As a check, the distance *CD* is also measured, which should be equal to the two corresponding marks on the baulks at the surface.

Through *A*, a line *AE* is set out parallel to *CB*, by usual methods. At *A*, a line *AB* is then drawn perpendicular to *AE*. The perpendicular *AB* = (*x*) should be of such length that *B* is in line with *CD*

Now since *A* is very nearly in line with *CD*, the angle *ACD* (= α) is extremely small. Hence,

$$\sin \alpha = \alpha \text{ (radians)} = \frac{AD}{CD} \sin \theta \qquad \ldots(i)$$

Hence $$AB = \text{deviation } x = CA \sin \alpha = CA \cdot \alpha = CA \frac{AD}{CD} \sin \theta \qquad \ldots(ii)$$

If the distance *x* is measured perpendicular to *AE* at *A*, the point *B* will be exactly in line with *CD* and can subsequently be adopted as the instrument station.

Transferring the Level Underground

The surface alignment should be followed with a net work of levels connecting the shafts. Whenever possible, the longitudinal section along the whole course of the surface alignment should be obtained. The bench marks thus established near each shaft should be checked carefully before the levels are transferred underground. At the ends of the tunnels, the levels can be transferred inside the tunnels by usual methods of levelling. For transferring the levels underground through shafts, various apparatus such as steel bands, chains, and specially constructed rods have been used for transferring the levels down vertical shafts.

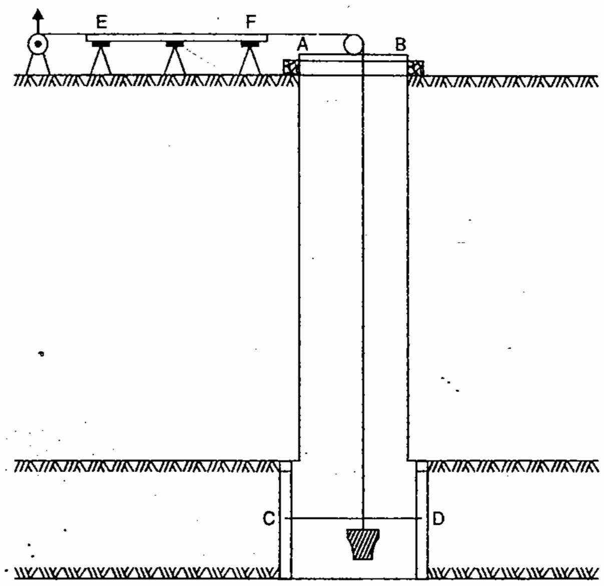


FIG. 7.8. TRANSFERRING LEVELS UNDERGROUND.

Fig. 7.8 illustrates a typical arrangement for transferring the levels underground through deep shafts. A fine steel wire loaded with a weight of 5 to 15 kg is passed over a pulley and is lowered down the shaft. Two fine wires *AB* and *CD* are stretched horizontally across the shaft at its top and bottom, and touching the suspended wire. A suitable mark is made against these points both at top and at the bottom. In order to ascertain the distance between these marks, the wire is wound up, without removing the stretching weight, and the distance between the marks on the wire is obtained as it passes over the surface of a horizontal plank *EF* suitably supported on trestles. The attached weight does not affect the measurement since the tension is constant throughout the whole operation. Thus, the levels of the marks at the bottom of the shafts can be ascertained. The level is then set up near the bottom of the shaft, and a permanent bench mark is establihsed.

Example. 7.3. *Explain the use of 'Weisbach triangle' for setting out underground. The centre line of a tunnel is represented by two plumb lines C and D, 4 metres apart, hanging vertically in a shaft, the whole circle bearing of the line CD being 80° 40′ 15″. A theodolite is set up underground at a point A, distant 3.902 m and roughly east of the nearer plumb line D, and the observed value of the angle CAD is found to be 16′ 12″. Calculate the bearing of the line CA and the perpendicular distance of A from the centre line of the tunnel.*

Solution. (Fig. 7.7)

$$\sin \alpha = 3.092 \frac{\sin 16' 12''}{4}$$

$\therefore$ $\alpha = 948'' = 15' 48''$

$\therefore$ Azimuth of $CA = 80° 40' 15'' + 15' 48'' = \mathbf{80° 56' 03''}$

and $x = AC \sin \alpha = AC. \alpha$, when α is in radians

$$= (4 + 3.902) \frac{948}{206265} = \mathbf{0.0363 \ m.}$$

7.6. SUSPENSION MINING COMPASS

It basically consists of a compass box connected with a suspension frame. The string of the suspension frame is set along the dip of the strata and its slope is measured with the help of a large diameter clinometer with plumb bob. Fennel Kessel manufactures two variations: (*i*) Kassel type and (*ii*) Freiberg type.

Fig. 7.9 shows the photograph of the Kassel type mining compass. The compass is connected by *hinges* with suspension frame which has the advantage of easy packing and taking less space in the container. The clamping screw of the knife edged magnetic needle is placed on the brim of the compass ring. The horizontal circle is divided at intervals of 1 degree and figured every 10 degrees. The clinometer, made from light metal, has diameter of 9.4 inch and is graduated to $\frac{1}{3}$ degree.

Freiberg type compass with clinometer is shown in Fig. 7.10. The functions of the mining compass of Freiberg type are exactly the same as of the Kassel type. Its mechanical features depart in two things from the Kassel type, *viz.*, the rigid connection of the compass suspension with the frame and the clamping screw to be placed centrically, under the compass box.

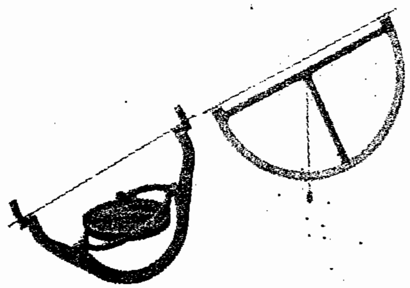


FIG. 7.10. FREIBERG TYPE MINING SUSPENSION COMPASS WITH CLINOMETER.

7.7. BRUNTON'S UNIVERSAL POCKET TRANSIT

Brunton's Universal pocket transit is one of the most convenient and versatile instrument for preliminary surveying on the surface or underground. It is suitable for forestry, geological and mining purposes, and for simple contour and tracing work. The main part of Brunton Pocket Transit is the magnetic compass with a 5 cm long magnetic needle pivoting on an agate cap. Special pinion arrangement provides for the adjustment of the local variation of the declination with a range of ± 30°. For accurate centring purposes a circular spirit bubble is built in. A clinometer connected with a tubular spirit bubble covers measurement of vertical angles within a range of ± 90° . Fig. 7.11 shows the photograph of Brunton Universal pocket transit along with box containing various accessories..

The Brunton pocket transit comprises a wide field of application for which it is equipped with the following special accessories :

1. ***Camera tripod*** for measurement of horizontal and vertical angles.
2. ***Plane table*** for using the compass as an alidade.
3. ***Protractor base plate*** for protracting work in the field or in the office.
4. ***Suspension plate*** for use of the instrument as a mining compass.
5. ***Brackets*** for suspension plate

Measurement of horizontal angles

Horizontal and vertical angles can be measured by using the camera tripod with the ball joint. For measuring horizontal angles, the compass box has to be screwed on the ball joint until the locking pin will fit into the socket which is imbedded in the compass case. For more precise centring, a plumb bob can be fastened at the plumb hook of the tripod. Accurate setting of the instrument is accomplished with a circular spirit bubble. The north end of the needle indicates magnetic bearing on the compass graduation.

Measurement of vertical angle

For measuring vertical angles, the compass has to be fitted in the ball joint. The observations have to be carried out with completely opened mirror by sighting through the hole of the diopter ring and the pointer. Before readings can be taken, the tubular bubble which is connected with the clinometer arm has to be centered by turning the small handle mounted at the back of the compass. Using the instrument in this vertical position, it is necessary to lock the needle to prevent the agate cap and the pivot from being damaged.

Use as a mining compass

Brunton compass can be fitted on the suspension plate and be used as mining compass. The compass is correctly positioned on the plate when the locking pin fits into the socket. Then, the North-South line of the compass is parallel to the longitudinal axis of the suspension plate.

For vertical angle measurements, the hook hinges have be fitted. The brackets prevent the suspension outfit from sliding along the rope. Before readings of vertical circle can be taken, accurate centring of the clinometer arm bubble is necessary.

Use with plane table

The compass in connection with the protector base plate can be used for protecting work in the field or in the office. The parallelism of the base plate edges and the line of sight of the compass is secured when the locking pin on the plate fits accurately into the socket. This combination gives the possibility to employ the compass as an alidade for minor plane table surveys.

7.8. MOUNTAIN COMPASS-TRANSIT

A mountain compass-transit (also known as compass theodolite) basically consists of a compass with a telescope. Both these are mounted on a levelling head which can be mounted on a tripod. For movement of the instrument about vertical axis, a clamp and tangent screw is used. For measurement of vertical angles, the telescope can rotate about the trunnion axis, provided with a clamp and slow motion screw. The instrument is levelled with respect to a circular bubble mounted on the upper plate, and a longitudinal bubble tube mounted on the telescope. Fig. 7.12 shows the photograph of a compass transit by Breithaupt Kassel. The instrument is suitable for compass traversing, reconnaissance, contour works, and for the purposes of forest departments. The eccentric telescope admits steep sights (in mountainous area), being provided with stadia hairs for optical distance measurings (tacheometric surveying). A telescope reversion spirit level suits the determination of the station-height as well as auxillary levelling. The vertical circle is graduated to 1° and reading with vernier can be taken upto 6′. The compass ring is graduated to 1° and reading can be estimated to 6′.

8

Triangulation

8.1. GEODETIC SURVEYING

The object of the Geodetic surveying is to determine very precisely the relative or absolute positions on the earth's surface of a system of widely separated points. The relative positions are determined in terms of the lengths and azimuths of the lines joining them. The absolute positions are determined in terms of latitude, longitude, and elevation above mean sea level. However, the distinction between Geodetic surveying and Plane surveying is fundamentally one of extent of area rather than of operations. The precise methods of geodesy are followed in the field work of extensive plane trigonometrical surveys also. Since the area embraced by a geodetic survey form an appreciable portion of the surface of the earth, the sphericity of the earth is taken into consideration while making the computation. The geodetic points so determined furnish the most precise control to which a more detailed survey of intervening country may be referred. Geodetic work is usually undertaken by the State Agency. In India, it is done by the Survey of India Department.

Triangulation. The horizontal control in Geodetic survey is established either by triangulation or by precise traverse. In triangulation, the system consists of a number of inter-connected triangles in which the length of only one line, called the *base line*, and the angles of the triangles are measured very precisely. Knowing the length of one side and the three angles, the lengths of the other two sides of each triangle can be computed. The apexes of the triangles are known as the *triangulation stations* and the whole figure is called the *triangulation system* or *triangulation figure*. The defect of triangulation is that it tends to accumulate errors of length and azimuth, since the length and azimuth of each line is based on the length and azimuth of the preceding line. To control the accumulation of errors, *subsidiary bases* are also selected. At certain stations, astronomical observations for azimuth and longitude are also made. These stations are called *Laplace Stations*.

The *objects* of Geodetic Triangulation are :

(1) To provide the most accurate system of horizontal control points on which the less precise triangles may be based, which in turn may form a framework to which cadastral, topographical, hydro-graphical, engineering and other surveys may be referred.

(2) To assist in the determination of the size and shape of the earth by making observations for latitude, longitude and gravity.

8.2. CLASSIFICATION OF TRIANGULATION SYSTEM

The basis of the classification of triangulation figures is the accuracy with which the length and azimuth of a line of the triangulation are determined. Triangulation systems of different accuracies depend on the extent and the purpose of the survey. The accepted grades of triangulation are :

(1) First order or Primary Triangulation
(2) Second order or Secondary Triangulation
(3) Third order or Tertiary Triangulation

(1) First-Order or Primary Triangulation

The first order triangulation is of the highest order and is employed either to determine the earth's figure or to furnish the most precise control points to which secondary triangulation may be connected. The primary triangulation system embraces the vast area (usually the whole of the country). Every precaution is taken in making linear and angular measurements and in performing the reductions. The following are the general specifications of the primary triangulation :

1.	Average triangle closure	: Less than 1 second
2.	Maximum triangle closure	: Not more than 3 seconds
3.	Length of base line	: 5 to 15 kilometres
4.	Length of the sides of triangles	: 30 to 150 kilometres
5.	Actual error of base	: 1 in 300,000
6.	Probable error of base	: 1 in 1,000,000
7.	Discrepancy between two measures of a section	: 10 mm $\sqrt{\text{kilometres}}$
8.	Probable error of computed distance	: 1 in 60,000 to 1 in 250,000
9.	Probable error in astronomic azimuth	: 0.5 seconds

(2) Second Order or Secondary Triangulation

The secondary triangulation consists of a number of points fixed within the framework of primary triangulation. The stations are fixed at close intervals so that the sizes of the triangles formed are smaller than the primary triangulation. The instruments and methods used are not of the same utmost refinement. The general specifications of the secondary triangulation are :

1.	Average triangle closure	: 3 sec
2.	Maximum triangle closure	: 8 sec
3.	Length of base line	: 1.5 to 5 km
4.	Length of sides of triangles	: 8 to 65 km
5.	Actual error of base	: 1 in 150,000
6.	Probable error of base	: 1 in 500,000
7.	Discrepancy between two measures of a section	: 20 mm $\sqrt{\text{kilometres}}$
8.	Probable error of computed distance	: 1 in 20,000 to 1 in 50,000
9.	Probable error in astronomic azimuth	: 2.0 sec.

(3). Third-Order or Tertiary Triangulation

The third-order triangulation consists of a number of points fixed within the framework of secondary triangulation, and forms the immediate control for detailed engineering and other surveys. The sizes of the triangles are small and instrument with moderate precision may be used. The specifications for a third-order triangulation are as follows :

1.	Average triangle closure	: 6 sec
2.	Maximum triangle closure	: 12 sec
3.	Length of base line	: 0.5 to 3 km
4.	Length of sides of triangles	: 1.5 to 10 km
5.	Actual error of base	: 1 in 75,0000
6.	Probable error of base	: 1 in 250,000
7.	Discrepancy between two measures of a section	: 25 mm $\sqrt{\text{kilometres}}$
8.	Probable error of computed distance	: 1 in 5,000 to 1 in 20,000
9.	Probable error in astronomic azimuth	: 5 sec.

8.3. TRIANGULATION FIGURES OR SYSTEMS

A *triangulation figure* is a group or system of triangles such that any figure has one side, and only one, common to each of the preceding and following figures. The common Figures or Systems are :

(1) Single chain of triangles [Fig. 8.1 (*a*)]
(2) Double chain of triangles [Fig. 8.1 (*b*)]
(3) Central point Figures [Fig. 8.1 (*c*)]
(4) Quadrilaterals [Fig. 8.1 (*d*)].

(i) Single chain of triangles **:**

This figure is used where a narrow strip of terrain is to be covered. Though the system is rapid and economical, it is not so accurate for primary work since the number of conditions to be fulfilled in the figure adjustment is relatively small. Also, it is not possible to carry the solution of triangles through the figures by two independent routes. If the accumulation of errors is not be become excessive, base lines must be introduced frequently.

(ii) Double chain of triangles **:**

It is used to cover greater area.

(iii) Centred figures **:**

Centred figures are used to cover area, and give very satisfactory results in flat country. The centred figures may be quadrilaterals, pentagons, or hexagons with central stations. The system provides the desired checks on the computations. However, the progress of work is slow due to more settings of the instrument.

(iv) Quadrilaterals **:**

The quadrilateral with four corner stations and observed diagonal forms the *best figures*. They are best suited for hilly country. Since the computed lengths of the sides can be carried through the system by different combinations of sides and angles, the system is the most accurate.

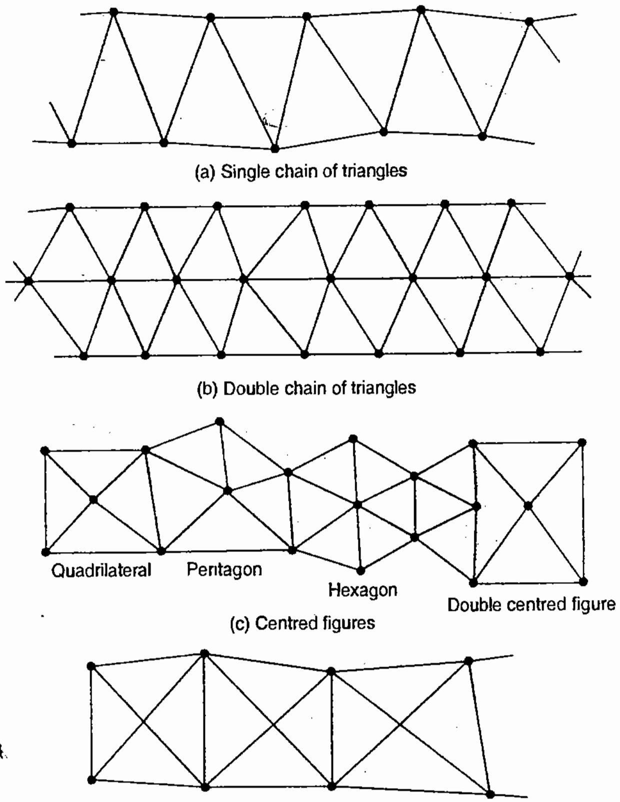


FIG. 8.1. TRIANGULATION FIGURES.

Criteria for selection of the figure :

The following factors should be considered while selecting a particular figure :

(1) The figure should be such that the computations can be done through two independent routes.

(2) The figure should be such that at least one, and preferably both routes should be well-conditioned.

(3) All the lines in a figure should be of comparable length. Very long lines should be avoided.

(4) The figure should be such that least work may secure maximum progress.

(5) Complex figures should not involve more than about twelve conditions.

Framework of a large country :

In very extensive survey, the primary triangulation is laid in two series of chains of triangles, which are usually placed roughly north and south, and east and west respectively. The enclosed area between the parallel and perpendicular series is filled by secondary and tertiary triangulation figures. This system is known as the *grid iron system*, and has been adopted for France, Spain, Austria and India. In another system, called the *central system*, the whole area of the survey may be covered by a network of primary triangulation extending outwards in all directions from the initial base line. The central system has been adopted for United Kingdom.

8.4. THE STRENGTH OF FIGURE

Well-conditioned Triangle. There are various triangulation figures and the accuracy attained in each figure depends upon (*i*) the magnitude of the angles in each individual triangle, and (*ii*) the arrangement of the triangles. Regarding (*i*), the shape of the triangle should be such that any error in the measurement of angle shall have a minimum effect upon the lengths of the calculated side. Such a triangle is then called *well-conditioned triangle*.

In a triangle, one side is known from the computations of the adjacent triangle. The error in the other two sides will affect the rest of the triangulation figure. In order that these two sides be equally accurate, they should be equal in length. This can be attained by making the triangle isosceles.

To find the magnitude of the angle of a triangle, let A, B and C be the three angles, and a, b and c be the three opposite sides of an isosceles triangle ABC. Let AB be the known side and BC and CA be the sides of equal length to be computed. Evidently, $\angle A = \angle B$

By sine formula, $$a = c\,\frac{\sin A}{\sin C} \qquad \ldots(i)$$

Let δA = the error in the measurement of angle A

and δa_1 = corresponding error in the side a.

Differentiating (*i*) partially, we get

$$\delta a_1 = \frac{c \cos A \,.\, \delta A}{\sin C}$$

$$\therefore \quad \frac{\delta a_1}{a} = \frac{\cos A}{\sin A}\,\delta A = \delta A \,.\, \cot A \qquad \ldots(ii)$$

Similarly, let δC = the error in the measurement of angle C

and δa_2 = corresponding error in the side a

Differentiating (*i*) partially, we get

$$\delta a_2 = -c \cdot \frac{\sin A \cos C\, \delta C}{\sin^2 C}$$

$$\therefore \quad \frac{\delta a_2}{a} = -\frac{\cos C}{\sin C}\,\delta C = -\,\delta C \cot C \qquad \ldots(iii)$$

If δA and δC = probable errors in angles = $\pm\,\beta$

the probable fraction error $\left(i.e., \frac{\delta a}{a}\right)$ in the side $\bar{a} = \pm \beta \sqrt{\cot^2 A + \cot^2 C}$

This is minimum when $\cot^2 A + \cot^2 C$ is minimum.

But $C = 180° - A - B = 180° - 2A$

$\therefore$ $\cot^2 A + \cot^2 2A$ should be minimum.

Differentiating $\cot^2 A + \cot^2 2A$ with respect to A and equating it to zero, we get, after reduction,

$$4 \cos^2 A + 2 \cos^2 A - 1 = 0$$

From which $A = 56° 14'$ approx.

Hence the best shape of a triangle is isosceles with base angles equal to 56° 14'. However, from practical considerations, an equilateral triangle is the most suitable. In general, however, triangles having an angle smaller than 30° or greater than 120° should be avoided.

Criterion of Strength of Figure

The strength of figure is a factor to be considered in establishing a triangulation system for which the computations can be maintained within a desired degree of precision. The U.S. Coast and Geodetic Survey has developed a very rapid and convenient method of evaluating the strength of a triangulation figure. The method is based on an expression for the square of the probable error (L^2), that would occur in the sixth place of the logarithm of any side, if the computations were carried from a known side through a single chain of triangles after the net had been adjusted for the side and angle conditions. The expression for L^2 is

$$L^2 = \frac{4}{3} d^2 \frac{D - C}{D} \Sigma [\delta_A^2 + \delta_A \delta_B + \delta_B^2] \quad ...[8.1\ (a)]$$

If R represents the terms in the equation affected by the shape of figure, then

$$R = \frac{D - C}{D} \Sigma [\delta_A^2 + \delta_A \delta_B + \delta_B^2] \quad ...(8.1)$$

and

$$L^2 = \frac{4}{3} d^2 R \quad ...[8.1\ (b)]$$

where d = probable error of an observed direction, in seconds.

D = number of directions observed (forward and/or back), excluding those along the known or fixed line.

δ_A = difference per second in the sixth place of logarithm of the sine of the distance angle A of each triangle in the chain used.

δ_B = same as δ_A but for the distance angle B.

$\Sigma (\delta_A^2 + \delta_A \delta_B + \delta_B^2)$ = summation of values for the particular chain of triangles through which the computation is carried from the known line to the line required. The value of $\Sigma (\delta_A^2 + \delta_A \delta_B + \delta_B^2)$ for a triangle is given in Table 8.1.

The *distance angles A* and *B* of a triangle are the angles opposite to (*i*) known side and (*ii*) the side which is to be computed. The third angle of the triangle, which

TABLE 8.1

VALUES OF $\delta_A^2 + \delta_A \delta_B + \delta_B^2$

	10°	12°	14°	16°	18°	20°	22°	24°	26°	28°	30°	35°	40°	45°	50°	55°	60°	65°	70°	75°	80°	85°	90°
0																							
10	428	359																					
12	359	295	253																				
14	315	253	214	187																			
16	284	225	187	162	143																		
18	262	204	168	143	126	113																	
20	245	189	153	130	113	100	91																
22	232	177	142	119	103	91	81	74															
24	221	167	134	111	95	83	74	67	61														
26	213	160	126	104	89	77	68	61	56	51													
28	206	153	120	99	83	72	63	57	51	47	43												
30	199	148	115	94	79	68	59	53	48	43	40	33											
35	188	137	106	85	71	60	52	46	41	37	33	27	23										
40	179	129	99	79	65	54	47	41	36	32	29	23	19	16									
45	172	124	93	74	60	50	43	37	32	28	25	20	16	13	11								
50	167	199	89	70	57	47	39	34	29	26	23	18	14	11	9	8							
55	162	115	86	67	54	44	37	32	27	24	21	16	12	10	8	7	5						
60	159	112	83	64	51	42	35	30	25	22	19	14	11	9	7	5	4	4					
65	155	109	80	62	49	40	33	28	24	21	18	13	10	7	6	5	4	3	2				
70	152	106	78	60	48	38	32	27	23	19	17	12	9	7	5	4	3	2	2	1			
75	150	104	76	58	46	37	30	25	21	18	16	11	8	6	4	3	2	2	1	1	1		
80	147	102	74	57	45	36	29	24	20	17	15	10	7	5	4	3	2	1	1	1	0	0	
85	145	100	73	55	43	34	28	23	19	16	14	10	7	5	3	2	2	1	1	0	0	0	0
90	143	98	71	54	42	33	27	22	19	16	13	9	6	4	3	2	1	1	1	0	0	0	0
95	140	96	70	53	41	32	26	22	18	15	13	9	6	4	3	2	1	1	0	0	0	0	
100	138	95	68	51	40	31	25	21	17	14	12	8	6	4	3	2	1	1	0	0	0		
105	136	93	67	50	39	30	25	20	17	14	12	8	5	4	2	2	1	1	0	0			
110	134	91	65	49	38	30	24	19	16	13	11	7	5	3	2	2	1	1	1				
115	132	89	64	48	37	29	23	19	15	13	11	7	5	3	2	2	1	1					
120	129	88	62	46	36	28	22	18	15	12	10	7	5	3	2	2	1						
125	127	86	61	45	35	27	22	18	14	12	10	7	5	4	3	2							
130	125	84	59	44	34	26	21	17	14	12	10	7	5	4	3								
135	122	82	58	43	33	26	21	17	14	12	10	7	5	4									
140	119	80	56	42	32	25	20	17	14	12	10	8	6										
145	116	77	55	41	32	25	21	17	15	13	11	9											
150	112	75	54	40	32	26	21	18	16	15	13												
152	111	75	53	40	32	26	22	19	17	16													
154	110	74	53	41	33	27	23	21	19														
156	108	74	54	42	34	28	25	22															
158	107	74	54	43	35	30	27																
160	107	74	56	45	38	33																	
162	107	76	59	48	42																		
164	109	79	63	54																			
166	113	86	71																				
168	122	98																					
170	143																						

is not used in the sine proportion, and which is opposite to the third side is called the *azimuth angle.*

C = number of angles and side conditions to be satisfied in the net from the known line to the side in equation.

C is computed from the following formula :

$$C = (n' - s' + 1) + (n - 2s + 3)$$

where
n = total number of lines
n' = number of lines observed in both directions
s = total number of stations
s' = number of occupied stations
$(n' - s' + 1)$ = *number of angle conditions*
and $(n' - 2s + 3)$ = *number of side conditions.*

The *relative strength* of figure can be computed quantitatively in terms of the factor *R. Lower the value of R, stronger the figure.* By means of computed strengths of figure, alternate routes of computation can be compared and the best route chosen. The value of R computed for the strongest chain of triangles is called R_1 and that for the second strongest chain R_2. *Since the strength of a figure is almost exactly equal to the strength of the strongest chain,* R_1 *is a measure of the strength of figure.*

For the angles measured with the same precision, the strength of figure thus depends upon three factors :

(1) Number of directions observed.

(2) The number of geometrical conditions imposed by the shape of the figures, together with the number of stations occupied in the field.

(3) The sizes of distance angles used in computation.

The U.S. Coast and Geodetic Survey recommends maximum value of R shown in Table 8.2.

TABLE 8.2

	First order		Second order		Third order	
	R_1	R_2	R_1	R_2	R_1	R_2
Single independent figure						
Desirable	15	...	25	80	25	120
Maximum	25	80	40	120	50	150
Net between bases						
Desirable	80	...	100	...	125	...
Maximum	110	...	130	...	175	...

Example 8.1. *The probable error of direction measurement is 1.25 seconds. Compute the maximum value of R if the maximum probable error desired is*

(a) 1 in 25,000 and (b) 1 in 10,000.

Solution

(*a*) Since L is the probable error of a logarithm, it represents the logarithm of the ratio of the true value and a value containing the probable error.

In this case, L = the sixth place in $\log\left(1 \pm \frac{1}{25000}\right)$

= the sixth place in log (1 ± 0.00004)

From seven figure log table,

log 1.0000 = 0.000000

Difference for 4 = 173

$\therefore$ log (1 + 0.00004) = 0.0000173

The sixth place in the log = 17

Hence $L = \pm 17$; $L^2 = 289$

Also $d = 1.25$

Now $$L^2 = \frac{4}{3} d^2 R \qquad ...[8.1\ (b)]$$

or $$R_{max} = \frac{3}{4}\frac{L^2}{d^2} = \frac{3}{4}\frac{289}{(1.25)^2} = \mathbf{139}$$

(*b*) In this case, L = the sixth place in $\log\left(1 \pm \frac{1}{10000}\right)$

= the sixth place in log (1 ± 0.0001)

$\therefore$ $= \pm 43$...(since log 1.0001 = 0.0000434)

$\therefore$ $L^2 = 1849$

$d = 1.25$

$d^2 = 1.5625$

$$\therefore \quad R_{max} = \frac{3}{4}\frac{L^2}{d^2} = \frac{3}{4} \times \frac{1869}{1.5625} = \mathbf{888.}$$

Example 8.2. *Compute the value of C and* $\frac{D-C}{D}$ *for the various nets shown in Fig. 8.2 (a) to 8.2 (d). The heavy lines are the bases of known length. Directions are not observed where lines are dotted.*

Solution.

(*a*) $C = (n' - s' + 1) + (n - 2s + 3)$

where
n = total number of lines = 13
s = total number of stations = 7
n' = number of lines observed in both directions = 10
s' = number of occupied stations = 7
D = total directions observed – 2 = {(13 × 2) – 3} – 2 = 21

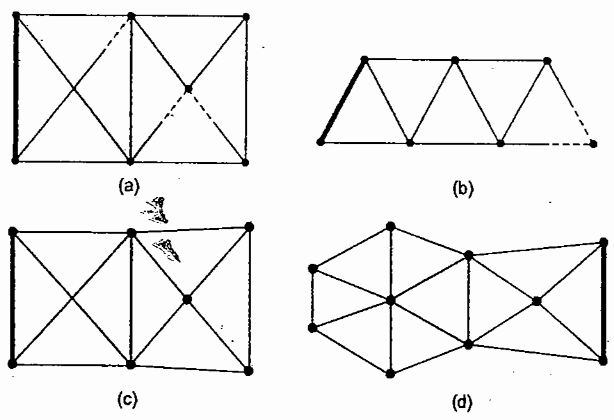


FIG. 8.2

$\therefore \quad C = (10 - 7 + 1) + (13 - 14 + 3) = 4 + 2 = 6$

$$\frac{D-C}{D} = \frac{21-6}{21} = \frac{15}{21} = \mathbf{0.714}$$

(*b*) $n = 11$

$s = 7$

$n' = 9$

$s' = 6$

$D = \{(11 \times 2) - 2\} - 2 = 18$

$C = (9 - 6 + 1) + (11 - 14 + 3) = 4$

$$\frac{D-C}{D} = \frac{18-4}{18} = \mathbf{0.778}$$

(*c*) $n = 13$

$s = 7$

$n' = 13$

$s' = 7$

$D = (13 \times 2) - 2 = 24$

$C = (13 - 7 + 1) + (13 - 14 + 3) = 7 + 2 = 9$

$$\frac{D-C}{D} = \frac{24-9}{24} = \mathbf{0.625}.$$

(*d*) $n = 19$

$s = 10$

$n' = 19$

$s' = 10$

$C = (19 \times 2) - 2 = 36$

$\therefore \quad D = (19 - 10 + 1) + (19 - 20 + 3) = 10 + 2 = \mathbf{12}$

$$\frac{D-C}{D} = \frac{36-12}{36} = \frac{24}{36} = \mathbf{0.667}$$

Example 8.3. *Compute the strength of the figure ABCD for each of the routes by which the length BD can be computed from the known side AC. All the stations were occupied, and all the angles were measured.*

Solution.

Here, D = the number of directions observed (not including the fixed side AC) = 10

n = total number of lines = 6

n' = total number of lines observed in both directions = 6

s = the number of stations = 4

s' = the number of stations occupied = 4

Hence $C = (n' - s' + 1) + (n - 2s + 3)$

$= (16 - 4 + 1) + (6 - 8 + 3) = 3 + 1 = 4$

and $\dfrac{D-C}{D} = \dfrac{10-4}{10} = 0.60$

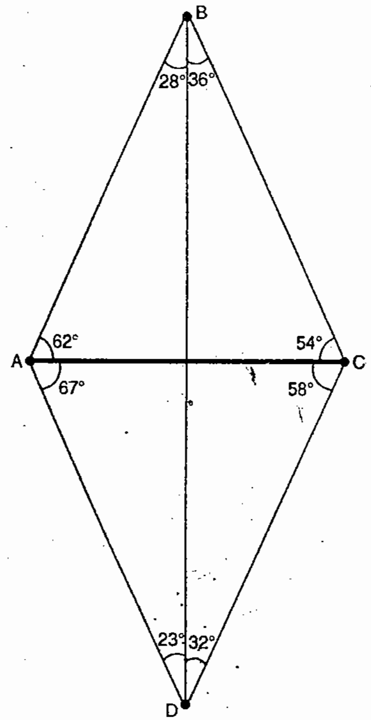


FIG. 8.3.

(*a*) ***Strength of figure by route 1 using Δs ACD and ADB.***

Common side = AD

For triangle ACD, distance angles are 55° and 58°

$\therefore$ From Table 8.1, $\delta_A^2 + \delta_A\delta_B + \delta_B^2 = 6$

For triangle ABD, distance angles are 28° and 129°

$\therefore \quad \delta_A^2 + \delta_A\delta_B + \delta_B^2 = 12$

Hence $\Sigma(\delta_A^2 + \delta_A\delta_B + \delta_B^2) = 6 + 12 = 18$

$\therefore \quad R_1 = \dfrac{D-C}{D}\,\Sigma(\delta_A^2 + \delta_A\delta_B + \delta_B^2)$

$= 0.6 \times 18 = 10.8 \approx 11.$

(*b*) ***Strength of figure by route 2 using Δs ACD and DCB***

Common side = CD

For triangle ACD, distance angles = 55° and 67°

$\therefore \quad \delta_A^2 + \delta_A\delta_B + \delta_B^2 = 4.6$

For triangle DCB, distance angles = 36° and 112°

$\therefore \quad \delta_A^2 + \delta_A\delta_B + \delta_B^2 = 6.6$

$\therefore \quad \Sigma(\delta_A^2 + \delta_A\delta_B + \delta_B^2) = 4.6 + 6.6 = 11.2$

and $R_2 = 0.6 \times 11.2 \approx 7$

(*c*) ***Strength of figure by route 3 using Δs ACB and ABD***

Common side = AB

For triangle ACB, distance angles = 64° and 54°

$\therefore \quad \delta_A^2 + \delta_A\delta_B + \delta_B^2 = 5$

For triangle ABD, distance angles = 23° and 129°

$\therefore \quad \delta_A^2 + \delta_A\delta_B + \delta_B^2 = 19$

$\therefore \quad \Sigma\,(\delta_A^2 + \delta_A\delta_B + \delta_B^2) = 5 + 19 = 24$ and $R_3 = 0.6 \times 24 = 14.4 \approx 14$

(d) Strength of figure by route 4 using Δs ACB and BCD

Common side = BC

For triangle ACB, distance angles = 64° and 62°

$\therefore \quad \delta_A^2 + \delta_A\delta_B + \delta_B^2 = 3.7$

For triangle BCD, distance angles = 32° and 112°

$\therefore \quad \delta_A^2 + \delta_A\delta_B + \delta_B^2 = 9.4$

$\therefore \quad \Sigma\,(\delta_A^2 + \delta_A\delta_B + \delta_B^2) = 3.7 + 9.4 = 13.1$

and $\quad R_4 = 0.6 \times 13.1 \approx 8$

Since the highest strength is represented by the lowest value of R, the best route for computing the value of BD is (*b*) in which $R_2 = 7$.

ROUTINE OF TRIANGULATION SURVEY

The routine of triangulation survey generally consists of the following operations :

(1) Reconnaissance
(2) Erection of signals and towers
(3) Measurement of base lines
(4) Measurement of horizontal angles
(5) Astronomial observations at Laplace stations, and
(6) Computations.

8.5. RECONNAISSANCE

Since the basic principle of surveying is *working from whole to part*, reconnaissance is very important in all types of surveys. The reconnaissance survey requires great skill, experience and judgement on the part of the party chief. Since the economy and accuracy of the whole triangulation system depends upon an efficient reconnaissance, it includes the following operations :

1. Examination of the country to be surveyed.
2. Selection of suitable sites for base lines.
3. Selection of suitable positions for triangulation stations.
4. Determination of intervisibility and height of stations.
5. Collection of miscellaneous information regarding communication of water, food, labour and guides etc.

Whenever possible, help should be taken from the existing maps. If the maps are not available, a rapid preliminary reconnaissance is undertaken to ascertain the general location of possible schemes of triangulation suitable for that topography. Later on, main reconnaissance is done to examine these schemes. Main reconnaissance is conducted as a very rough triangulation

and plotted as the work advances. The plotting may be done by protracting the angles. The essential features of the topography are also sketched in. The relative strength and cost of various triangulations or schemes are then studied and a final scheme is then selected. Since the reconnaissance is a sort of rapid survey, the following instruments are generally used for the survey :

(1) A small theodolite and sextant for measurement of angles.
(2) Prismatic compass for the measurement of bearings.
(3) Aneroid barometer for ascertaining elevations.
(4) Steel tape.
(5) Good telescope or powerful field glass.
(6) Heliotropes for testing intervisibility.
(7) Drawing instruments and materials.
(8) Guyed ladders, ropes, creepers etc., for climbing trees.

Selection of Triangulation Stations

The selection of triangulation stations is based upon the following considerations :

1. The triangulation stations should be intervisible For this purpose, they should be placed upon the most elevated ground (such as tops of hills etc.) so that long sights through undisturbed atmosphere may be secured.

2. They should form well-shaped triangles. As far as possible, the triangles should be either isosceles with base angles of about 56° or equilateral. In general, however, no angle should be smaller than 30° or greater than 120°.

3. The stations should be easily accessible, and should be such that supplies of food and water are easily available, and camping ground or nearest suitable accommodation is available.

4. They should be so selected that the length of sight is neither too small nor too large. Small length of sight will result in errors due to centring and bisection while large line of sight will make the signal too indistinct for accurate bisection.

5. They should be in commanding situation so as to serve as the control of the subsidiary triangulation and for possible future extension of the principal system. The stations of the subsidiary triangulation should be such that they are useful for detail surveys.

6. In heavily wooden country, the stations should be so located that the cost of clearing and cutting, and of building towers is minimum.

7. The stations should be situated so that lines of sight do not pass over towns, factories, furnaces etc. nor graze any obstruction, so that the effects of irregular atmospheric refraction is avoided.

Intervisibility and Height of Stations

As stated earlier, the stations should be selected in commanding position so that they are intervisible. In general, the reconnaissance party can ascertain whether the proposed stations are intervisible by direct observations through strong field glasses either at the ground level or from the top of trees or guyed ladder. However, if the distance between the stations is more and difference in elevation is less, calculations are necessary to determine whether it is necessary to elevate the stations to get the intervisibility or not. Generally, it is necessary to raise both the instrument as well as the signal to overcome the curvature of the earth and to clear all the intervening obstructions. The height of the instrument as well as the signal depends upon the following factors :

1. The distance between the stations.
2. The relative elevation of stations.
3. The profile of the intervening ground.

1. The Distance between the Stations

If there is no obstruction due to intervening ground, the distance of the visible horizon from a station of known elevation above datum is given by

$$h = \frac{D^2}{2R}(1 - 2m) \quad ...(8.2)$$

where h = height of the station above datum

D = distance to the visible horizon

R = mean radius of the earth

m = mean co-efficient of refraction

= 0.07 for sights over land, and = 0.08 for sights over sea.

If the values of D and R are substituted in proper units, the value of h corresponding to $m = 0.07$ is given by

$h = 0.574\ D^2$, where h is in feet and D is in miles

and $h = 0.06728\ D^2$, where h is in metres and D is in km.

2. Relative Elevation of Stations

If there is no obstruction due to intervening ground, the formula $h = \frac{D^2}{2R}(1 - 2m)$ may be used to get the necessary elevation of a station at distance, so that it may be visible from another station of known elevation.

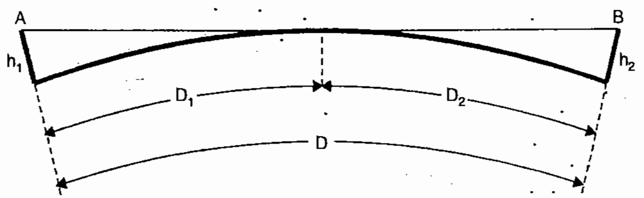


FIG. 8.4.

For example, let h_1 = known elevation of station A above datum

h_2 = required elevation of B above datum

D_1 = distance from A to the point of tangency

D_2 = distance from B to the point of tangency

D = the known distance between A and B.

Then, $h_1 = 0.06728\ D_1^2$

$$D_1 = \sqrt{\frac{h_1}{0.06728}} = 3.8553\ \sqrt{h_1} \quad ...(i)\ [8.2\ (a)]$$

where D_1 is in km and h_1 is in metres.

Knowing D_1, D_2 is given by

$$D_2 = D - D_1 \quad ...(ii)$$

Knowing D_2, h_2 is calculated from the relation

$$h_2 = 0.06728\ D_2^2 \text{ metres} \quad ...(iii)\ [8.2\ (b)]$$

Thus, the required elevation h_2 is determined. If the actual ground level at B is known it can be ascertained whether it is necessary to elevate the station B above the ground, and if so, the required height of tower can be calculated. However, while making the above calculations, the line of sight should not graze the surface at the point of tangency but should be above it by 2 to 3 metres.

3. Profile of the Intervening Ground

In the reconnaissance, the elevations and positions of peaks in the intervening ground between the proposed stations should be determined. A comparison of their elevations should be made to the elevation of the proposed line of sight to ascertain whether the line of sight is clear off the obstruction or not. The problem can be solved by using the principles discussed in the factors (1) and (2) above, or by a solution suggested by Captain G.T. McCaw. The former method will be clear from the worked out examples.

Captain G.T. McCaw's Method

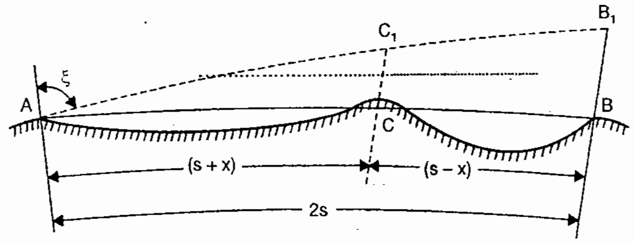


FIG. 8.5.

In Fig. 8.5, let h_1 = height of station A above datum

h_2 = height of station B above datum

h = height of line of sight at the obstruction C

$2s$ = distance between the two stations A and B

$(s + x)$ = distance of obstruction C from A

$(s - x)$ = distance of obstruction C from B

ζ = zenith distance from A to B

The height h of the line of sight at the obstruction is given by

$$h = \tfrac{1}{2}(h_2 + h_1) + \tfrac{1}{2}(h_2 - h_1)\frac{x}{s} - (s^2 - x^2)\operatorname{cosec}^2 \zeta \left(\frac{1 - 2m}{2R}\right) \quad ...(8.3)$$

The value of $\operatorname{cosec}^2 \zeta$ can be taken approximately equal to unity. However, if more accuracy is required, it may be computed from the expression,

$$\operatorname{cosec}^2 \zeta = 1 + \frac{(h_2 - h_1)^2}{4s^2} \qquad ...[8.3\ (a)]$$

The expression $\frac{1-2m}{2R} = 0.574$, if x, s and R are substituted in miles, and h_1, h_2 and h are in feet

and $\frac{1-2m}{2R} = 0.06728$ if, x, s and R are in km and h_1, h_2 and h are in metres.

Station Marks

The triangulation station should be permanently marked with copper or bronze tablets. The name of station and the year in which it is set should be stamped on the tablet. The following are the *essentials* of good construction of station marks :

(1) The mark should be distinctive and indestructible. Two marks should be provided, one visible on the surface and the other buried vertically below. The mark may be set on firm rock, or on concrete monument.

(2) Two or three reference marks, similar in material and shape to the station mark, should be installed. The distance and bearings of these references marks from the station mark and from each other should be recorded on them.

(3) At each station where a tall signal tower is needed, an azimuth mark should be established at some distance away from the station mark. The azimuth mark should be of the same size and character as the reference mark.

Example 8.4. *Two triangulation stations A and B are 60 kilometres apart and have elevations 240 m and 280 m respectively. Find the minimum height of signal required at B so that the line of sight may not pass near the ground than 2 metres. The intervening ground may be assumed to have a uniform elevation of 200 metres.*

Solution. (Fig 8.4)

Minimum elevation of line of sight = 200 + 2 = 202 m

Let us take this elevation as the datum

∴ Height of A above this datum = h_1 = 240 − 202 = 38 m

The tangent distance D_1 corresponding to h_1 is given by Eq. 8.2 (*a*) :

$$D_1 = 3.8553\sqrt{h_1} = 3.8553\sqrt{38} = 23.766 \text{ km.}$$

∴ Distance of B from the point of tangency

$$= D_2 = D - D_1 = 60 - 23.766 = 36.234 \text{ km.}$$

The elevation h_2 (of B above the datum) corresponding to the distance D_2 is given by

$$h_2 = 0.06728\ D_2^2 = 0.06728\ (36.234)^2 = 88.33 \text{ m}$$

∴ Elevation of line of sight at B = 202 + 88.33 = 290.33 m

Ground level at B = 280 m

∴ Minimum height of signal above ground at B = 290.33 − 280 m = **10.33** m.

Example 8.5. *The altitude of two proposed stations A and B 130 km apart are respectively 220 m and 1160 m. The altitudes of two points C and D on the profiles between them are respectively 308 m and 632 m, the distances being AC=50 km and AD=90 km. Determine whether A and B are intervisible, and if necessary, find the minimum height of a scaffolding at B, assuming A as the ground station.*

Solution.

Let *acedb* be the visible horizon (level line) and a horizontal sight Ab_1 through A meet the horizon tangentially in e. Ao, Co, Do and Bo are the vertical lines through A, C, D and B respectively, O being the centre of the earth.

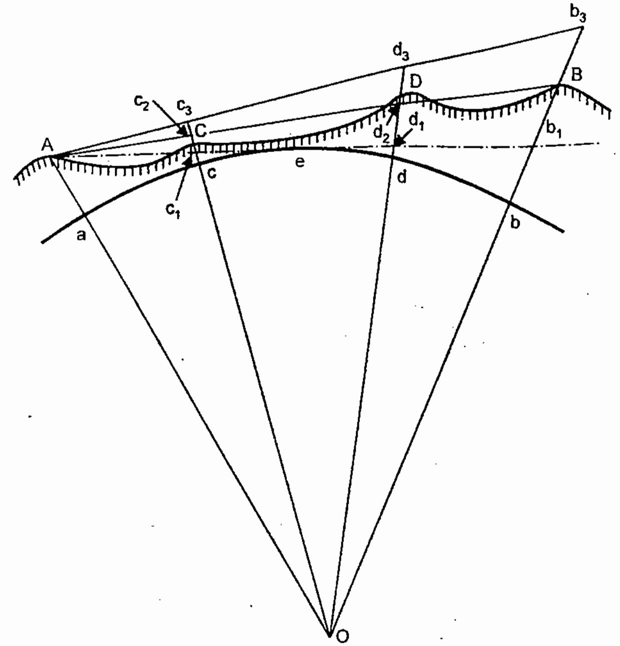


FIG. 8.6

The distance Ae to the visible horizon from station A of an altitude 220 metres is given by

$$D = Ae = 3.8553\sqrt{h} = 3.8553\sqrt{220} = 57.18 \text{ km.}$$

Let a, c, d and b be the points in which the vertical lines through A, C, D and B cuts the level line.

Now AC = 50 km ; AD = 90 km ; AB = 130 km

∴ $ce = Ae - AC = 57.18 - 50 = 7.18$ km

$ed = AD - Ae = 90 - 57.18 = 32.82$ km

$eb = AB - Ae = 130 - 57.18 = 72.82$ km.

Let c_1, d_1 and b_1 be the points in which a horizontal line through A cut the vertical lines through C, D and B respectively. The corresponding heights cc_1, dd_1 and bb_1 are given by

$$cc_1 = 0.06728\ (ce)^2 = 0.06728\ (7.18)^2 = 3.49 \text{ m}$$

$$dd_1 = 0.06728\ (ed)^2 = 0.06728\ (32.82)^2 = 72.47 \text{ m}$$

and $$bb_1 = 0.06728\ (eb)^2 = 0.06728\ (72.82)^2 = 356.77 \text{ m}$$

Now $$Bb = \text{Elev. of } B = 1160 \text{ m}$$

∴ $$Bb_1 = Bb - bb_1 = 1160 - 356.77 = 803.23 \text{ m}$$

Let AB be the line of sight.

Now from triangles Ac_1c_2, Ad_1d_2 and Ab_1B

$$c_1c_2 = Bb_1 \frac{Ac_1}{Ab_1} = 803.23 \times \frac{50}{130} = 308.93 \text{ m}$$

and $$d_1d_2 = Bb_1 \frac{Ad_1}{Ab} = 803.23 \times \frac{90}{130} = 556.08 \text{ m}$$

Elevation of line of sight at C = elevation of $c_2 = cc_1 + c_1c_2 = 3.49 + 308.93 = 312.42$ m

Elevation of line of sight at D = elevation of $d_2 = dd_1 + d_1d_2 = 72.47 + 556.08 = 628.55$ m

Elevation of $C = 308$ m and that of $D = 632$ m

Thus, the line of sight clears the peak C, but fails to clear the peak D by $632 - 628.55 = 3.45 \text{ m} = d_2D$.

Let Ad_3 be the new line of sight, such that

$$Dd_3 = 3 \text{ metres (minimum)}$$

Hence $$d_2d_3 = d_3D + d_2D = 3 + 3.45 = 6.45 \text{ m}$$

Hence $$Bb_3 = d_2d_3 \frac{AB}{Ad_2} = 6.45 \times \frac{130}{90} = 9.32 \text{ m} \approx 9.5 \text{ m (say)}.$$

Hence minimum height of scaffold at B = **9.5 m.**

Example 8.6. *The altitudes of two proposed stations A and B, 100 km apart, are respectively 420 m and 700 m. The intervening obstruction situated at C, 70 km from A has an elevation of 478 m. Ascertain if A and B are intervisible, and, if necessary, find by how much B should be raised so that the line of sight must nowhere be less than 3 m above the surface of the ground.*

Solution (Fig. 8.7).

Let $aceb$ be the visible horizon and a horizontal sight Ab_1 through A meet the horizon tangentially in e.

The distance Ae to the visible horizon from station A of an altitude 420 metres is given by

$$D = Ae = 3.8553\sqrt{h} = 3.8553\sqrt{420} = 79.01 \text{ km}$$

Now $$AC = 70 \text{ km and } AB = 100 \text{ km}$$

∴ $$ec = Ae - AC = 79.01 - 70 = 9.01 \text{ km}$$

and $$eb = AB - Ae = 100 - 79.01 = 20.99 \text{ km}$$

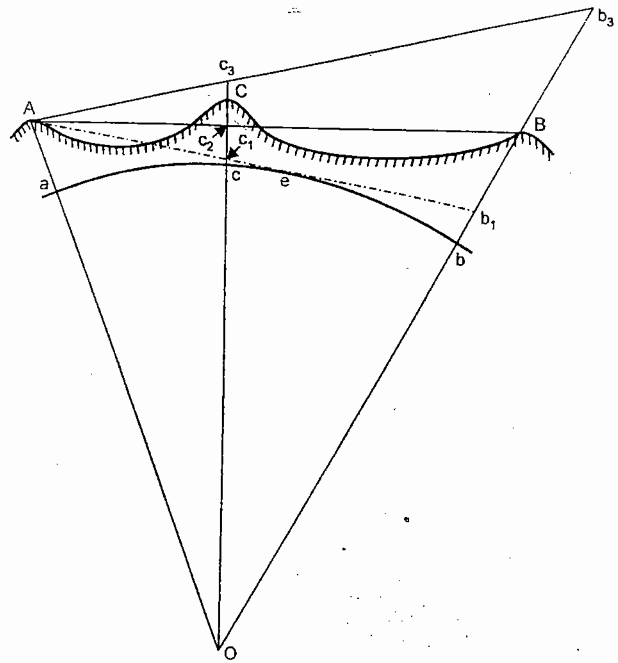


FIG. 8.7

The corresponding heights cc_1 and bb_1 are given by

$$cc_1 = 0.06728\ (ec)^2 = 0.06728\ (9.01)^2 = 5.46 \text{ m}$$

and $$bb_1 = 0.06728\ (eb)^2 = 0.06728\ (20.99)^2 = 29.64 \text{ m}$$

New $$Bb = \text{Elev. of } B = 700$$

∴ $$Bb_1 = Bb - bb_1 = 700 - 29.64 = 670.36 \text{ m}$$

Now, from similar triangles Ac_1c_2 and Ab_1B,

$$c_1c_2 = Bb_1 \frac{Ac_1}{Ab_1} = 670.36 \times \frac{70}{100} = 469.25 \text{ m}$$

∴ Elevation of line sight at C = elevation of c_2

$$= cc_1 + c_1c_2 = 5.46 + 469.25 = 474.71 \text{ m}$$

∵ Elevation of $C = 478$ m

Hence the line of sight fails to clear the peak by

$$c_2C = 478 - 474.71 = 3.29 \text{ m}$$

In order that the line of sight should at least be 3 m above the ground anywhere, the line of sight should be raised by $(3.29 + 3) = 6.29$ **m.**

That is, $$c_2c_3 = 6.29 \text{ m}$$

Hence $b_3B = c_2c_3 \dfrac{AB}{Ac_2} = 6.29 \times \dfrac{100}{70} = 8.99 \text{ m} \approx 9.0 \text{ m}$

Hence height of scaffold at $B = \mathbf{9.0}$ **m.**

Solution by Captain McCaw's Method :

The height h of the line of sight at the obstruction is given by

$$h = \tfrac{1}{2}(h_2 + h_1) + \tfrac{1}{2}(h_2 - h_1)\frac{x}{s} - (s^2 - x^2) \times \left(\frac{1-2m}{2R}\right)$$

Hence $h_1 = 420$ m

$h_2 = 700$

$2s = 100$ km or $s = 50$ km

$s + x = 70$ km

∴ $x = 70 - s = 70 - 50 = 20$ km

$\dfrac{1-2m}{2R} = 0.06728$

Substituting these values, we get

$$h = \tfrac{1}{2}(420 + 700) + \tfrac{1}{2}(700 - 420)\frac{20}{50} - (2500 - 400) \times 0.06728 = 560 + 56 - 141.29$$

$= 474.71$

Hence the line of sight fails to clear the peak C by $478 - 474.71 = 3.29$ m

In order that the line of sight should at least be 3 m above the ground anywhere, the line of sight should be raised by $(3.29 + 3) = 6.29$ m at C.

Hence corresponding height of scaffold at $B = 6.29 \times \dfrac{100}{70} = 8.99 \text{ m} \approx \mathbf{9.0}$ m.

8.6. SIGNALS AND TOWERS

(a) TOWERS

A tower is a structure erected over a station for the support of the instrument and observing party and is provided when the station, or the signal, or both are to be elevated. The amount of elevation depends on the character for the terrain and length of sight desired. The triangulation tower must be built in duplicate, securely founded and braced and guyed. The inner tower supports the instrument only and the outer supports the observer and the signal. The two towers should be entirely independent to each other. Towers may be of masonry, timber, or steel. For small heights, masonry structures are most suitable, but otherwise they are uneconomical. Timber scaffolds are most commonly used, and have been constructed to heights over 50 metres. Steel towers made of light sections are very portable and can be easily erected and dismantled. Mr. J.S. Bilby, of the United States Coast and Geodetic Survey, evolved such a portable type of tower (Fig. 8.8) made of steel sections and rods which can be assembled and dismantled very easily. The Bilby tower can raise the observer and the lamp to a height of 30 m or even 40 m with a beacon 3 m higher. Five men can erect the tower, weighing 3 tonnes, in 5 hours.

(b) SIGNALS

A signal is a device erected to define the exact position of an observed station. The signal may be classified as under:

(1) Daylight or nonluminous (opaque) signal ;

(2) Sun or luminous signal ; and

(3) Night signal.

A signal should fulfil the following requirements :

(*i*) It should be conspicuous (clearly visible against any back ground).

(*ii*) It should be capaple of being accurately centred over the station mark.

(*iii*) It should be suitable for accurate bisection.

(*iv*) It should be free from phase, or should exhibit little phase.

Non-luminous or Opaque Signals

Daylight or non-luminous signals consist of the various forms of mast, target or tin cone types, and are generally used for direct sights less than 30 kilometres. For sights under 6 kilometres, pole signals [Fig. 8.9 (*a*)] consisting of round pole painted black and white in alternate section and supported on a tripod or quadripod may be used. A target signal [Fig. 8.9 (*b*)] consists of a pole carrying two square or rectangular targets placed at right angles to each other. The targets are made of cloth stretched on wooden frames.

The signals should be of dark colour for visibility against the sky and should be painted white, or in white or black strips against a dark background. The top of the mast should carry a flag. To make the signal conspicuous, its height above the station should be roughly proportional to the length of the longest sight upon it. A height in the vertical plane corresponding to at least 30″ is necessary. The following rules may serve as a guide :

Diameter of signal in cm $= 1.3D$ to $1.9D$, where D is in kilomteres.

Height of signal in cm $= 13.3D$, where D is in kilometres.

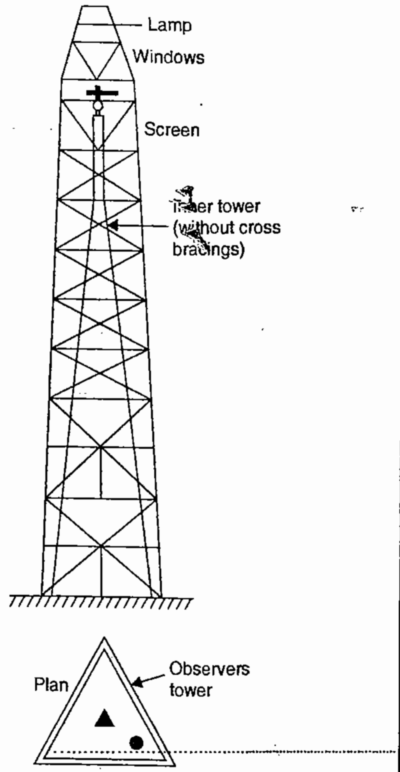


FIG. 8.8. BILBY STEEL TOWER.

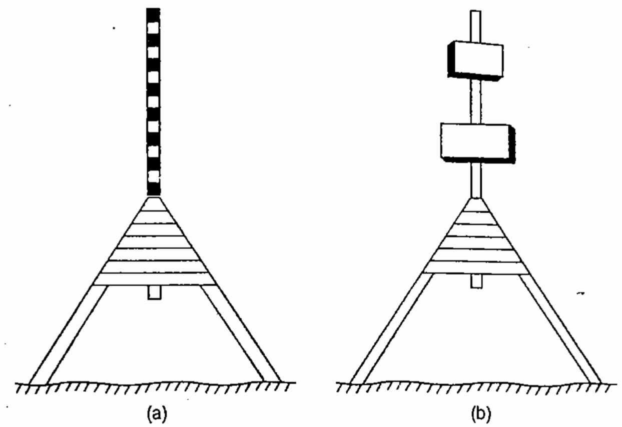


FIG. 8.9. NON-LUMINOUS SIGNALS

Luminous or Sun Signals

Sun signals are those in which the sun's ray's are reflected to the observing theodolite, either directly, as from a beacon, or indirectly from a signal target. They are generally used when the length of sight exceeds 30 km. The heliotrope and heliograph are the special instruments used as sun signals. The *heliotrope* consists of a plane mirror to reflect the sun's rays and a line of sight to enable the attendant to direct the reflected rays towards the observing station. The line of sight may be either telescopic or in the form of a sight vane with an aperture carrying crosswires. The heliotrope is centred over the station mark, and the line of sight is directed upon the distant station by the attendant at the heliotrope. Flashes are sent from the observing station to enable the direction to be established. Because of the motion of the sun, the heliotroper must adjust the mirror every minute on its axes. The reflected rays from a divergent beam have an angle equal to that subtended by the sun at the mirror *viz.* about 32 min. The base of the cone of the reflected rays has therefore, a diameter of about 10 metres in every kilometre of distance. Since the signal is visible from any point within this base, great refinement in pointing the heliotrope is unnecessary. However, in order that the signal may be visible, the error in alignment should be less than 16 minutes. Another form of heliotrope is the *'Galton Sun Signal'*

Night Signals

Night signals are used in observing the angles of a triangulation system at night. Various forms of night signals used are ;

(1) Various forms of oil lamps with reflectors or optical collimators for lines of sight less than 80 kilometres.

(2) Acetylene lamp designed by captain G.T. McCaw for lines of sight up to 80 kilometres.

Phase of Signals

Phase of signal is the error of bisection which arises from the fact that, under lateral illumination, the signal is partly in light and partly in shade. The observer sees only the illuminated portion and bisects it. *It is thus the apparent displacements of the signal.* The *phase correction* is thus necessary so that the observed angle may be reduced to that corresponding to the centre of the signal.

The correction can be applied under two conditions :

(*i*) When the observation is made on the bright portion.

(*ii*) When the observation is made on the bright line.

(i) When the observation is made on the bright portion.

Fig. 8.10 (*a*) shows the case when the observation is made on the bright portion *FD*.

Let A = position of the observer.

B = centre of the signal (in plan).

FD = visible portion of the illuminated surface.

AE = line of sight

E = mid-point of FD

β = phase correction

θ_1 and θ_2 = angles which the extremities of the visible portion make with AB

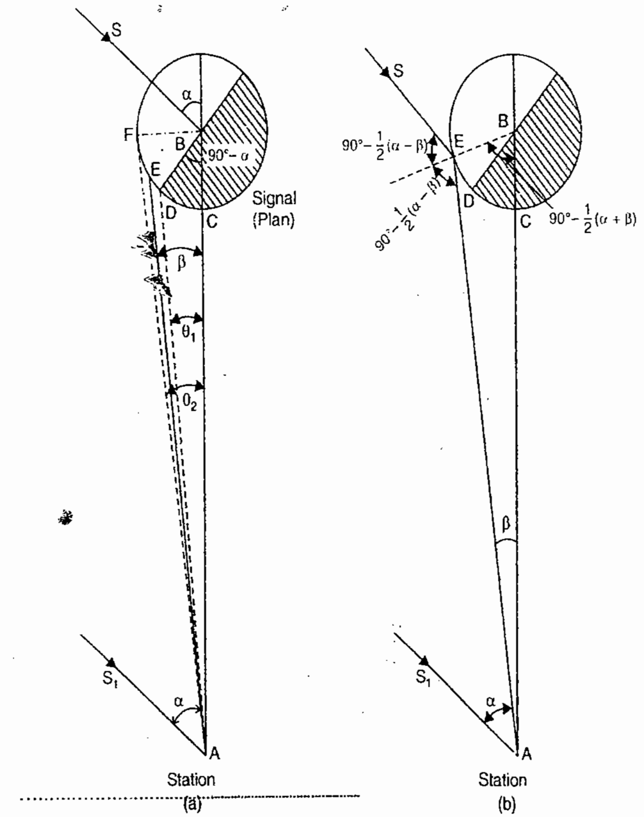


FIG. 8.10. PHASE CORRECTION.

α = the angle which the direction of sun makes with AB

r = radius of the signal

D = distance AB

The phase correction $\beta = \theta_1 + \frac{1}{2}(\theta_2 - \theta_1) = \frac{1}{2}(\theta_1 + \theta_2)$

But $\theta_2 = \dfrac{r}{D}$ radians

and $\theta_1 = \dfrac{r \sin (90° - \alpha)}{D} = \dfrac{r \cos \alpha}{D}$ radians

$\therefore \quad \beta = \frac{1}{2}\left\{ \dfrac{r \cos \alpha}{D} + \dfrac{r}{D} \right\} = \dfrac{r(1 + \cos \alpha)}{2D}$

or $\beta = \dfrac{r \cos^2 \frac{1}{2}\alpha}{D}$ radians

$= \dfrac{r \cos^2 \frac{1}{2}\alpha}{D \sin 1''}$ seconds $= \dfrac{206265\, r \cos^2 \frac{1}{2}\alpha}{D}$ seconds ...(8.4)

(ii) When the observation is made on the bright line

If Fig. 8.10 (*b*), let observation be made on the bright line formed by the reflected rays as indicated by the path *SE*. *AE* is the observed line of sight.

Let $\angle EAB = \beta =$ phase correction.

Since *SE* and S_1A are parallel,

$$\angle SEA = 180° - (\alpha - \beta)$$

$$\therefore \quad \angle BEA = 180° - \tfrac{1}{2}\angle SEA = 180° - \tfrac{1}{2}[180° - (\alpha - \beta)] = 90° + \tfrac{1}{2}(\alpha - \beta)$$

$$\angle EBA = 180° - (\beta + \angle BEA) = 180° - \beta - 90° - \tfrac{1}{2}(\alpha - \beta) = 90° - \tfrac{1}{2}(\alpha + \beta)$$

$$\approx 90 - \tfrac{1}{2}\alpha, \text{ since } \beta \text{ is small in comparison to } \alpha.$$

$$\therefore \quad \beta = \frac{r \sin (90° - \frac{1}{2}\alpha)}{D} \text{ radians} = \frac{r \cos \frac{1}{2}\alpha}{D} \text{ radians}$$

$$= \frac{r \cos \frac{1}{2}\alpha}{D \sin 1''} \text{ seconds} = \frac{206265\, r \cos \frac{1}{2}\alpha}{D} \text{ seconds} \qquad \ldots(8.5)$$

The effect of phase is more common in cylindrical signals and with square masts. In the target signal the phase arises from the shadow of the upper target falling upon the lower one. If a single target is used and set normal to the line of sight during observation, the phase may be avoided.

The phase correction is applied algebraically to the observed angle, according to the relative position of the sun and the signal.

Example 8.7. *Observations were made from instrument station A to the signal at B. The sun makes an angle of 60° with the line BA. Calculate the phase correction if (i) the observation was made on the bright portion, and (ii) the observation was made on the bright line. The distance AB is 9460 metres. The diameter of the signal is 12 cm.*

Solution. (*i*) ***Observation made on the bright portion.***

The correction β is given by

$$\beta = \frac{206265\, r \cos^2 \frac{1}{2}\alpha}{D} \text{ seconds} \quad \text{(Eq. 8.4)}$$

Here $\alpha = 60°$; $r = 6$ cm

$D = 9460$ m $= 9460 \times 10^2$ cm

$$\therefore \quad \beta = \frac{206265 \times 6 \times \cos^2 30°}{946000} \text{ seconds} = \mathbf{0.98 \text{ seconds.}}$$

(*ii*) ***Observation made on the bright line***

The correction β is given by

$$\beta = \frac{206265\, r \cos \frac{1}{2}\alpha}{D} \text{ seconds}$$

$$= \frac{206265 \times 6 \cos 30°}{946000} \text{ seconds} = \mathbf{1.13 \text{ seconds.}}$$

8.7. BASE LINE MEASUREMENT

The measurement of base line forms the most important part of the triangulation operations. The base line is laid down with great accuracy of measurement and alignment as it forms the basis for the computations of triangulation system. The length of the base line depends upon the grades of the triangulation. Apart from main base line, serveral other check bases are also measured at some suitable intervals. In India, ten bases were used, the lengths of the nine bases vary from 6.4 to 7.8 miles and that of the tenth base is 1.7 miles.

Selection of Site for Base Line. Since the accuracy in the measurement of the base line depends upon the site conditions, the following points should be taken into consideration while selecting the site :

1. The site should be fairly level. If, however, the ground is sloping, the slope should be uniform and gentle. Undulating ground should, if possible be avoided.

2. The site should be free from obstructions throughout the whole of the length. The line *clearing* should be cheap in both labour and compensation.

3. The extremities of the base should be intervisible at ground level.

4. The ground should be reasonably firm and smooth. Water gaps should be few, and if possible not wider than the length of the long wire or tape.

5. The site should suit extension to primary triangulation. This is an important factor since the error in extension is likely to exceed the error in measurement.

In a flat and open country, there is ample choice in the selection of the site and the base may be so selected that it suits the triangulation stations. In rough country, however, the choice is limited and it may sometimes be necessary to select some of the triangulation stations that are suitable for the base line site.

Standards of Length. The ultimate standard to which all modern national standards are referred is the international metre established by the Bureau International der Poids at Measures and kept at the Pavilion de Breteuil, Sevres, with copies allotted to various national surveys. The metre is marked on three platinum-iridium bars kept under standard conditions. One great disadvantage of the standard of length that are made of metal are that they are subject to very small secular change in their dimensions. Accordingly, the metre has now been standardized in terms of wavelength of cadmium light. The various national standards are as follows :

(*i*) ***Great Britian.*** The legal unit in Great Britain is the Imperial yard of bronze bar with gold plugs kept at the Board of Trade in London.

1 Imperial yard = 0.91439180 legal metres = 0.91439842 international metres.

(*ii*) ***The United States.*** The standard is copy no. 27 of the international metre. Primary triangulation is computed in metres and is converted in feet by the statutory ratio:

1 metre = 39.37 inch.

(*iii*) ***India.*** The triangulation is computed in term of the old 10 feet bar 'A' as it was in 1840–70 , having its length equal to 9.9999566 British feet at that time. The modern survey standards are however a nickel metre (of 1911) and a silica metre (of 1925) kept at Dehra Dun and standardized at the National Physical Laboratory.

Forms of Base Measuring apparatus :

There are two forms of base measuring apparatus :

(A) *Rigid Bars* (B) *Flexible apparatus.*

(A) Rigid Bars

Before the introduction of invar tapes, rigid bars were used for work of highest precision. The rigid bars may be divided into two classes :

(*i*) ***Contact apparatus,*** in which the ends of the bars are brought into successive contacts. Example : The Eimbeck Duplex Apparatus.

(*ii*) ***Optical apparatus,*** in which the effective lengths of the bars are engraved on them and observed by microscopes. Example : The Colby Apparatus and the Woodward Iced Bar Apparatus.

The rigid bars may also be divided into the following classes depending upon the *way* in which the uncertainties of temperature corrections are minimised :

(*i*) *Compensating base bars,* which are designed to maintain constant length under varying temperature by a combination of two more metals. Example : The Colby Apparatus.

(*ii*) *Bimetallic non-compensating base bars,* in which two measuring bars act as a bimetallic thermometer. Example : The Eimbeck Duplex Apparatus (U.S. Coast and Geodetic Survey), Borda's Rod (French system) and Bessel's Apparatus (German system).

(*iii*) *Monometallic base bars,* in which the temperature is either kept constant at melting point of ice, or is otherwise ascertained. Example : The Woodward Iced Bar Apparatus, and Struve's Bar (Russian system)

The Colby Apparatus

This is a compensating and optical type rigid bar apparatus designed by Maj-Gen. Colby to eliminate the effect of changes of temperature upon the measuring appliance. The apparatus was employed in the Ordinance Survey and the Indian Surveys. All the ten bases of G.T. of Survey of India were measured with the Colby apparatus. The apparatus (Fig. 8.11) consists of two bars, one of steel and the other of brass, each 10 ft long and riveted together at the centre of their length. The ratio of coefficients of linear expansion of these metals having been determined as 3 : 5. Near each end of the compound bar, a metal tongue is supported by double conical pivots held in forked ends of the bars. The tongue projects on the side away from the brass rod. On the extremities of these tongues, two minute marks *a* and *a'* are put, the distance between them being exactly

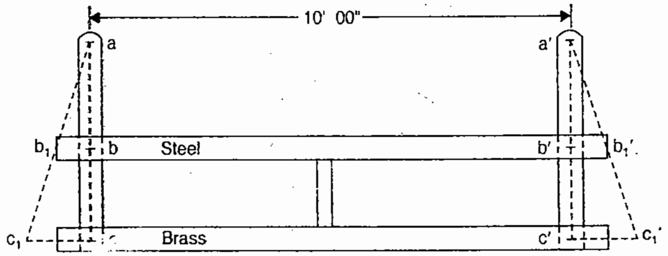


FIG. 8.11. THE COLBY APPARATUS

equal to 10' 0". The distance *ab* (or *a'b'*) to the junction with the steel is kept $\frac{3}{5}$ths of the distance *ac* (or *a'c'*) to the brass junction. Due to change in temperature, if the distance *bb'* of steel changes to b_1b_1' by an amount *x*, the distance *cc'* of brass will change to c_1c_1' by an amount $\frac{5}{3}x$ *thus unaltering the positions of dots a and a'*. The brass is coated with a spcial preparation in order to render it equally susceptible to change of temperature as the steel. The compound bar is held in the box at the middle of its length. A spirit level is also placed on the bar. In India, five compound bars were simultaneously employed in the field. The gap between the forward mark of one bar and the rear bar of the next was kept constant equal to 6" by means of a *framework* based on the same principles as that of the 10' compound bar. The framework consists of two microscopes, the distance between the cross-wires of which was kept exactly equal to 6". To start with, the cross-wires of the first microscope of the *framework* were brought into coincidence with the platinum dot, let into the centre of the mark of the one extremity of the base line. The platinum dot *a* of the first compound bar was brought into the coincidence with the cross-hairs of the second microscope. The cross-hairs of the first microscope of the second framework (consisting two microscopes 6" apart) are then set over the end *a'* of the first rod. The work is thus continued till a length of $(10' \times 5 + 5 \times 6") = 52' \ 6"$ is measured at a time with the help of 5 bars and 2 frameworks. The work is thus continued till the end of the base is reached.

(B) Flexible Apparatus

In recent years, the use of flexible instruments has increased due to the longer length that can be measured at a time without any loss in accuracy. The flexible apparatus consists of (*a*) steel or invar tapes, and (*b*) steel and brass wires. The flexible apparatus has the following *advantages* over the rigid bars :

(*i*) Due to the greater length of the flexible apparatus, a wider choice of base sites is available since rough ground with wider water gap can be utilised.

(*ii*) The speed of measurement is quicker, and thus less expensive.

(*iii*) Longer bases can be used and more check bases can be introduced at closer intervals.

Steel Tapes

Steel tapes are semi-tempered bands of tough, flexible steel which has a thermal coefficient of expansion of very nearly 0.00000645 per degree Fahrenheit. The temperature of a steel tape cannot be measured with sufficient accuracy by mercurial thermometer in the day time. Accurate results can, however, be obtained if the measurements are made at night or on cloudy or even hazy days when there is little radiant heat. At these times the tape and air temperatures are nearly the same so that the temperature of the tape can be accurately determined and corrections applied.

Invar Tapes and Wires

The research of Dr. Guillaume, of the French Bureau of Weights and Measures, led to the discovery of *invar,* the least expansible steel alloy containing about 36% nickel. The coefficient of thermal expansion is the lowest of all the known metals and alloys and seldom exceeds 0.0000005 per dergree F. However, the temperature coefficient not only varies with the percentage of nickel, but also with the thermal and mechanical treatment given to each tape. Every tape has its own coefficient which must be separately determined. Another peculiar thing with invar is that it undergoes some secular change in its length which increases slowly with time specially in the first few years. Due to this reason, invar can never be used for permanent standards. The instability, however, can be reduced by a process of artificial ageing, which consists in annealing them by exposure for several days to the temperature of boiling water. The coefficient of expansion of invar tapes also show slight variation with time, and should be determined both before and after a base line measurement. Invar is much softer than steel and must be handled very carefully. It should be wound upon a large reel or drum. The tensile strength of invar varies from 100000 to 125000 lb/in^2 with an elastic modulus of 22×10^6 ib/in^2. Invar tapes can be obtained in lengths of 100 ft to 300 ft in the 1/4 in. width, and in the 6 mm width between 24 and 30 to 100 metres. The metric tape is usually divided to millimetres for 2 lengths of 10 cm at each end. In the 100 ft tape, the ends are divided to 1/25 in. or 1/100 ft. Due to their high cost, they are not used for ordinary work.

Equipment for base line measurement :

The equipment for base line measurement by flexible apparatus consists of the following:

1. Three standardised tapes : out of the three tapes one is used for field measurement and the other two are used for standardising the field tape at suitable intervals.
2. Straining device, marking tripods or stakes, and supporting tripods or staking.
3. A steel tape for spacing the tripods or stakes.
4. Six thermometers, four for measuring the temperature of the field tape and two for standardising the four thermometers.
5. A sensitive and accurate spring balance.

The Field Work

The field work for the measurement of base line is carried out by two parties :

(1) *The setting out party* consisting of two surveyors and a number of porters have the duty to place the measuring tripods in alignment in advance of the measurement, and at correct intervals.

(2) *The measuring party* consisting of two observers, recorder, leveller and staffman, for actual measurements.

The base line is cleared off the obstacles and is divided into suitable sections 1/2 to 1 kilometre in length and is accurately aligned with a transit. Whenever the alignment changes, stout posts are driven firmly in the ground. The setting out party then places measuring tripods in alignments in advance of the measurement which can be done by *two methods* :

(1) Measurement on Wheeler's methods by Wheeler's base line apparatus.

(2) Jaderin's method.

(*i*) **Wheeler's base line apparatus** (Fig. 8.12)

The marking stakes are driven on the line with their tops about 50 cm above the surface of the ground, and at distance apart slightly less than the length of the tape. On

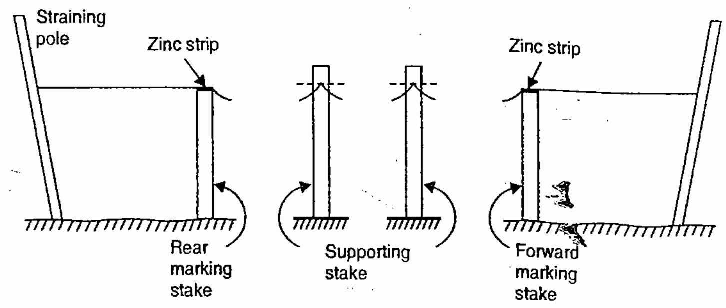


FIG. 8.12. WHEELER'S BASE LINE APPARATUS.

the tops of the marking stakes, strips of zinc, 4 cm in width, are nailed for the purpose of scribing off the extremities of the tapes. Supporting stakes are also provided at intervals of 5 to 15 metres, with their faces in the line. Nails are driven in the sides of the support stakes to carry hooks to support the tape. The points of supports are set either on a uniform grade between the marking stakes or at the same level. A weight is attached to the other end of the straining tripod to apply a uniform pull. To measure the length, the rear end of the tape is connected to the straining pole and the forward end of the spring balance to the other end of which a weight is attached. The rear end of the tape is adjusted to coincide with the mark on the zinc strip at the top of the rear marking stake by means of the adjusting screw of the slide. The position of the forward end of the tape is marked on the zinc strip at the top of the forward marking stake after proper tension has been applied. The work is thus continued. The thermometers are also observed.

(*ii*) **Jaderin's method**

In this method, introduced by Jaderin, the measuring tripods are aligned and set at a distance approximately equal to the length of the tape. The ends of the tapes are attached to the straining tripods to which weights are attached. The spring balance is used to measure the tension. The rear mark of the tape is adjusted to coincide with the mark on rear measuring tripod. The mark on the forward measuring tripod is then set at the forward mark of the tape. The tape is thus suspended freely and is subjected to constant tension. An aligning and levelling telescope is also sometimes fitted to the measuring tripod. The levelling observations are made by a level and light staff fitted with a rubber pad

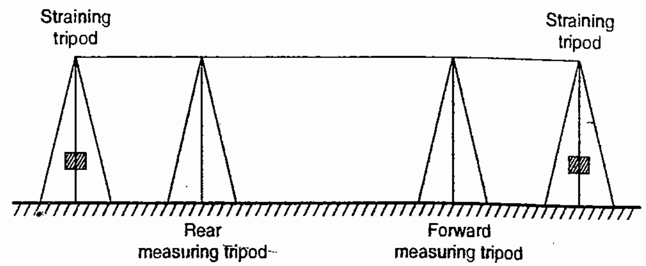


FIG. 8.13. JADERIN'S METHOD.

for contact with the tripod heads. The tension applied should not be less than 20 times the weight of the tape.

Measurements by Steel and Brass Wires : Principle of Bimetallic Thermometer

The method of measurement by steel and brass wires is based on Jaderin's application of the principle of bimetallic thermometer to the flexible apparatus. The steel and brass wires are each 24 m long and 1.5 to 2.6 mm in diameter. The distance between the measuring tripods is measured first by the steel wire and then by the brass wire by Jaderin's method as explained above (Fig. 8.13) with reference to invar tape or wire. Both the wires are nickel plated to ensure the same temperature conditions for both. From the measured lengths given by the steel and brass wires, the temperature effect is eliminated as given below :

Let L_s = distance as computed from the absolute length of the steel wire.

L_b = distance computed from the absolute length of the brass wire.

α_s = co-efficient of expansion for steel.

α_b = co-efficient of expansion for brass.

D = corrected distance.

T_m = mean temperature during measurement.

T_s = Temperature at standardisation

$T = T_m - T_s$ = temperature increase.

Now $$D = L_s(1 + \alpha_s T) = L_b(1 + \alpha_b T) \quad ...(1)$$

or $$T(L_b\,\alpha_b - L_s\,\alpha_s) = L_s - L_b$$

$\therefore$ $$T = \frac{L_s - L_b}{L_b\,\alpha_b - L_s\,\alpha_s} \quad ...(2)$$

Substituting this value of T in (1) for steel wire, we get

$$D = L_s\left\{1 + \frac{\alpha_s(L_s - L_b)}{L_b\,\alpha_b - L_s\,\alpha_s}\right\}$$

$\therefore$ Correction for steel wire $= D - L_s = + \dfrac{L_s\,\alpha_s(L_s - L_b)}{L_b\alpha_b - L_s\,\alpha_s}$

$$\approx + \frac{\alpha_s(L_s - L_b)}{\alpha_b - \alpha_s} \text{ with sufficient accuracy.}$$

Similarly, correction for brass wire $= D - L_b \approx + \dfrac{\alpha_b(L_s - L_b)}{\alpha_b - \alpha_s}$

The corrections can thus be applied without measuring the temperature in the field. The method has however been superseded by the employment of invar tapes or wires.

8.8. CALCULATION OF LENGTH OF BASE : TAPE CORRECTIONS

After having measured the length, the correct length of the base is calculated by applying the following corrections :

1. Correction for absolute length
2. Correction for temperature
3. Correction for pull or tension
4. Correction for sag

5. Correction for slope
6. Correction for alignment
7. Reduction to sea level.
8. Correction to measurement in vertical plane

1. Correction for Absolute Length

If the *absolute length* (or actual length) of the tape or wire is not equal to its *nominal* or *designated length*, a correction will have to be applied to the measured length of the line. If the absolute length of the tape is greater than the nominal or the designated length, the measured distance will be too short and the correction will be additive. If the absolute length of the tape is lesser than the nominal or designated length, the measured distance will be too great and the correction will be subtractive.

Thus, $$C_a = \frac{L.\,c}{l} \quad ...(8.6)$$

where C_a = correction for absolute length

L = measured length of the line

c = correction per tape length

l = designated length of the tape

C_a will be of the same sign as that of c.

2. Correction for Temperature

If the temperature in the field is *more* than the temperature at which the tape was standardised, the length of the tape *increases*, measured distance becomes *less*, and the correction is therefore, *additive*. Similarly, if the temperature is *less*, the length of the tape *decreases*, measured distance becomes *more* and the correction is *negative*. The temperature correction is given by

$$C_t = \alpha(T_m - T_0)L \quad ...(8.7)$$

where α = coefficient of thermal expansion

T_m = mean temperature in the field during measurement

T_0 = temperature during standardisation of the tape

L = measured length.

If, however, steel and brass wires are used simultaneously, as in Jaderin's Method, the corrections are given by

$$C_t\ (\text{brass}) = \frac{\alpha_b(L_s - L_b)}{\alpha_b - \alpha_s} \quad ...[8.8\ (a)]$$

and $$C_t\ (\text{steel}) = \frac{\alpha_s(L_s - L_b)}{\alpha_b - \alpha_s} \quad ...[8.8\ (b)]$$

To find the new standard temperature T_0' which will produce the nominal length of the tape or band

Some times, a tape is not of standard or designated length at a given standard temperature T_0. The tape/band will be of the designated length at a new standard temperature T_0'.

Let the length at standard temperature T_0 be $l \pm \delta l$, where l is the designated length of the tape.

Let ΔT be the number of degrees of temperature change required to change the length of the tape by $= \delta l$

Then $$\delta l = (l \pm \delta l)\, \alpha\, \Delta T$$

$\therefore$ $$\Delta T = \frac{\delta l}{(l \pm \delta l)\, \alpha} \simeq \frac{\delta l}{l\, \alpha}$$

(Neglecting δl which will be very small in comparison to l)

If T_0' is the new standard temperature at which the length of the tape will be exactly equal to its designated length l, we have

$$T_0' = T_0 \pm \Delta T$$

or $$T_0' = T_0 \pm \frac{\delta l}{l\alpha} \quad \ldots(8.9)$$

See example 8.15 for illustration.

3. Correction for Pull or Tension

If the pull applied during measurement is *more* than the pull at which the tape was standardised, the length of the tape *increases*, measured distance becomes *less*, and the correction is *positive*. Similarly, if the pull is less, the length of the tape *decreases*, measured distance becomes more and the correction is negative.

If C_p is the correction for pull, we have

$$C_p = \frac{(P - P_0)\, L}{AE} \quad \ldots(8.10)$$

where P = Pull applied during measurement (N)
P_0 = Standard pull (N)
L = Measured length (m)
A = Cross-sectional area of the tape (cm^2)
E = Young's Modulus of Elasticity (N/cm^2)

The pull applied in the field should be less than 20 times the weight of the tape.

4. Correction for Sag : When the tape is stretched on supports between two points, it takes the form of a horizontal catenary. The horizontal distance will be less than the distance along the curve. The difference between horizontal distance and the measured length along catenary is called the *Sag Correction*. For the purpose of determining the correction, the curve may be assumed to be a parabola.

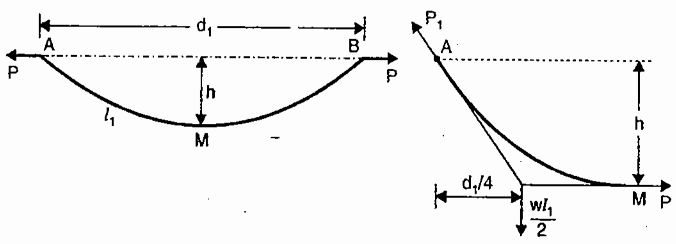


FIG. 8.14. SAG CORRECTION

Let l_1 = length of the tape (in metres) suspended between A and B
M = centre of the tape
h = vertical sag of the tape at its centre
w = weight of the tape per unit length (N/m)
C_{s1} = Sag correction in metres for the length l_1
C_s = Sag correction in metres per tape length l
$W_1 = wl_1$ = weight of the tape suspended between A and B
d_1 = horizontal length or span between A and B.

The relation between the curved length (l_1) and the chord length (d_1) of a very flat parabola, $\left(i.e., \text{ when } \frac{h}{l_1} \text{ is small}\right)$ is given by

$$l_1 = d_1 \left[1 + \frac{8}{3} \left(\frac{h}{d_1} \right)^2 \right]$$

Hence $$C_{s1} = d_1 - l_1 = -\frac{8}{3} \frac{h^2}{d_1} \quad \ldots(1)$$

The value of h can be found from statics [Fig. 8.14 (*b*)]. If the tape were cut at the centre (M), the exterior force at the point would be tension P. Considering the equilibrium of half the length, and taking moments about A, we get

$$Ph = \frac{wl_1}{2} \times \frac{d_1}{4} = \frac{wl_1\, d_1}{8}$$

or $$h = \frac{wl_1\, d_1}{8P} \quad \ldots(2)$$

Substituting the value of h in (1), we get

$$C_{s1} = \frac{8}{3} \frac{1}{d_1} \left(\frac{wl_1\, d_1}{8P} \right)^2 = \frac{d_1}{24P^2} (wl_1)^2 \approx \frac{l_1\, (wl_1)^2}{24P^2} = \frac{l_1 W_1^2}{24P^2} \quad \ldots(8.11)$$

If l is the total length of tape and it is suspended in n equal number of bays, the Sag Correction (C_s) per tape length is given by

$$C_s = n\, C_{s1} = \frac{nl_1\, (wl_1)^2}{24\, P^2} = \frac{l\, (wl_1)^2}{24\, P^2} = \frac{l\, (wl)^2}{24\, n^2\, P^2} = \frac{lW^2}{24n^2\, P^2} \quad \ldots(8.12)$$

where C_s = tape correction per tape length
l = total length of the tape
W = total weight of the tape
n = number of equal spans
P = pull applied

If L = the total length measured
and N = the number of whole length tape
then : Total Sag Correction $= NC_s$ + Sag Correction for any fractional tape length.

Note. Normally, the mass of the tape is given. In that case, the weight W (or wl) is equal to mass $\times\, g$, where the value of g is taken as 9.81. For example, if the mass of tape is 0.8 kg, $W = 0.8 \times 9.81 = 7.848$ N.

It should be noted that the Sag Correction is always negative. If however, the tape was standardised on catenary, and used on flat, the correction will be equal to 'Sag Correction for standard pull – sag correcion at the measured pull', and will be positive if the measured pull in the field is more than the standard pull.

For example, let the tape be standardised in catenary at 100 N pull.

If the pull applied in the field is 120 N, the Sag Correction will be = Sag Correction for 100 N pull – Sag Correction for 120 N pull

$$= \frac{l_1 W_1^2}{24\,(100)^2} - \frac{l_1 (W_1)^2}{24\,(120)^2} = \frac{l_1 W_1^2}{24}\left[\frac{1}{(100)^2} - \frac{1}{(120)^2}\right]$$

and is evidently positive

If the pull applied in the field is 80 N, the Sag Correction will be

$$= \frac{l_1 W_1^2}{24\,(100)^2} - \frac{l_1 W_1^2}{24\,(80)^2} = \frac{l_1 W_1^2}{24}\left[\frac{1}{(100)^2} - \frac{1}{(80)^2}\right]$$ *and is evidently negative.*

If, however the pull applied in the field is equal to the standard pull, no Sag Correction is necessary. See Example 8.11.

Equation 8.12 gives the Sag Correction when the ends of the tape are at the same level. If, however, the ends of the tape are not at the same level, but are at an inclination θ with the horizontal, the Sag Correction given is by the formula,

$$C_s' = C_s \cos^2\theta \left(1 + \frac{wl}{P}\sin\theta\right) \qquad \text{...[8.13 (a)]}$$

when tension P is applied at the higher end ;

and $$C_s' = C_s \cos^2\theta \left(1 - \frac{wl}{P}\sin\theta\right) \qquad \text{...[8.13 (b)]}$$

when tension P is applied at the lower end.

If, however, θ is small, we can have

$$C_s' = C_s \cos^2\theta \qquad \text{...(8.14)}$$

irrespective of whether the pull is applied at the higher end or at the lower end. *It should be noted that equation 8.14 includes the corrections both for sag and slope, i.e. if equation 8.14 is used, separate correction for slope is not necessary.* See Example 8.13.

Normal Tension. Normal tension is the pull which, when applied to the tape, equalises the correction due to pull and the correction due to sag. Thus, at normal tension or pull, the effects of pull and sag are neutralised and no correction is necessary.

The correction for pull is $C_p = \frac{(P_n - P_0)\, l_1}{AE}$ (additive)

The correction for sag $C_{s1} = \frac{l_1 (wl_1)^2}{24\, P_n^2} = \frac{l_1 W_1^2}{24\, P_n^2}$ (subtractive)

where P_n = the *normal pull* applied in the field.

Equating numerically the two, we get

$$\frac{(P_n - P_0)\, l_1}{AE} = \frac{l_1 W_1^2}{24\, P_n^2}$$

$$P_n = \frac{0.204\, W_1 \sqrt{AE}}{\sqrt{P_n - P_0}} \qquad \text{...(8.15)}$$

The value of P_n is to be determined by trial and error with the help of the above equation.

5. Correction for Slope or Vertical Alignment

The distance measured along the slope is always greater than the horizontal distance and hence the correction is always *subtractive*.

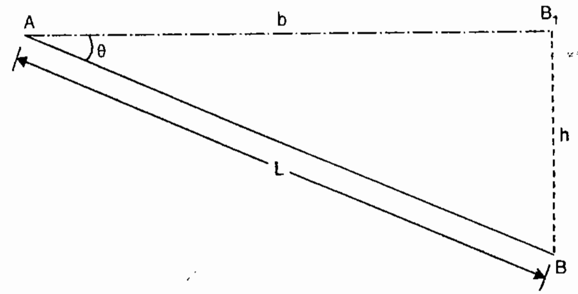


FIG. 8.15. CORRECTION FOR SLOPE.

Let

$AB = L$ = inclined length measured

AB_1 = horizontal length

h = difference in elevation between the ends

C_V = slope correction, or correction due to vertical alignment

Then $$C_V = AB - AB_1 = L - \sqrt{L^2 - h^2}$$

$$= L - L\left(1 - \frac{h^2}{2L^2} - \frac{h^4}{8L^4}\right) = \frac{h^2}{2L} + \frac{h^4}{8L^3} + \ldots$$

The second term may safely be neglected for slopes flatter than about 1 in 25.

Hence, we get : $$C = \frac{h^2}{2L} \text{ (subtractive)} \qquad \text{...(8.16)}$$

Let L_1, L_2 ...etc.= length of successive uniform gradients

h_1, h_2, ...etc.= differences of elevation between the ends of each.

The total slope correction $= \frac{h_1^2}{2L_1} + \frac{h_2^2}{2L_2} + \ldots\ldots = \Sigma \frac{h^2}{2L}$

If the grades are of uniform length L, we get total slope correction $= \frac{\Sigma h^2}{2L}$

If the angle (θ) of slope is measured instead of h, the correction is given by

$$C_V = L - L\cos\theta = L(1 - \cos\theta) = 2L\sin^2\frac{\theta}{2} \qquad \text{...(8.17)}$$

Effect of measured value of slope θ

Usually, the slope θ of the line is measured instrumentally, with a theodolite. In that case the following modification should be made to the measured value of the slope. See Fig. 8.16.

Let h_1 = height of the instrument at A

h_2 = height of the target at B

α = measured vertical angle

θ = slope of the line AB

l = measured length of the line

Then $\theta = \alpha + \delta\alpha$.

From $\Delta A_1B_1B_2$, by sine rule, we get

$$\sin \delta\alpha = \frac{(h_1 - h_2) \sin (90° + \alpha)}{l} = \frac{(h_1 - h_2) \cos \alpha}{l}$$

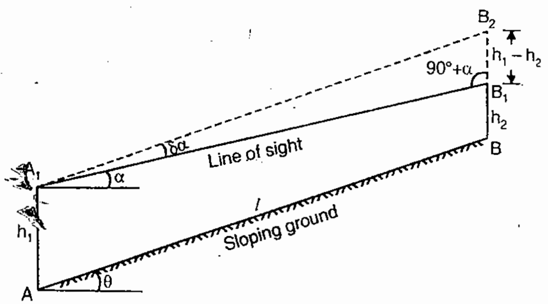


FIG. 8.16

$$\therefore \quad \delta\alpha'' = \frac{206265 (h_1 - h_2) \cos \alpha}{l} \quad ...(8.18)$$

The sign of $\delta\alpha$ will be obtained by the above expression itself.

6. Correction for horizontal alignment

(*a*) **Bad ranging or misalignment**

If the tape is stretched out of line, measured distance will always be *more* and hence the *correction* will be *negative*. Fig. 8.17 shows the effect of wrong alignment, in which $AB = (L)$ is the measured length of the line, which is along the wrong alignment while the correct alignment is AC. Let d be the perpendicular deviation.

Then $$L^2 - l^2 = d^2$$

or $$(L + l)(L - l) = d^2$$

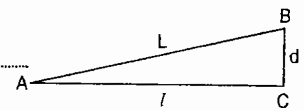


FIG. 8.17

Assuming $L = l$ and applying it to the first parenthesis only, we get

$$2L(L - l) \simeq d^2$$

or $$L - l \simeq \frac{d^2}{2L}$$

$$\text{Hence correction } C_h = \frac{d^2}{2L} \quad ...(8.19)$$

It is evident that smaller the value of d is in comparison to L, the more accurate will be the result.

(*b*) **Deformation of the tape in horizontal plane**

If the tape is not pulled straight and the length L_1 of the tape is out of the line by amount d,

$$\text{Then,} \quad C_h = \frac{d^2}{2L_1} + \frac{d^2}{2L_2} \quad ...(8.20)$$

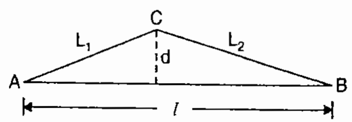


FIG. 8.18

(*c*) **Broken base**

Due to some obstructions etc., it may not be possible to set out the base in one continuous straight line. Such a base is then called a *broken base*.

In Fig. 8.19, let AC = straight base

AB and BC = two sections of the broken base

β = exterior angle measured at B.

$AB = c$; $BC = a$; and $AC = b$.

The correction (C_h) for horizontal alignment is given by

$$C_h = (a + c) - b$$

....(subtractive)

The length b is given by the sine rule

$$b^2 = a^2 + c^2 + 2\,ac \cos \beta$$

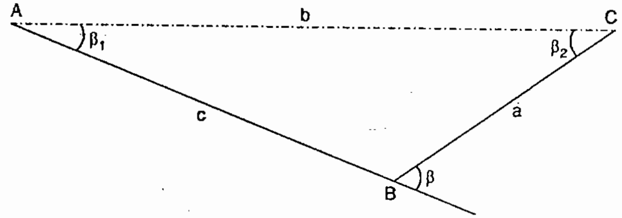


FIG. 8.19. CORRECTION FOR HORIZONTAL ALIGNMENT

or $$a^2 + c^2 - b^2 = -2\,ac \cos \beta$$

Adding $2ac$ to both the sides of the above equation, we get

$$a^2 + c^2 - b^2 + 2ac = 2ac - 2ac \cos \beta \quad \text{or} \quad (a + c)^2 - b^2 = 2ac(1 - \cos \beta)$$

$$\therefore \quad (a + c) - b = \frac{2ac(1 - \cos \beta)}{(a + c) + b} = \frac{4ac \sin^2 \frac{1}{2}\beta}{(a + c) + b}$$

$$\therefore \quad C_h = (a + c) - b = \frac{4ac \sin^2 \frac{1}{2}\beta}{(a + c) + b} \quad ...[8.21(a)]$$

Taking $\sin \frac{1}{2}\beta \approx \frac{1}{2}\beta$ and expressing β in minutes, we get

$$C_h = \frac{ac\,\beta^2 \sin^2 1'}{(a + c) + b} \quad ...[8.21(b)]$$

Taking $b \approx (a + c)$ we get

$$C_h = \frac{ac\,\beta^2 \sin^2 1'}{2(a + c)} \quad ...[8.21]$$

$$= \frac{ac\,\beta^2}{(a + c)} \times 4.2308 \times 10^{-8} \quad ...[8.21(c)]$$

where $\frac{1}{2} \sin^2 1' = 4.2308 \times 10^{-8}$.

7. Reduction to Mean Sea Level

The measured horizontal distance should be reduced to the distance at the mean sea level, called the *Geodetic distance*. If the length of the base is reduced to mean sea level, the calculated length of all other triangulation lines will also be corresponding to that at mean sea level.

Now correction of sag $= C_s = \frac{nl_1(wl_1)^2}{24\,P^2} = \frac{nl_1 W_1^2}{24\,P^2} = \frac{3 \times 10 \times (6.24)^2}{24\,(100)^2} = \mathbf{0.00487}$ **m.**

Example 8.10. *A steel tape 20 m long standardised at 55° F with a pull of 10 kg was used for measuring a base line. Find the correction per tape length, if the temperature at the time of measurement was 80 °F and the pull exerted was 16 kg. Weight of 1 cubic cm of steel = 7.86 g, Wt. of tape = 0.8 kg and E = 2.109 × 10⁶ kg/cm². Coefficient of expansion of tape per 1°F = 6.2 × 10⁻⁶.*

Solution. Correction for temperature $= 20 \times 6.2 \times 10^{-6}(80 - 55) = 0.0031$ m (additive)

Correction for pull $= \frac{(P - P_0)L}{AE}$

Now, weight of tape $= A\,(20 \times 100)(7.86 \times 10^{-3})$ kg $= 0.8$ kg (given)

$\therefore \quad A = \frac{0.8}{7.86 \times 2} = 0.051$ sq. cm

Hence, $C_p = \frac{(16 - 10)\,20}{0.051 \times 2.109 \times 10^6} = 0.00112$ (additive)

Correction for sag $= \frac{l_1(wl_1)^2}{24\,P^2} = \frac{20(0.8)^2}{24\,(16)^2} = 0.00208$ m (subtractive)

$\therefore$ **Total correction** $= +0.0031 + 0.00112 - 0.00208 = +\mathbf{0.00214}$ **m**

Example 8.11. *A nominal distance of 30 m was set out with a 30 m steel tape from a mark on the top of one peg to a mark on the top of another, the tape being in catenary under a pull of 100 N and at a mean temperature of 70 °F. The top of one peg was 0.25 m below the top of the other. The top of the higher peg was 460 m above mean sea level. Calculate the exact horizontal distance between the marks on the two pegs and reduce it to mean sea level, if the tape was standardised at a temperature of 60° F in catenary under a pull of (a) 80 N. (b) 120 N and (c) 100 N.*

Take radius of earth = 6370 km.

Density of tape = 7.86 g/cm³

Section of tape = 0.08 sq. cm.

Co-efficient of expansion = 6 × 10⁻⁶ per 1° F

Young's modulus = 2 × 10⁷ N/cm²

Solution.

(*i*) Correction for standardisation ... *nil.*

(*ii*) Correction for slope $= \frac{h^2}{2L} = \frac{(0.25)^2}{2 \times 30} = 0.0010$ m (subtractive)

(*iii*) Temperature correction $= L\alpha\,(T_m - T_0) = 30 \times 6 \times 10^{-6}\,(70 - 60)$
$= 0.0018$ m (additive)

(*iv*) Tension correction $= \frac{(P - P_0)\,L}{AE}$

(*a*) When $P_0 = 80$ N, tension correction $= \frac{(100 - 80)\,30}{0.08 \times 2 \times 10^7} = 0.0004$ m (additive).

(*b*) When $P_0 = 120$ N, tension correction $= \frac{(100 - 120)\,30}{0.08 \times 2 \times 10^7} = 0.0004$ m (subtractive)

(*c*) When $P_0 = 100$ N, tension correction = zero

(*v*) Sag correction $= \frac{LW^2}{24\,P^2}$

Now mass of tape per metre run

$= (0.08 \times 1 \times 100) \times \frac{7.86}{1000}$ kg $= 0.06288$ kg/m

$\therefore$ Weight of tape per metre run $= 0.06288 \times 9.81 = 0.6169$ N/m

$\therefore$ Total weight of tape $= 0.6169 \times 30 = 18.51$ N.

(*a*) When $P_0 = 80$ N

Sag correction $= \frac{30 \times (18.51)^2}{24\,(80)^2} - \frac{30\,(18.51)^2}{24\,(100)^2} = 0.0669 - 0.04283 = 0.02407$ (additive).

(*b*) When $P_0 = 120$ N

Sag correction $= \frac{30\,(18.51)^2}{24\,(120)^2} - \frac{30\,(18.51)^2}{24\,(100)^2} = 0.02974 - 0.04283$
≈ -0.0131 m (*i.e.*, subtractive).

(*c*) When $P_0 = 100$ N $= P$, sag correction is zero.

Final correction

(*a*) Total correction $= -0.0010 + 0.0018 + 0.0004 + 0.02407$ m $= +\mathbf{0.02527}$ m.

(*b*) Total correction $= -0.0010 + 0.0018 - 0.0004 - 0.0131$ m $= -\mathbf{0.0127}$ m.

(*c*) Total correction $= -0.0010 + 0.0018 + 0 + 0 = +\mathbf{0.0008}$ **m.**

Example 8.12. *It is desired to find the weight of the tape by measuring its sag when suspended in catenary with both ends level. If the tape is 20 m long and the sag amounts to 20.35 cm at the mid span under a tension of 100 N, what is the weight of the tape?*

Solution. (Fig. 8.14). From expression (2) of sag, we have

$$h = \frac{wl_1\,d_1}{8P}$$

But $h = 20.35$ cm (given)

Taking $l_1 = d_1$ (approximately), we get

$$h = \frac{wl_1^2}{8P}$$

or $w = \frac{8Ph}{l_1^2} = \frac{8 \times 100}{20 \times 20} \times \frac{20.35}{100}$ N/m $= \mathbf{0.407}$ **N/m**

Mass of tape $= \frac{0.407}{9.81} = 0.0415$ **kg/m = 41.5** g/m.

Example 8.13. *Derive an expression for correction to be made for the effect of sag and slope in base line measurement, introducing the case where the tape or wire is supported at equidistant points between measuring pegs or tripods.*

Solution. (Fig. 8.22)

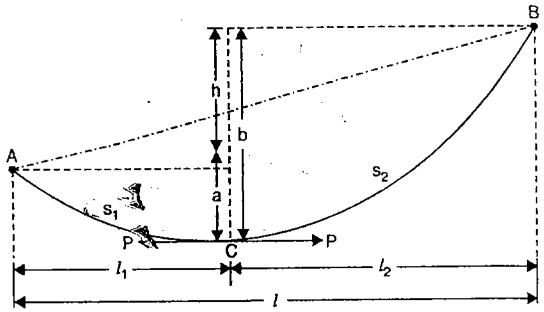


FIG. 8.22.

In Fig. 8.22, let the tape be supported at A and B, and let C be the lowest point where the tension is horizontal having value equal to P. Let the two horizontal lengths be l_1 and l_2 such that $l_1 + l_2 = l$. Let s_1 and s_2 be the lengths along the curve such that $s_1 + s_2 = s$ = total length along the curve. Let a = difference in elevation between A and C and b = difference in elevation between B and C. Let $h = b - a$ = difference in level between B and A. Treating approximately the curve to be parabola, the equations are :

$$y = k_1 x^2, \text{ for } CA \quad \text{and} \quad y = k_2 x^2, \text{ for } CB$$

where the origin is C in both the cases.

Now, when $x = l_1$, $y = a$; $\therefore \quad k_1 = \dfrac{a}{l_1^2}$

When $x = l_2$, $y = b$; $\therefore \quad k_2 = \dfrac{b}{l_2^2}$

Hence the equations are : ....

$$y = \frac{ax^2}{l_1^2} \text{ for } CA \quad \text{and} \quad y = \frac{bx^2}{l_2^2} \text{ for } CB$$

$$\therefore \quad \frac{dy}{dx} = \frac{2ax}{l_1^2} \text{ for } CA \quad \text{and} \quad \frac{dy}{dx} = \frac{2bx}{l_2^2} \text{ for } CB$$

Thus, the length of the curve

$$s = s_1 + s_2 = \int_0^{l_1} \left\{ 1 + \left(\frac{2\,a\,x}{l_1^2} \right)^2 \right\} dx + \int_0^{l_2} \left\{ 1 + \left(\frac{2bx}{l_2^2} \right)^2 \right\} dx$$

$$= \left[l_1 + l_2 + \frac{2}{3} \left(\frac{a^2}{l_1} + \frac{b^2}{l_2} \right) \right] = l + \frac{2}{3} \left(\frac{a^2}{l_1} + \frac{b^2}{l_2} \right) \qquad \ldots(1)$$

Again, from the statics of the figure, we get

$$P \times a = \frac{wl_1^2}{2} \text{ for } CA, \text{ and } P \times b = \frac{wl_2^2}{2} \; ; \; \therefore \quad P = \frac{wl_1^2}{2a} = \frac{wl_2^2}{2b} \qquad \ldots(2)$$

and

$$\frac{a}{l_1^2} = \frac{b}{l_2^2} \qquad \ldots(3)$$

Substituting these values in (1), we get

$$s - l = \frac{2}{3} \left\{ \left(\frac{w}{2P} \right)^2 l_1^3 + \left(\frac{w}{2P} \right)^2 l_2^3 \right\} = \frac{1}{6} \frac{w^2}{P^2} (l_1^3 + l_2^3) \qquad \ldots(8.26)$$

Now, writing $l_1 = \frac{1}{2} l - e$ and $l_2 = \frac{1}{2} l + e$, we get

$(s - l)$ = (sag + level) correction

$$= \frac{1}{6} \frac{w^2}{P^2} [(\tfrac{1}{2} l - e)^3 + (\tfrac{1}{2} l + e)^3] = \frac{w^2}{6P^2} \left\{ \frac{l^3}{4} + \frac{3}{4} l (l_2 - l_1)^2 \right\}$$

$$= \frac{w^2 l^3}{24\,P^2} + \frac{w^2}{8P^2} \frac{l^2 (l_2 - l_1)^2}{l} = \frac{l(wl)^2}{24P^2} + \frac{w^2}{8P^2} \frac{(l_2^2 - l_1^2)^2}{l} \qquad \ldots(4)$$

Now from (3), $\quad \dfrac{b-a}{a} = \dfrac{l_2^2 - l_1^2}{l_1^2}$

From (2), $\quad \dfrac{w^2}{4P^2} = \dfrac{a^2}{l_1^4}$

$$\therefore \quad \frac{w^2}{8P^2} \frac{(l_2^2 - l_1^2)^2}{l} = \frac{a^2}{2l_1^4} \left\{ \frac{b-a}{a} l_1^2 \right\}^2 \frac{1}{l} = \frac{(b-a)^2}{2l} = \frac{h^2}{2l}$$

Substituting in (4), we get

$$(s - l) = \frac{l(wl)^2}{24P^2} + \frac{h^2}{2l}$$

Thus, the total correction is the sum of the separate corrections for sag and slope.

Example 8.14. *A flexible, uniform, inextensible tape of total weight 2W hangs freely between two supports at the same level under a tension T at each support. Show that horizontal distance between the supports is*

$$\frac{H}{w} \log_e \frac{T+W}{T-W}$$

where H *= horizontal tension at the centre of the tape and* w *= weight of tape per unit length.*

Solution : Fig. 3.23 (*a*) shows the whole tape, being hung from two supports A and B. Let O be the lowest point, which is the origin of co-ordinates. Fig. 8.23 (*b*) shows a portion OM of the tape, of length s, such that the horizontal tension at O is H, and the tension P at point M makes an angle ψ with the x-axis. Resolving forces vertically and horizontally for this portion of tape,

$$P \sin \psi = w \,.\, s \qquad \ldots(1)$$

$$P \cos \psi = H \qquad \ldots(2)$$

$$\therefore \quad \tan \psi = \frac{w \,.\, s}{H}$$

(From 1 and 2)

Differentiating with respect to x,

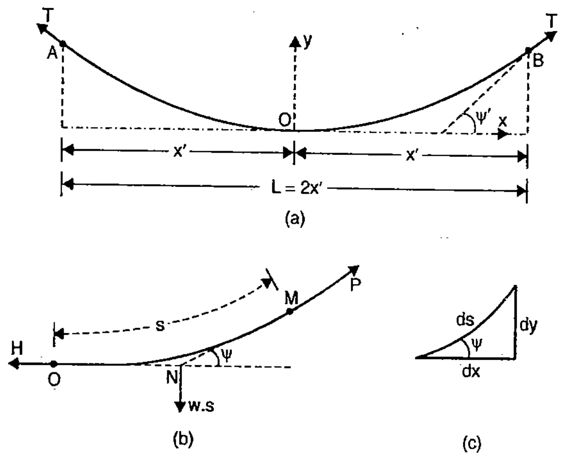


FIG. 8.23.

$$\sec^2 \psi \frac{d\psi}{dx} = \frac{w}{H}\frac{ds}{dx} \qquad ...(3)$$

Now, from the elemental triangle [Fig. 3.40 (c)]

$$\frac{ds}{dx} = \sec \psi$$

$$\therefore \quad \sec^2 \psi \cdot \frac{d\psi}{dx} = \frac{w}{H} \sec \psi$$

or $$\sec \psi \cdot \frac{d\psi}{dx} = \frac{w}{H} \qquad ...(4)$$

Let x' be half the length of tape, and ψ' be the inclination of tangent at the end. Integrating Eq. (4) from O to B, we get

$$\int_0^{\psi'} \sec \psi \, d\psi = \int_0^{x'} \frac{w}{H} dx$$

$$\therefore \quad \left[\log_e (\sec \psi + \tan \psi)\right]_0^{\psi'} = \frac{w}{H} x'$$

or $$x' = \frac{H}{w}\left(\log_e \frac{\sec \psi' + \tan \psi'}{1+0}\right)$$

or $$x' = \frac{H}{w} \log_e (\sec \psi' + \tan \psi') \qquad ...(5)$$

Again, resolving vertically for one-half of the tape,

$$T \sin \psi' = W \quad \text{or} \quad \sin \psi' = \frac{W}{T}.$$

$$\therefore \quad \cos \psi' = \sqrt{1 - \sin^2 \psi'} = \frac{\sqrt{T^2 - W^2}}{T}$$

Also, $$\tan \psi' = \frac{W}{\sqrt{T^2 - W^2}}$$

Substituting the values in Eq. (5), we get

$$x' = \frac{H}{w} \log_e \left[\frac{T}{\sqrt{T^2 - W^2}} + \frac{W}{\sqrt{T^2 - W^2}}\right] = \frac{H}{w} \log_e \left(\frac{T + W}{\sqrt{T^2 - W^2}}\right)$$

$$= \frac{H}{w} \log_e \sqrt{\frac{T+W}{T-W}} = \frac{1}{2}\frac{H}{w} \log_e \frac{T+W}{T-W}$$

The total horizontal distance $= 2x'$

$$= \frac{H}{w} \log_e \frac{T+W}{T-W} \qquad \text{(Hence proved)}$$

Example 8.15. *A field tape, standardised at 18°C measured 100.0056 m. Determine the temperature at which it will be exactly of the nominal length of 100 m. Take $\alpha = 11.2 \times 10^{-6}$ per °C.*

Solution : Given $\delta l = 0.0056$ m ; $T_0 = 18°$ C

New standard temperature $T_0' = T_0 \pm \frac{\delta l}{l\alpha} = 18° - \frac{0.0056}{100 \times 11.2 \times 10^{-6}} = 18° - 5° =$ **13° C**

Example 8.16. *A distance AB measures 96.245 m on a slope. From a theodolite set at A, with instrument height of 1.400 m, staff reading taken at B was 1.675 m with a vertical angle of 4° 30′ 40″. Determine the horizontal length of the line AB. What will be the error if the effect were neglected.*

Solution : Given $h_1 = 1.400$ m; $h_2 = 1.675$ m; $\alpha = 4° 30' 20''$; $l = 96.245$ m

$$\delta\alpha'' = \frac{206265\,(h_1 - h_2) \cos \alpha}{l} = \frac{206265\,(1.400 - 1.675) \cos 4° 30' 20''}{96.245}$$

$$= -588'' = -0° 09' 48''$$

$$\therefore \quad \theta = \alpha + \delta\alpha = 4° 30' 20'' - 0° 09' 48'' = 4° 20' 32''$$

$\therefore$ Horizontal length $L = l \cos\theta = 96.245 \cos 4° 20' 32'' = 95.966$ m.

If the effect were neglected, $L = 96.245 \cos 4° 30' 40'' = 95.947$ m

$\therefore$ Error = **0.019 m**

Example 8.17. *(a) Calculate the elongation at 400 m of a 1000 m mine shaft measuring tape hanging vertically due to its own mass. The modulus of elasticity is 2×10^5 N/mm², the mass of the tape is 0.075 kg/m and the cross-sectional area of the tape is 10.2 mm².*

(b) If the same tape is standardised as 1000.00 m at 175 N tension, what is the true length of the shaft recorded as 999.126 m ?

Solution

(a) Taking $M = 0$, we have

$$s_x = \frac{mgx}{2AE}(2l - x) = \frac{0.075 \times 9.81 \times 400\,(2000 - 400)}{2 \times 10.2 \times 2 \times 10^5} = \mathbf{0.115\ m}$$

(b) $$s = \frac{gx}{AE}\left[M + \frac{m}{2}(2l - x) - \frac{P_0}{g}\right]$$

Here $x = 999.126$, $M = 0$ and $P_0 = 175$

$$\therefore \quad s = \frac{9.81 \times 999.126}{10.2 \times 2 \times 10^5}\left[0 + \frac{0.075}{2}(2 \times 1000 - 999.126) - \frac{175}{9.81}\right]$$

$$= \mathbf{0.095\ m}$$

8.9. MEASUREMENT OF HORIZONTAL ANGLES

Instrument. The instruments for geodetic survey require great degree of refinement. In earlier days of geodetic surveys, the required degree of refinement was obtained by making greater diameter of the horizontal circles. The greater theodolite of *Ordinance Survey* has a diameter of 36″. These large diameter theodolites were replaced by the micrometer theodolites (similar in principle to the old 36″and 24″ instruments) such as the Troughton and Simm's 12″ or the Parkhurst 9″. However, more recently the tendency has been to replace the micrometer theodolites by others of the double reading type (glass arc) such as the Wild, Zeiss and Tavistock having diameters of $5\frac{1}{2}''$ and 5″ respectively.

The distinguished features of the double reading theodolite with optical micrometers are as follows:

(*i*) They are small and light
(*ii*) The graduations are on glass circle and are much finer.
(*iii*) The mean of the two readings on opposite sides of the circle is read directly in an auxiliary eye-piece generally besides the telescope. This saves the observing time, and also saves distrubance of the instrument.
(*iv*) No adjusments for micrometer run are necessary.
(*v*) It is completely water proof and dust proof.
(*vi*) It is electrically illuminated.

There are two types of instruments used in the triangulation of high precision :

1. The repeating theodolite.
2. The direction theodolite.

1. The Repeating Theodolite

The characteristic feature of the repeating theodolite is that it has a double vertical axis (two centres and two clamps). It has two or more verniers to read to 20, 10 or 5 seconds. The ordinary transit is the repeating theodolite. The vernier theodolite by M/s Vickers Instruments Ltd. and the Watts Microptic theodolite no 1, fall under this category, and have been illustrated fully in Author's 'Surveying Vol. 1.'

2. The Direction Theodolite

The direction theodolite has only one vertical axis, and a single horizontal clamp-and-tangent screw which controls the rotation about the vertical axis. Optical micrometers are used to read fractional parts of the smallest divisions of the graduated circle. The direction theodolite is used for very precise work needed in the first order or second-order triangulation survey. There are various direction theodolites in use, and the following will be illustrated here:

(1) Wild T-3 Precision theodolite
(For Wild T-2, see Author's Surveying Vol 1)
(2) Wild T-4 Universal Theodolite.

The Wild T-3 Precision Theodolite

Fig. 8.24 shows the Wild T-3 precision theodolite meant for primary triangulation. Both the horizontal and vertical circles are made of glass. The graduation interval of horizontal circle is 4′ and that of the vertical circle is 8′. The readings can be taken on the optical micrometer direct to 0.2″ and by estimation to 0.02″. The following is the technical data:

Magnification 24, 30 or 40 ×
Clear objective/glass diameter 2.36 in. (60 mm)
Shortest focussing distance 15 ft. (4.5 m)
Normal range ... 20–60 miles (32 km to 96 km)
Field of view at 1000 ft.......29 ft (8.84 m)
Length of telescope 10.2 in. (260 mm)
Sensitivity of alidade level, 7″ per 2 mm
Sensitivity of collimation level 12″ per 2 mm

Coincidence adjustment of vertical circle level to 0.2″
Diameter of horizontal circle 5.5 in. (140 mm)
Graduation intveral of horizontal circle 4′
Diameter of vertical circle 3.8 in. (97 mm)
Graduation interval of vertical circle 8′
Graduation interval of micrometer drum 0.2″.

The vertical axis system consists of the axle bush and the vertical axis turning therein on ball bearings, which is automatically centered by the weight of the instrument. The glass circle is mounted on the outer side of the axle bush and is oriented as desired by drive knob. Since there is only one set of clamp and tangent screws for the motion about vertical axis, the angles are measured by direction method only. This is, therefore, a direction theodolite.

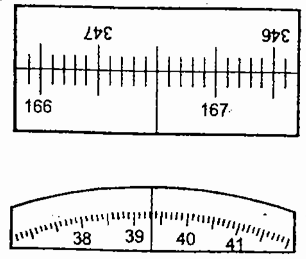


FIG. 8.25.

The micrometers for reading the horizontal and vertical circles are both viewed in the same eye-piece which lies at the side of the telescope. In the field of view of the micrometer appear the circle graduations from two parts of the circle 180° apart, separated by a horizontal line. The horizontal circle is divided in 4′ interval. The appearance of field of view is shown in Fig. 8.25, in which the top window shows the circle readings. A vertical line in the bottom half of the window serves as an index from which the coarse readings are taken. The lower window is graduated to seconds readings and carries a pointer. Coincidence system is used to take the readings. To read the micrometer, micrometer knob is turned so that the two sets of graduations in the upper window appear to one another, and finally coincide. The seconds readings will then be given by the scale and pointer in the lower window. *The reading on the second scale in the bottom window is one-half of the proper readings.* Hence, the number of seconds which are read on this scale must either be doubled, or opposite graduations in the upper window should be brought into coincidence twice and the two readings on the seconds scale added together as illustrated in Fig. 8.25.

To view the horizontal circle reading, an inverted knob is turned in a clockwise direction ; to view the vertical circle reading, this knob is turned in the reverse direction. Thus the same eye-piece can be used for taking the readings of both the circles.

The Wild T-4 Universal Theodolite (Fig. 8.26).

The Wild T-4 is a theodolite of utmost precision for first order triangulation, the determination of geographic positions and taking astronomical observations. The instrument has a horizontal circle of 250 mm (9.84″) which is almost double the diameter of that of T-3 model. The readings can thus be taken with greater accuracy. The theodolite is of the *'broken telescope'* type, that is, the image formed in the telescope is viewed through an eyepiece placed at one end of the trunnion axis which is made hollow. The graduation interval on horizontal circle is 2′ with direct reading to 0.1″ on optical micrometer. The other technical data is as follows :

Telescope power : 65 ×

Clear objective glass aperture : 60 mm (2.36″)

Azimuth (horizontal) circle on glass : 360°

Diameter of scale : 250 mm (9.84″)

Interval between divisions : 2′

Direct reading to 0.1″

Elevation (vertical) circle on glass 360°

Diameter of scale : 145 mm (5″.71)

Interval between divisions : 4′

Direct readings to 0.2″

Setting circle, for telescope angle of sight

Interval of division 1°

Scale reading microscope interval 10′

Angles can be estimated to 1′

Sensitivity of suspension level 1″

of elevation circle level 5″

of Horrebow level (both)– 2″

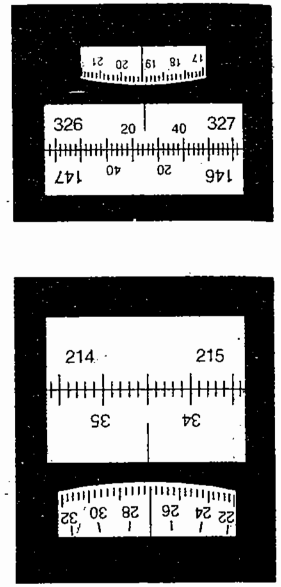


FIG. 8.27.

The vertical and azimuth circles are both equipped with a reading micrometer which gives automatically the arithmetic mean of two diametrically opposed readings. Fig. 8.27 shows the examples of circle readings.

The eye-piece is equipped with the so-called longitude micro-meter for accurate recording of a star's transit. The reversal of the horizontal axis and telescope is carried out by a special hydraulic arrangement which ensures freedom from vibration. Electrical light- ing, to illuminate both circles and field, is built into the body.

Methods of Observation of Horizontal Angles

There are two general methods of observing angles in triangulation :

(1) The Repetition method, and

(2) The Direction method, or reiteration method, or the method of series.

In the *direction method*, the several angles at a station are measured in terms of the direction of their side from that of an initial station. In the *repetition method,* each angle is measured independently by multiplying it mechanically on the circle, the result being obtained by dividing the multiple angle by the number of repetitions. The repetition method is adopted when a repetition theodolite, *i.e.*, the vernier theodolite with a slow motion screw for the lower plate is available. The direction theodolites are equipped with optical micrometers and are much more accurate. Therefore, the repetition method is confined to secondary and tertiary work only while the direction method is used for the primary work.

(A) The method of Repetition

To measure the angle *PQR* at the station *Q*, the following procedure is followed (Fig. 8.28) :

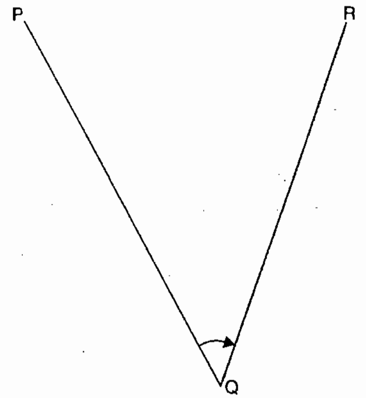


FIG. 8.28

(1) Set the instrument at *Q* and level it. With the help of upper clamp and tangent screw, set 0° reading on vernier *A*. Note the reading of vernier *B*.

(2) Loose the lower clamp and direct the telescope towards the point *P*. Clamp the lower clamp and bisect point *P* accurately by lower tangent screw.

(3) Unclamp the upper clamp and turn the instrument *clockwise* about the inner axis towards *R*. Clamp the upper clamp and bisect *R* accurately with the upper tangent screw. Note the reading of verniers *A* and *B* to get the approximate value of the angle *PQR*.

(4) Unclamp the lower clamp and turn the telescope clockwise to sight *P* again. Bisect *P* accurately by using the lower tangent screw. *It should be noted that the vernier reading will not be changed in this operation since the upper plate is clamped to the lower.*

(5) Unclamp the upper clamp, turn the telescope clockwise and sight *R*. Bisect *R* accurately by upper tangent screw.

(6) Repeat the process until the angle is repeated the required number times (usually 3). The average angle with the face left will be equal to final the reading divided by three.

(7) Change face and make 3 more repetitions as described above. Find the average angle with face right, by dividing the final reading by three.

(8) The average horizontal angle is then obtained by taking the average of the two angles obtained with face left and face right.

Sets by Method of Repetition for High Precision

For measuring an angle to the highest degree of precision, several *sets* of repetitions are usually taken. There are two methods of taking a single set :

(*a*) *First method.* (1) Keeping the telescope normal throughout, measure the angle *clockwise* by 6 repetitions. Obtain the *first value* of the angle by dividing the final reading by 6. (2) Invert the telescope and measure the angle *counter-clockwise* by 6 repetitions. Obtain the *second value* of the angle by dividing the final reading by 6. (3) Take the mean of the first and second value to get the average value of the angle by *first set.*

Take as many sets in this way as may be desired. For first order work, five or six sets are usually required. The final value of the angle will be obtained by taking the mean of the values obtained by different sets.

(*b*) *Second method.* (1) Measure the angle clockwise by six repetitions, the first three with the telescope normal and the last three with telescope inverted. Find the *first value* of the angle by dividing the final by six.

(2) Without altering the reading obtained in the sixth repetition, measure the explement of the angle (*i.e.*, 360° – *PQR*) *clockwise* by six repetitions, the first three with telescope inverted and the last three with telescope normal. Take the reading which should theoretically

be equal to zero (for the initial value). If not, note the error and distribute half the error to the *first value* of the angle. *The result is the corrected value of the angle by the first set.* Take as many sets as are desired and find the average angle. For more accurate work, the initial reading at the beginning of each set may not be set to zero but to different values.

Note. *During an entire set of observations, the transit should not be relevelled.*

Elimination of Error by Method of Repetition

The following errors are eliminated by method of repetition :

(1) Errors due to eccentricity of verniers and centres are eliminated by taking both vernier readings.

(2) Errors due to inadjustments of line of collimation and the trunnion axis are eliminated by taking both face readings.

(3) The errors due to inaccurate graduations are eliminated by taking the readings at different parts of the circle.

(4) Errors due to inaccurate bisection of the object, eccentric centring etc., may be to some extent counter-balanced in different observations.

It should be noted, however, that in repeating angles, operations such as sighting and clamping are multiplied, and hence opportunities for error are multiplied. The limit of precision in the measurement of an angle is ordinarily reached after the fifth or sixth repetition.

Errors due to slip, displacement of station signals, and want of verticality of the vertical axis etc. are not eliminated since they are all cumulative.

(B) The Direction Method : In the direction method, the signals are bisected successively and a value is obtained for each direction at each of several rounds of observations. One of the triangulation stations which is likely to be always clearly visible may be selected as the *initial* or the *reference station.* Let *A* be adopted as the initial station to measure the angles *AOB*, *BOC*, *COD* at *O* (Fig. 8.29) with instrument having more than one micrometer. One of the micrometer is set to 0° and with the telescope direct (or normal), *A* is bisected and all the micrometers read. Each of the stations *B, C, D* are then bisected successively, and all the micrometers read. The stations are then again bisected in the opposite direction as *C*, *B* and *A* and all the micrometers are read after each bisection. Thus, two values are obtained for each angle when the telescope is normal. The telescope is then inverted and the observations are repeated. This constitutes one *set* in which four values of each angle are obtained. The micrometer originally at 0° is now brought to a new reading equal to $\frac{360°}{mn}$ (where *m* is the number of micrometers and *n* is the number of the sets), and a

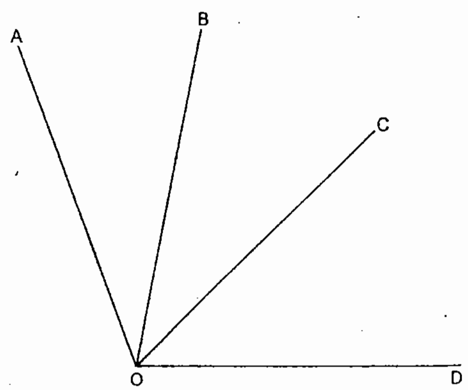


FIG. 8.29.

second set is observed in the same manner on a different part of the circle. The number of *sets* (or *positions*, as is sometimes called) depends on the accuracy required. For first order triangulation, sixteen such sets are required with a direction theodolite, while for second order triangulation *four* and for third order triangulation *two.* With more refined graduations, however, six to eight sets are sufficient for the geodetic work.

Elimination of errors by direction method

The following errors are eliminated by direction method :

(1) The errors due to the eccentricity of vertical axis and of the microscopes are eliminated by reading all the micrometers.

(2) The errors due to the imperfect adjustments of the line of collimation and horizontal axis are eliminated by taking both face observations, *i.e.*, by taking half the observations with the telescope direct and half with the telescope reversed.

(3) The errors due to graduations are eliminated by reading the values of each angle on different parts of the circle. This is done by changing or shifting 'zero' at the beginning of each set.

(4) The errors due to manipulation, twist of the instrument and station due to effect of sun and wind, and slip due to defective clamps. are eliminated by taking half the observations from left to right (*i.e.*, in clockwise direction) and the other half from right to left (*i.e.*, in the anti-clockwise direction).

(5) The accidental errors due to bisection and reading are eliminated by taking number of observations.

8.10. SATELLITE STATION : REDUCTION TO CENTRE

In order to secure well-conditioned triangle or better visibility, objects such as church spires, steeples, flag poles, towers etc. are sometimes selected as the triangulation stations. When the observations are to taken from such a station, it is impossible to set up an instrument over it. In such a case, a subsidiary station, known as a *satellite station* or *eccentric station* or *false station* is selected as near to the main station as possible, and observation are taken to the other triangulaton stations with the same precision as would have been used in the measurement of angles at the true station. These angles are later corrected and reduced to what they would have been if the true station was occupied. The operation of applying the corrections

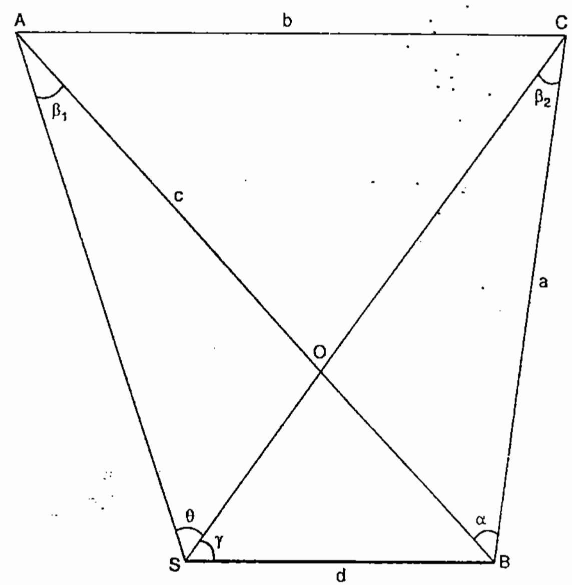


FIG. 8.30.

due to the eccentricity of the station is generally known as *'reduction of centre.'* The distance between the true station and the satellite station is determined either by method of trigonometrical levelling or by triangulation. Satellite stations should be avoided as far as possible in primary triangulation.

In Fig. 8.30, let A, B, C = triangulation stations

S = satellite station for B.

$d = BS$ = eccentric distance between B and S, determined by trigonometrical levelling or by triangulation.

$\theta = \angle ASC$ = observed angle at a S.

α = True angle at B.

$\gamma = \angle CSB$ = observed angle at S.

$\beta_1 = \angle SAB$.

$\beta_2 = \angle SCB$.

$AC = b$, $AB = c$ and $BC = a$

O = point of intersection of lines AB and CS.

(1) The angles CAB and ACB are known by observations to B from A and C respectively. The length of the side AC is known by computations from the adjacent triangle. The sides AB and BC can then be calculated by applying sine rule to the triangle ABC.

Thus, $$BC = a = \frac{b \sin CAB}{\sin ABC} \qquad ...(1)$$

and $$AB = c = \frac{b \sin ACB}{\sin ABC} \qquad ...(2)$$

In the above expressions, $\angle ABC$ may be taken equal to $180° - \angle BAC - \angle BCA$, *at the first instance* to calculate the sides AB and BC.

(2) Knowing the sides AB and BC, and the eccentric distance SB, triangles ABS and CBS can be solved by sine rule to get the values of the angles β_1 and β_2 respectively.

Thus, from triangle ABS $$\sin \beta_1 = \frac{SB \sin ASB}{BC} = \frac{d \sin(\theta + \gamma)}{a}$$

And, from triangle CBS, $$\sin \beta_2 = \frac{SB \sin BSC}{BC} = \frac{d \sin \gamma}{a}$$

Since BS is very small in comparsion to BA and BC, the angles β_1 and β_2 are extremely small, and we may write

$$\beta_1 \text{ (seconds)} = \frac{\sin \beta_1}{\sin 1''} = \frac{d \sin(\theta + \gamma)}{c \sin 1''} = \frac{d \sin(\theta + \gamma)}{c} \times 206265 \qquad ...[8.27\ (a)]$$

and $$\beta_2 \text{ (seconds)} = \frac{\sin \beta_2}{\sin 1''} = \frac{d \sin \gamma}{a \sin 1''} = \frac{d \sin \gamma}{a} \times 206265 \qquad ...[8.27\ (b)]$$

(3) After having calculated the angles β_1 and β_2, the observed angle θ at S is reduced to that at B as follows :

$$\angle ABC = \alpha = \angle AOC - \beta_2 = (\beta_1 + \theta) - \beta_2 = \theta + \beta_1 - \beta_2 \qquad ...(1)$$

$$= \theta + \frac{d \sin(\theta + \gamma)}{c \sin 1''} - \frac{d \sin \gamma}{a \sin 1''}$$

The above expression for the true angle α does not cover all the four possible cases corresponding to the four positions of the satellite station S, as shown by S_1, S_2, S_3 and S_4 in Fig. 8.31 (*a*)].

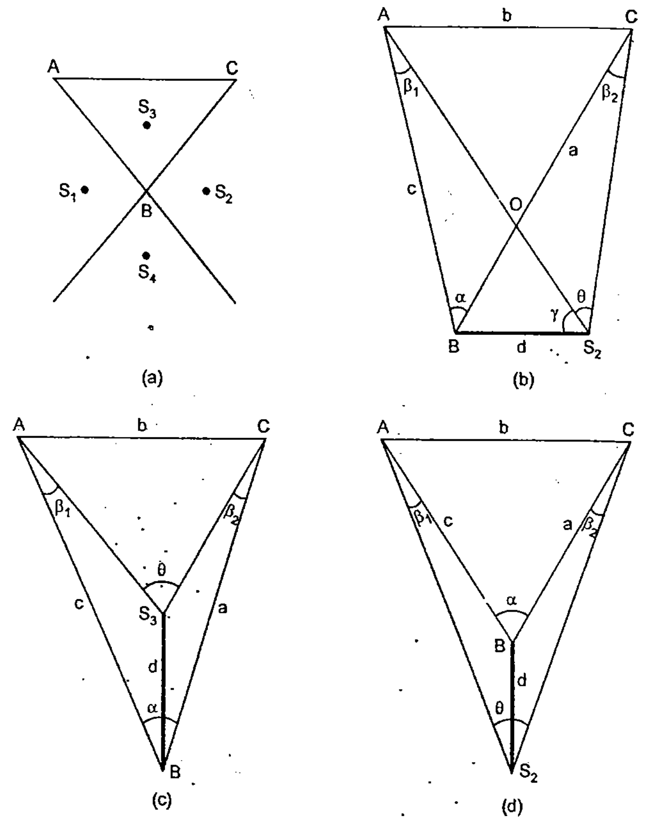


FIG. 8.31

Case I. Position S_1 to the left of B [Fig. 8.31 (*a*) and Fig. 8.30]

The true angle $\alpha = \theta + \beta_1 - \beta_2$ ...(1)

Case II. Position S_2 to the right of B [Fig. 8.31 (*b*)]

The true angle $\alpha = \angle AOC - \beta_1 = (\theta + \beta_2) - \beta_1 = \theta - \beta_1 + \beta_2$ ...(2)

Case III. Position S_3 between AC and B [Fig. 8.31(*c*)]

The true angle $\alpha = \theta - \beta_1 - \beta_2$ ...(3)

Case IV. Position S_4 [Fig. 8.31 (*d*)]

The true angle $\alpha = \theta + \beta_1 + \beta_2$ ...(4)

To ascertain the signs of corrections (*i.e.* β_1 and β_2) when a number of angles are observed from the satellite *S*, it is convenient to assume *SB* as an arbitrary meridian. The observed angles are then reduced to this meridian and the corrections are computed from the formula,

$$\beta(\text{in seconds}) = \frac{d \sin \theta}{D \sin 1''} \quad ...(8.28)$$

where θ = observed angle reduced to the assumed meridian

D = distance from the true station to the observed station

The sign of β will be the same as the sign of sin θ.

Thus, in Fig. 8.32 let *S* = satellite station

B = true station

A_1, A_2, A_3, A_4 etc. = observed stations

$\theta_1, \theta_2, \theta_3, \theta_4$ etc. are the angles to A_1, A_2, A_3, A_4 respectively, reduced to *SB* by taking *SB* as the reference meridian.

$\beta_1, \beta_2, \beta_3, \beta_4$ = correction corresponding to A_1, A_2, A_3, A_4 ; D_1, D_2, D_3, D_4 = distances of A_1, A_2, A_3, A_4 etc. from *B* respectively.

Then $\beta_1 \text{ (seconds)} = \dfrac{d \sin \theta_1}{D_1 \sin 1''}$

$\beta_2 \text{ (seconds)} = \dfrac{d \sin \theta_2}{D_2 \sin 1''}$

etc. etc.

Since $\sin \theta_1$ is positive, the correction β_1 is also positive. *i.e.*, the bearing BA_1 is obtained by adding β_1 to the bearing SA_1 (*i.e.* θ_1). Similarly, since θ_2 is in the second quadrant, β_2 is positive *i.e.* to get the bearing of BA_2, β_2 will be added to θ_2. Knowing the bearings of BA_1 and BA_2 the angle A_1BA_2 can be calculated. If θ_3 and θ_4 are in third and fourth quadrant β_3 and β_4 will be applied negatively. See example 8.20 for illustration.

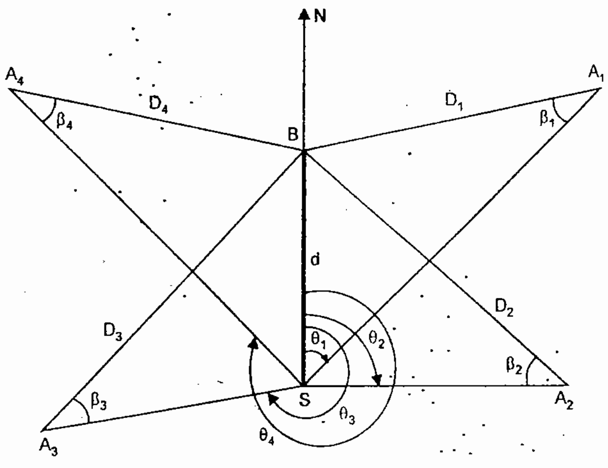


FIG. 8.32

Eccentricity of Signal : When observations are made upon a signal which is out of centre, it is essential to correct the angles. The correction can be found exactly in the same way as that of a satellite station.

Thus, in Fig. 8.30, if *S* is the signal for the main station *B*, all observed angles *SAC* and *SCA* are to be corrected for β_1 and β_2 respectively to get the true angles *BAC* and *BCA* respectively. The values β_1 and β_2 can be found by measuring the distance *BS* and knowing the length of *AC* from the adjacent triangle.

Thus $\beta_1 = \dfrac{d \sin (\theta + \gamma)}{c \sin 1''}$ and $\beta_2 = \dfrac{d \sin \gamma}{a \sin 1''}$.

Example 8.18. *From an eccentric station S, 12.25 metres to the west of the main station B, the following angles were measured*

$\angle BSC = 76° 25' 32''$; $\angle CSA = 54° 32' 20''$

The stations S and C are to the opposite sides of the line AB. Calculate the correct angle ABC if the lengths AB and BC are 5286.5 and 4932.2 m respectively.

Solution. [Fig. 8.31 (*d*)]

Here $\angle BSC = \gamma = 76° 25' 32''$

$\angle CSA = \theta = 54° 32' 20''$

$AB = c = 5286.5$ m

$BC = a = 4932.2$ m

$BS = d = 12.25$ m

From Eq. 8.27 (*a*), we have

$$\beta_2 = \frac{d \sin (\theta + \gamma)}{c} \times 206265 \text{ seconds}$$

$$= \frac{12.25 \sin (54° 32' 20'' + 76° 25' 32'')}{5286.5} \times 206265 \text{ seconds}$$

$= 360.92$ seconds $= 6' 0''.92$

Similarly, from Eq. 8.27 (*b*), we have

$$\beta_1 = \frac{d \sin \gamma}{a} \times 206265 \text{ seconds} = \frac{12.25 \sin 76° 25' 32''}{4932.2} \times 206265$$

$= 497.98$ seconds $= 8' 17''.98$

Now the corrected angle $ABC = \theta + \beta_1 - \beta_2$

$= 54° 32' 20'' + 6' 0''.92 - 8' 17''.98 = \mathbf{54° 30' 2''.94}$.

Example 8.19. *In measuring angles from a triangulation station B, it was found necessary to set the instrument at a satellite station S, due south of the main station B and at a distance of 12.2 metres from it. The line BS approximately bisects the exterior angle ABC. The angles ASB and BSC were observed to be 30° 20' 30" and 29° 45' 6" respectively. When the station B was observed, the angles CAB and ACB were observed to be 59° 18' 26" and 60° 26' 12" respectively. The side AC was computed to be 4248.5 metres from the adjacent triangle. Determine the correct value of the angle ABC.*

Solution. [Fig. 8.31 (*d*)]

In triangle *ABC*, $\angle CAB = 59° 18' 26''$

$\angle ACB = 60° 26' 12''$

$\therefore \angle ABC = 180° - (59° 18' 26'' + 60° 26' 12'') = 60° 15' 22''$ approximately

$\therefore \quad AB = AC \dfrac{\sin ACB}{\sin ABC} = 4248.5 \dfrac{\sin 60° 26' 12''}{\sin 60° 15' 22''} = 4256.1$ m

and $\quad BC = AC \dfrac{\sin CAB}{\sin ABC} = 4248.5 \dfrac{\sin 59° 18' 26''}{\sin 60° 15' 22''} = 4207.7$ m

Now from ΔABS, $\sin \beta_1 = BS \dfrac{\sin ASB}{AB}$

Since β_1 is extremely small, we have

$$\beta_1 = \frac{\sin \beta_1}{\sin 1''} = BS \frac{\sin ASB}{AB} \times 206265 \text{ seconds}$$

$$= 12.2 \frac{\sin 30° 20' 30''}{4256.1} \times 206265 \text{ seconds} = 298.67 \text{ seconds} = 4' 58''.67$$

Similarly, from ΔCBS,

$$\sin \beta_2 = BS \frac{\sin BSC}{BC} \quad \text{or} \quad \beta_2 = BS \frac{\sin BSC}{BC} \times 206265 \text{ seconds}$$

$$= 12.2 \frac{\sin 29° 45' 6''}{4207.7} \times 206265 \text{ seconds}$$

$$= 296.78 \text{ seconds} = 4' 56''.78$$

Now the correct angle $ABC = \angle ASC + \beta_1 + \beta_2$

$= (30° 20' 30'' + 29° 45' 6'') + 4' 58''.67 + 4' 56''.78 =$ **60° 15′ 31″.45** .

Example 8.20. *From a satellite station S, 5.8 metres from the main triangulation station A, the following directions were observed :*

A	*0°*	*0′*	*0″*
B	*132°*	*18′*	*30″*
C	*232°*	*24′*	*6″*
D	*296°*	*6′*	*11″*

The lengths AB, AC and AD were computed to be 3265.5 m, 4022.2 m and 3086.4 m respectively

Determine the directions of AB, AC and AD.

Solution. (Fig. 8.33)

The correction to any direction is given by

$$\beta = \frac{d \sin \theta}{D \sin 1''} \text{ seconds} \qquad \ldots(8.28)$$

(*a*) ***For the line AB*** :

θ = angle reduced to the direction SA

$= 132° 18' 30''$

$d = AS = 5.8$ m

$D = AB = 3265.5$

$\therefore \quad \beta = \dfrac{5.8 \sin 132° 18' 30''}{3265.5} \times 206265$ seconds

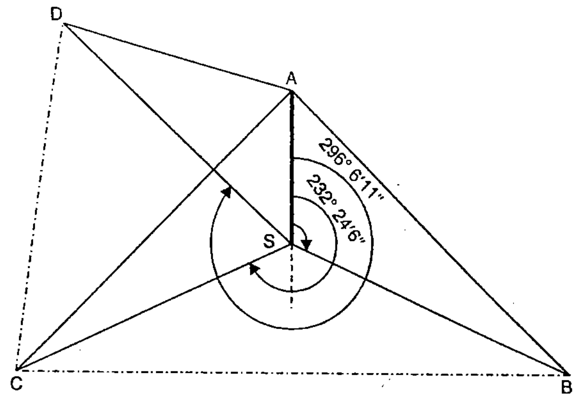


FIG. 8.33

$= +270''.9 = +4' 30''.9$

$\therefore$ Direction of AB = direction of $SB + \beta$

$= 132° 18' 30'' + 4' 30''.9$

$=$ **132° 23′ 0″.9**

(*b*) ***For the line AC*** :

θ = angle reduced to the direction S

$= 232° 24' 6''$

$D = AC = 4022.2$ m

$\therefore \quad \beta = \dfrac{5.8 \sin 232° 24' 6''}{4022.2} \times 206265$ seconds

$= -235.7$ seconds

$= -3' 55''.7$

$\therefore$ Direction of AC = Direction of $SC + \beta$

$= 232° 24' 6'' - 3' 55''.7$

$=$ **232° 20′ 4″.3**

(*c*) ***For the line AD*** :

θ = angle reduced to the direction $SA = 296° 6' 11''$

$D = AD = 3086.4$ m

$\beta = \dfrac{5.8 \sin 296° 6' 11''}{3086.4} \times 206265$ seconds

$= -348.1$ seconds

$= -5' 48''.1$

$\therefore$ Direction of AD = direction of $SD + \beta$

$= 296° 6' 11'' - 5' 48''.1$

$= \mathbf{296° 0' 22''.9}$

8.11. EXTENSION OF BASE : BASE NET

The base lines are usually much shorter than the average length of the triangle sides. This is mainly due to two reasons :

(*i*) it is often not possible to get a favourite site for a longer base, and

(*ii*) it is difficult and expensive to measure long base lines. Hence, in connecting the comparatively short base line to the main triangulation, badly conditioned figure must be avoided by expanding the base in a series of stages. The group of triangles meant for extending the base is known as the *base net*.

There are a great variety of the extension layouts, but the following important points should be kept in mind in selecting the one :

(*i*) Small angles opposite the known side must be avoided.

(*ii*) The net should have sufficient redundant lines to provide three or four side equations within the figure.

(*iii*) Subject to the above, it should provide the quickest extension with the fewest stations.

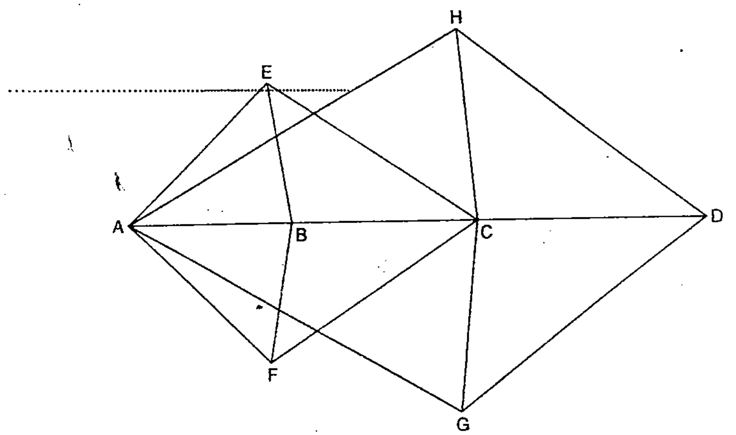


FIG. 8.34. EXTENSION OF BASE

Fig. 8.34 represents such a base net in which it is required to extend the base *AB*. The following steps are necessary :

(1) Select two stations *E* and *F* to the either side *AB* such that *AEB* and *AFB* are well-conditioned triangles.

(2) In the line *AB*, prolonged very accurately with the help of a theodolite, choose a favourable position *C* from which *E* and *F* are both visible, and which forms well-shaped triangles *AEC* and *AFC*. Thus if the angles at *A, E, F* and *C* are about 45°, and those at base about 90°, both sets of the triangles will be well-conditioned.

(3) In the triangle ***AEB, AB*** is measured, and hence *EB* can be calculated by measuring all the three angles, *A, E* and *B*. From triangle *EBC*, *BC* can be computed from the known side *EB* and the measured angles at *E, B* and *C*.

(4) Similarly, *BC* can also be calculated from triangles *AFB* and *FBC*. Thus, two values of *BC* are obtained.

Two more values of $BC = (AC - AB)$ can be obtained by computations from triangles *AEC* and *AFC*.

Thus, the base *AB* is extended to *AC*. Similar procedure can be adopted if further extensions to *D* etc. are required.

Fig. 8.35 shows various typical forms of base extensions.

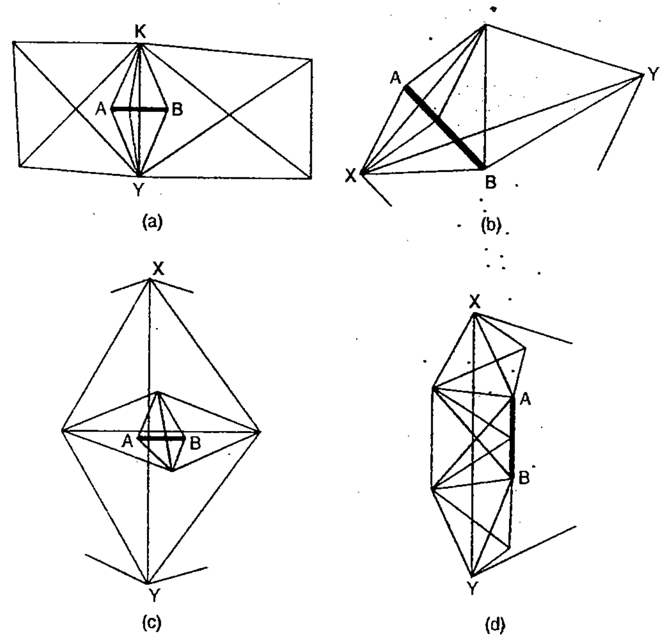


FIG. 8.35. BASE EXTENSION.

PROBLEMS

1. How do you determine the intervisibility of triangulation stations?

Two triangulation stations A and B are 40 km apart and have elevations of 178 m and 175 m respectively. Find the minimum height of signal required at B so that the line of sight may not pass nearer the ground than 3 metres. The intervening ground may be assumed to have a uniform elevation of 150 metres.

2. The altitudes of two proposed stations A and B, 80 km apart are respectively 225 m and 550 m. The intervening obstructions situated at C, 40 km from A has an elevation of 285 m. Ascertain if A and B are intervisible, and if necessary, find by how much B should be raised so that the line of sight must nowhere be less than 3 m above the surface of the ground.

3. The altitudes of two proposed triangulation stations A and C, 65 miles apart, are respectively 703 ft and 3520 ft above sea level datum, while the heights of two eminences B and D on the profile between A and C are respectively 1170 and 2140 ft, the distance AB and AD being respectively 24 miles and 45 miles.

Ascertain if A and C are intervisible and, if necessary, determine a suitable height for a scaffold at C, given that A is a ground station. The earth's mean radius may be taken as 3960 miles, and coefficient of refraction 0.07. (*U.L.*)

4. What is meant by a satellite station and reduction to centre ? Derive expression for reducing the angles measured at the satellite stations to centre.

5. What is meant by the eccentricity of signal ? How would you correct the observation when made upon an eccentric signal ?

6. On occupying a ground station A of a triangulation survey, it was evident that some elevation of the theodolite would be neccessary, in order to sight the signals at adjacent stations: P on the left and Q on the right. It was found, however, that these stations could be seen from a ground station B, south-west of A, so that AB approximately bisects the angle PBQ.

Where upon, B was adopted as a false station and the distance AB was carefully measured, being 2.835 m, while the angles PBA and ABQ were observed to be 28° 16′ 35″ and 31° 22′ 20″ respectively. The side PQ was computed to be 994.87 metres in the adjacent triangle, and when A was under observation, the interior angles at P and Q were found to have mean value of 62° 34′ 15″ and 57° 39′ 20″ respectively. Determine accurately the magnitude of the angle PAQ.

7. Directions are observed from a satellite station S, 62.18 m from staton C, with the following results :

A, 0° 0′ 0″ ; B, 71° 54′ 32″; C, 296° 12′ 2″.

The approximate lengths of AC and BC are respectively 8041 m and 10864 m. Calculate the angle ACB.

8. In a quadrilateral $ABCD$ in clockwise order, forming part of a triangulation, a church spire was observed as the central station O. Accordingly, a satellite station S was selected 6.71 metres from O, and inside the triangle BOC. The following table gives the approximate distance from the central station and the angles observed from S.

Observed station	*Horizontal angle at S measured clockwise from O*	*Distance (metres)*
A	30° 45′ 30″	OA = 5532
B	98° 32′ 00″	OB = 6789
C	210° 10′ 40″	OC = 3914
D	320° 14′ 15″	OD = 4670
O	360° 00′ 00″	

Calculate the four central angles at O.

9. Discuss the effect of phase in sighting a sun signal and show with sketches how it may be eliminated or reduced.

Derive formulae for the correction to be applied to cylindrical signals (*a*) when the bright portion is bisected and (*b*) when the bright line is bisected. (*U.L*)

10. What is meant by 'base net' ? Explain how you would extend a base line.

11. (*a*) What are the principal objects to be kept in view in selecting the ground for a base line in large survey ? Enumerate in sequence the operations necessary before the measurement of the base line commences. State the correction to be applied in base line measurements.

(*b*) Explain how you would prolong a given base line. (*U.L.*)

12. Show that in base line measurement with tapes and wires in flat catenary with supports at different levels, the total correction will be $-(x + c)$, where x is the parabolic approximation for sag between the level supports and c, the level or slope correction taken permissibly to the first approximation. (*U.L.*)

13. Find the sag correction for 30 m steel tape under a pull of 80 N in three equal spans of 10 m each. Mass of one cubic cm of steel = 7.86 g/cm^3. Area of cross-section of the tape = 0.10 sq. cm.

14. A steel tape is 30 m long at a temperature of 65°F when lying horizontally on the ground. Its sectional area is 0.082 sq. cm, its mass 2 kg and coefficient of expansion 65×10^{-7} per 1° F. The tape is stretched over three equal spans. Calculate actual length between the end graduations under the following conditions : temp. 85° F, pull 180 N. Take $E = 2.07 \times 10^7$ N/cm^2 .

15. A 30 m steel tape was standardized on the flat and was found to be exactly 30 m under no pull at 66° F. It was used in catenary to measure a base of 5 bays. The temperature during the measurement was 92° F and pull exerted during the measurement was 100 N. The area of cross-section of the tape was 0.08 sq. cm. The specific mass of steel is 7.86 g/cm^3.

$$\alpha = 0.0000063 \text{ per } 1° \text{ F and } E = 2.07 \times 10^7 \text{ N/cm}^2.$$

Find the true length of the line.

16. A base line for a triangulation is to be measured with a steel tape. Give a complete list of the necessary apparatus with sketches and describe how you would carry out the measurment. Give approximate dimensions of the tape you would use. What kind of steel should it be made of ? Give your reasons. Write down a complete list of corrections which must be applied to the measued length, indicating whether these corrections are additive or subractive. (*A.M.I.C.E.*)

17. A copper transmission line, $\frac{1}{2}$ in. in diameter, is stretched between the two points, 1000 ft apart, at the same level, with a tension of $\frac{1}{2}$ ton, when the temperature is 90° F. It is neccessary to define its limiting positions when the temperature varies. Making use of the corrections for sag, temperature, and elasticity normally applied to base line measurements by tape in catenary, find the tension at a temperature of 14° F and the sag in the two cases. Young's modulus for copper 10×10^6 lb/in^2, its density 555 lb/ft^3 and its coefficient of linear expeansion 9.3×10^{-6} per ° F. (*U.L.*)

18. A nominal distance of 100 ft was set out with a 100 ft steel tape from a mark on the top of one peg to a mark on the top of another, the tape being in catenary under a pull of 20 lb and at a mean temperature of 70° F. The top of one peg was 0.56 ft below the top of the other. The tape has been standardized in catenary under a pull of 25 lb at a temperature of 62° F.

Calculate the exact horizontal distance between the marks on the two pegs and reduce it to mean sea level. The top of the higher peg was 800 ft above mean sea level.

Radius of the earth	$= 20.9 \times 10^6$ ft.
Density of tape	$= 0.28$ lb/in^3
Section of tape	$= 0.125$ in. $\times 0.05$ in.
Co-efficient of expansion	$= 6.25 \times 10^{-6}$ per 1°F
Young's modulus	$= 30 \times 10^6$ lb/in^3

(*A.M.I.C.E.*)

ANSWERS

1. 6.96 m.
2. 17 m.
3. 19.07 ft.
6. 59° 48′ 37″.58
7. 71° 44′ 59″
8. 67° 47′ 44″ ; 111° 32′ 20″ ; 110° 03′ 23″ ; 70° 36′ 33″ .
13. 0.0116 m.
14. 30.005 m.
15. 30.005 m.
17. 1142 lb ; 84.5 and 82.8 ft.
18. 99.9807 ft.

9

Survey Adjustments and Theory of Errors

9.1. INTRODUCTION : KINDS OF ERRORS

Errors of measurement are of three kinds : (*i*) mistakes, (*ii*) systematic errors, and (*iii*) accidental errors.

(*i*) **Mistakes.** Mistakes are errors that arise from inattention, inexperience, carelessness and poor judgment or confusion in the mind of the observer. If a mistake is undetected, it produces a serious effect on the final result. Hence every value to be recorded in the field must be checked by some independent field observation.

(*ii*) **Systematic Error.** A systematic error is an error that under the same conditions, will always be of the same size and sign. A systematic error always follows some definite mathematical or physical law, and a correction can be determined and applied. Such errors are of constant character and are regarded as *positive* or *negative* according as they make the result *too great* or *too small*. Their effect is therefore, *cumulative*.

If undetected, systematic errors are very serious. Therefore :

(1) All the surveying equipments must be designed and used so that whenever possible systematic errors will be *automatically* eliminated and (2) all systematic errors that cannot be surely eliminated by this means must be evaluated and their relationship to the conditions that cause them must be determined. For example, in ordinary levelling, the levelling instrument must first be adjusted so that the line of sight is as nearly horizontal as possible when bubble is centred. Also the horizontal lengths for backsight and foresight from each instrument position should be kept as nearly equal as possible. In precise levelling, every day, the actual error of the instrument must be determined by careful peg test, the length of each sight is measured by stadia and a correction to the result is applied.

(*iii*) **Accidental Error.** Accidental errors are those which remain after mistakes and systmatic errors have been eliminated and are caused by a combination of reasons beyond the ability of the observer to control. They tend sometimes in one direction and some times in the other, *i.e.,* they are equally likely to make the apparent result too large or too small.

An accidental error of a single determination is the difference between (1) the true value of the quantity and (2) a determination that is free from mistakes and systematic errors. *Accidental error represent the limit of precision in the determination of a value. They obey the laws of chance and, therefore, must be handled according to the mathematical laws of probability.*

The theory of errors that is discussed in this chapter deals only with the accidental errors after all the known errors are eliminated and accounted for.

9.2. DEFINITIONS

The following are some of the terms which shall be used :

1. **Independent Quantity.** An observed quantity may be classified as (*i*) *independent* and (*ii*) *conditioned.* An independent quantity is the one whose value is independent of the values of other quantities. It bears no relation with any other quantity and hence change in the other quantities does not affect the value of this quantity. Example : reduced levels of several bench marks.
2. **Conditioned Quantity.** A conditioned quantity is the one whose value is dependent upon the values of one or more quantities. Its value bears a rigid relationship to some other quantity or quantities. *It is also called a dependent quantity.* For example, in a triangle ABC, $\angle A + \angle B + \angle C = 180°$. In this *conditioned equation,* any two angles may be regarded as independent and the third as dependent or conditioned.
3. **Direct Observation.** An observation is the numerical value of a measured quantity, and may be either direct or indirect. A *direct observation* is the one made directly on the quantity being determined, *e.g.*, the measurement of a base, the single measurement of an angle etc.
4. **Indirect Observation.** An indirect observation is one in which the observed value is deduced from the measurement of some related quantities, *e.g.*, the measurement of angle by repetition (a multiple of the angle being measured.)
5. **Weight of an Observation.** The weight of an observation is a number giving an indication of its precision and trustworthiness when making a comparison between several quantities of different worth. Thus, if a certain observation is of weight 4, it means that it is four times as much reliable as an observation of weight 1. When two quantities or observations are assumed to be equally reliable, the observed values are said to be of equal weight or of unit weight. Observations are called weighted when different weights are assigned to them. Observations are required to be weighted when they are made with unequal care and under dissimilar conditions. Weights are assigned to the observations or quantities observed in direct proportion to the number of observations.
6. **Observed Value of a Quantity.** An observed value of a quantity is the value obtained when it is corrected for all the known errors.
7. **True Value of Quantity.** The true value of a quantity is the value which is absolutely free from all the errors. The true value of a quantity is indeterminate since the true error is never known.
8. **Most Probable Value.** The most probable value of a quantity is the one which has more chances of being *true* than has any other. It is deduced from the several measurements on which it is based.
9. **True Error.** A true error is the difference between the true value of a quantity and its observed value.

10. **Most Probable Error.** The most probable error is defined as that quantity which added to, and subtracted from, the most probable value fixes the limits within which it is an even chance the true value of the measured quantity must lie.
11. **Residual Error.** A residual error is the difference between the most probable value of a quantity and its observed value.
12. **Observation Equation.** An observation equation is the relation between the observed quantity and its numerical value.
13. **Conditioned Equation.** A conditioned equation is the equation expressing the relation existing between the several dependent quantities.
14. **Normal Equation.** A normal equation is the one which is formed by multiplying each equation by the co-efficient of the unknown whose normal equation is to be found and by adding the equations thus formed. As the number of normal equations is the same as the number of unknowns, the most probable values of the unknown can be found from these equations.

9.3. THE LAWS OF ACCIDENTAL ERRORS

Investigations of observations of various types show that accidental errors follow a definite law, the *law of probability.* This law defines the occurrence of errors and can be expressed in the form of equation which is used to compute the probable value or the probable precision of a quantity. The most important features of accidental errors which usually occur are :

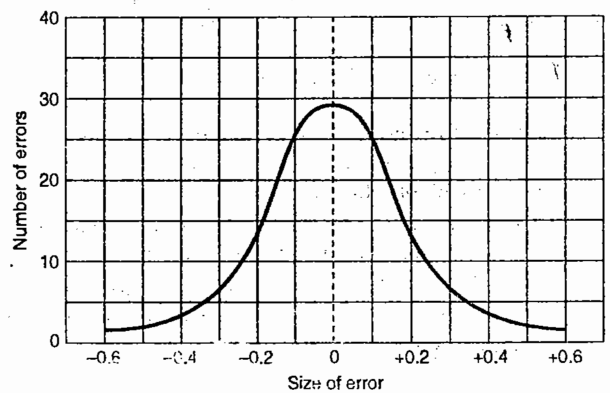


FIG. 9.1. PROBABILITY CURVE.

(*i*) Small errors tend to be more frequent than the large ones ; that is they are the *most probable.*

(*ii*) Positive and negative errors of the same size happen with equal frequency; that is, they are *equally probable.*

(*iii*) Large errors occurs infrequently and are impossible.

Probability Curve. The theory of probability describes these features by stating that the relative frequencies of errors of different extents can be represented by a curve as shown in Fig. 9.1.

This curve, called the *curve of error* or *probability curve,* forms the basis for the mathematical derivation of theory of errors.

The formula for probable error is difficult to derive. It is stated here categorically: *Probable error* of a single measurement is given by

$$E_s = \pm\, 0.6745 \sqrt{\frac{\Sigma v^2}{n-1}} \qquad \ldots(9.1)$$

where E_s = Probable error of single observation.

v = Difference between any single observation and the mean of the series.

n = Number of observations in the series.

Probable Error of an Average. Since the average of n measurements is the sum of the measurements divided by n, the probable error of the average of n measurements is $\frac{\sqrt{n}}{n}$ times the probable error of one measurement. Thus, *probable error of an average or mean* is given by

$$E_m = \frac{\sqrt{n}}{n} 0.6745 \sqrt{\frac{\Sigma v^2}{n-1}} = 0.6745 \sqrt{\frac{\Sigma v^2}{n(n-1)}} = \frac{E_s}{\sqrt{n}} \qquad \ldots(9.2)$$

where E_m = probable error of the mean.

Probable Error of a Sum. When a measurement is the result of the sums and differences of several (n) observations having different probable errors E_1, E_2 E_3 ..E_n, the probable error of the measurement is the square root of the sum of the squares of the probable errors of the several observations. Thus,

$$\text{Probable error of measurement} = \sqrt{E_1^2 + E_2^2 + E_3^3 + \ldots\ldots E_n^2} \qquad \ldots(9.3)$$

Most Probable Value. As defined earlier the most probable value of a quantity is the one which has more chances of being *true* than any other. It can be proved from the theory of errors that :

(*i*) The most probable value of a quantity is equal to the arithmetic mean if the observations are of equal weight.

(*ii*) The most probable value of a quantity is equal to the weighted arithmetic mean in case of observations of unequal weights.

Average Error. An average error in a series of observation of equal weight is defined as the arithmetical mean of separate errors, taken all with the same sign, either plus or minus.

Mean Square Error (m.s.e.). The mean square error is equal to the square root of the arithmetic mean of the squares of the individual errors.

$$\text{Thus, m.s.e.} = \pm \sqrt{\frac{v_1^2 + v_2^2 + v_3^2 \ldots\ldots}{n}} = \pm \sqrt{\frac{\Sigma v^2}{n}}$$

Example 9.1. *In carrying a line of levels across a river, the following eight readings were taken with a level under identical conditions :*

2.322, 2.346, 2.352, 2.306, 2.312, 2.300, 2.306, 2.326.

Calculate (i) the probable error of single observation.

(ii) the probable error of the mean

Solution

The computations for v and v^2 are arranged in the tabular form below :

Rod reading	v *(m)*	v^2
2.322	0.001	0.000001
2.346	0.025	0.000625
2.352	0.031	0.000961
2.306	0.015	0.000225
2.312	0.009	0.000081
2.300	0.021	0.000441
2.306	0.015	0.000225
2.326	0.005	0.000025
Mean : 2.321		$\Sigma v^2 = 0.002584$

From equation 9.1, $E_s = \pm 0.6745 \sqrt{\frac{0.002584}{8-1}} = \pm 0.01295$ metre

From equation 9.2, $E_m = \frac{E_s}{\sqrt{n}} = \pm \frac{0.01295}{\sqrt{8}} = \pm 0.00458$ metre.

9.4. GENERAL PRINCIPLES OF LEAST SQUARES

It is found from the probability equation that the most probable values of a series of errors arising from observations of equal weight are those for which the sum of the squares is a minimum. The fundamental law of least squares is derived from this. According to the principle of least squares, *the most probable value of an observed quantity available from a given set of observations is the one for which the sum of the squares of the residual errors is a minimum.* When a quantity is being deduced from a series of observations, the residual errors will be the difference between the adopted value and the several observed values.

Let V_1, V_2, V_3 etc. be the observed values

x = most probable value

Then,
$$\left.\begin{aligned} x - V_1 &= e_1 \\ x - V_2 &= e_2 \\ x - V_3 &= e_3 \\ &\ldots\ldots\ldots\ldots \\ x - V_n &= e_n \end{aligned}\right] \qquad \ldots(1)$$

where e's are the respective errors of the observed values.

If M = *arithmetic mean,* then

$$M = \frac{V_1 + V_2 + V_3 \ldots. + V_n}{n} = \frac{\Sigma V}{n} \qquad \ldots(2)$$

where n = number of observed values.

From equation (1),

$$nx - \Sigma V = \Sigma e$$

or $x = \frac{\Sigma V}{n} + \frac{\Sigma e}{n}$, but $\frac{\Sigma V}{n} = M$ from (2)

$$\therefore \quad x = M + \frac{\Sigma e}{n} \qquad \ldots(3)$$

If n is large and e is kept small by making precise measurement, $\frac{\Sigma e}{n}$ becomes practically infinitesimal with respect to M.

Hence $x \approx M$ ...(9.4)

Thus, the arithmetic mean is the true value where the number of observed value is very large.

Let $r_1, r_2, r_3.....r_n$ be the *residuals* (*i.e.* the difference between the mean values and the observed values). Thus,

$$\left.\begin{array}{l} M - V_1 = r_1 \\ M - V_2 = r_2 \\ M - V_3 = r_3 \\ \dots\dots\dots\dots \\ M - V_n = r_n \end{array}\right] \quad ...(4)$$

Adding the above,

$$nM - \Sigma V = \Sigma r$$

or $$M = \frac{\Sigma V}{n} + \frac{\Sigma r}{n}$$

Under the preceding conditions and by preceding equation

$$M = \frac{\Sigma V}{n}$$

and hence $$\frac{\Sigma r}{n} = 0 \quad ...(5)$$

Hence the sum of the residuals equals zero and the sum of plus residual equals the sum of the minus residuals.

Let N be any other value of the unknown other than the arithmetic mean. We have

$$\left.\begin{array}{l} N - V_1 = r_1' \\ N - V_2 = r_2' \\ N - V_3 = r_3' \\ \dots\dots\dots\dots \\ N - V_n = r_n' \end{array}\right] \quad ...(6)$$

Squaring equation (4), and adding, we get

$$\Sigma r^2 = nM^2 + \Sigma V^2 - 2M\,\Sigma V \quad ...(7)$$

Similarly, squaring equations (6) and adding, we get

$$\Sigma r'^2 = nN^2 + \Sigma V^2 - 2N\,\Sigma V \quad ...(8)$$

Substituting $nM = \Sigma V$ in equation (7), we get

$$\Sigma r^2 = M\Sigma V - 2M\Sigma V + \Sigma V^2 = \Sigma V^2 - M\Sigma V$$

$$= \Sigma V^2 - \frac{\Sigma V^2}{n}, \text{ by putting } M = \Sigma \frac{V}{n}$$

or $$\Sigma V^2 = \Sigma r^2 + \frac{\Sigma V^2}{n} \quad ...(9)$$

Substituting ΣV^2 of equation (9) in equation (8), we get

$$\Sigma r'^2 = nN^2 + \left(\Sigma r^2 + \frac{\Sigma V^2}{n}\right) - 2N\Sigma V = \Sigma r^2 + n\left(N^2 - 2N\frac{\Sigma V}{n} + \frac{\Sigma V^2}{n^2}\right) = \Sigma r^2 + n\left(N - \frac{\Sigma V}{n}\right)^2$$

As $\left(N - \frac{\Sigma V}{n}\right)^2$ is always positive, Σr^2 is less than $\Sigma r'^2$. *That is, the sum of the squares of the residuals found by the use of the arithmetic mean is a minimum.* This is, thus, the fundamental law of least squares.

9.5. LAWS OF WEIGHTS

From the method of least squares the following laws of weights are established :

(1) ***The weight of the arithmetic mean of the measurements of unit weight is equal to the number of observations.***

For example, let an angle A be measured six times, the following being the values:

∠A	Weight	∠A	Weight
30° 20′ 8″	1	30° 20′ 10″	1
30° 20′ 10″	1	30° 20′ 9″	1
30° 20′ 7″	1	30° 20′ 10″	1

∴ Arithmetic mean $= 30° 20' + \frac{1}{6}(8'' + 10'' + 7'' + 10'' + 9'' + 10'') = 30° 20' 9''$.

Weight of arithmetic mean = number of observations = 6.

(2) ***The weight of the weighted arithmetic mean is equal to the sum of the individual weights.***

For example, let an angle A be measured six times, the following being the values:

∠A	Weight	∠A	Weight
30° 20′ 8″	2	30° 20′ 10″	3
30° 20′ 10″	3	30° 20′ 9″	4
30° 20′ 6″	2	30° 20′ 10″	2

Sum of the individual weights = 2 + 3 + 2 + 3 + 4 + 2 = 16

$$\text{Weighted arithmetic mean} = 30° 20' + \frac{1}{16}[8'' \times 2) + (10'' \times 3) + (6'' \times 2) + (10'' \times 3) + (9'' \times 4) + (10'' \times 2)] = 30° 20' 9''.$$

Weight of the weighted arithmetic mean = 16

(3) ***The weight of algebraic sum of two or more quantities is equal to the reciprocal of the sum of reciprocals of individual weights.***

For example let. $\alpha = 42° 10' 20''$, weight 4

$\beta = 30° 40' 10''$, weight 2

Sum of reciprocals of individual weights $= \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$

∴ Weight of $\alpha + \beta\ (= 72° 50' 30'') = \frac{1}{\frac{1}{4} + \frac{1}{2}} = \frac{1}{\frac{3}{4}} = \frac{4}{3}$

Weight of $\alpha - \beta$ (= 11° 30′ 10″) $= \dfrac{1}{\frac{1}{4}+\frac{1}{2}} = \dfrac{1}{\frac{3}{4}} = \dfrac{4}{3}$

(4) ***If a quantity of given weight is multiplied by a factor, the weight of the result is obtained by dividing its given weight by the square of the factor.***

For example, let $\alpha = 42° 10' 20''$, weight 6.

Then, weight of 3α (= 126° 31′) $= \dfrac{6}{(3)^2} = \dfrac{6}{9} = \dfrac{2}{3}$.

(5) ***If a quantity of given weight is divided by a factor, the weight of the result is obtained multiplying its given weight by the square of the factor.***

For exmple, let $\alpha = 42° 10' 30''$, weight 4.

Then weight of $\dfrac{\alpha}{3}$ (= 14° 3′ 30″) $= 4(3)^2 = 36$.

(6) ***If an equation is multiplied by its own weight, the weight of the resulting equation is equal to the reciprocal of the weight of the equation.***

For example, let $A + B = 98° 20' 30''$, weight $\dfrac{3}{5}$

Then, weight of $\dfrac{3}{5}(A + B) = [59° 0' 18'']$ is equal to $\dfrac{1}{\frac{3}{5}}$ or $\dfrac{5}{3}$.

(7) ***The weight of an equation remains unchanged, if all the signs of the equation are changed or if the equation is added to or subtracted from a constant.***

For example let $A + B = 80° 20'$, weight 3.

Then weight of $180° - (A + B)$ or [99° 40′] is equal to 3.

Rules of assigning weightage to the field observations

The following rules may be employed in giving the weights to the various field observations :

(1) The weight of an angle varies directly as the number of the observations made for the measurement of that angle.

(2) Weights vary inversely as the length of various routes in the case of lines of levels.

(3) If an angle is measured a large number of times, its weight is inversely proportional to the square of the probable error.

(4) The corrections to be applied to various observed quantities are in inverse proportion to their weights.

9.6. DETERMINATION OF PROBABLE ERROR

We shall discuss the determination of the probable error (p.e.) of the following cases:

1. ***Direct observations of equal weight on a single unknown quantity***

(*a*) p.e. of single observation of unit weight.

(*b*) p.e. of single observation of weight *w*.

(*c*) p.e. of single arithmetic mean.

2 *Direct observations of unequal weight on a single unknown quantity.*

(*a*) p.e. of single observation of unit weight.

(*b*) p.e. of single observation of weight *w*.

(*c*) p.e. of weighted arithmetical mean.

3. *Computed quantities.*

Case 1. Direct Observation of Equal Weight on a Single Unknown Quantity

If observations on a single quantity are made with equal weights, its most probable value will be equal to the arithmetic mean. Knowing the most probable value, the residual error (v_1, v_2, v_3 etc ....) of each individual measurement can be found by subtracting the most probable value from each observed value. Then :

(a) Probable error (p.e.) of single observation of unit weight

$$= E_s = \pm 0.6745 \sqrt{\frac{\Sigma v^2}{n-1}} \qquad ...(9.1)$$

where $\Sigma v^2 = v_1^2 + v_2^2 + v_3^2 + ...v_n^2$; v = residual error

n = number of observations.

(b) Probable error of single observation of weight w

$$\frac{\text{p.e. of single observation of unit wt.}}{\sqrt{\text{weight}}} = \frac{E_s}{\sqrt{w}} \qquad ...(9.5)$$

(c) Probable error of the arithmetic mean $= E_m = \pm 0.6745 \sqrt{\dfrac{\Sigma v^2}{n(n-1)}} = \dfrac{E_s}{\sqrt{n}}$ ...(9.2)

Case 2. Direct Observations of Unequal Weights on a Single Quantity

When observations are made with unequal weights, the most probable value of the observed quantity is equal to the weighted arithmetic mean of the observed quantities.

From the principle of least squares, *the most probable values of the observed quantities (of unequal weights or precision) are those that render the sum of the weighted squares of the residual errors a minimum.*

Let V_1, V_2, V_3 ....be observed quantities with weight w_1, w_2, w_3 etc,

Then, by the above principle,

$$w_1 v_1^2 + w_2 v_2^2 + w_3 v_3^2 + ... + w_n v_n^2 = \text{a minimum}$$

where $v_1 = N - V_1$; $v_2 = N - V_2$; $v_3 = N - V_3$; .... ; $v_n = N - V_n$

where N is the most probable value of the quantity.

Hence $w_1(N - V_1)^2 + w_2(N - V_2)^2 + w_3(N - V_3)^2 .... + w_n(N - V_n)^2$ = a minimum

Hence
$$N = \frac{w_1V_1 + w_2V_2 + w_3V_3 ..... + w_nV_n}{w_1 + w_2 + w_3 + ... + w_n} \qquad ...(9.6)$$

which proves the proposition that the *most probable value of the observed quantity is equal to the weighted arithmetic mean of the observed quantities.*

Knowing the most probable value N of the quantity, the residual errors etc. of individual observation can be found by subtracting the most probable value from the observed quantities. Then *(a) Probable error (p.e.) of single observation of unit weight*

$$= E_s = \pm 0.6745 \sqrt{\frac{\Sigma wv^2}{n-1}} \quad \ldots(9.7)$$

(b) Probable error of single observation of weight w

$$= \frac{\text{p.e. of single observation of unit weight}}{\sqrt{\text{weight}}} = \frac{E_s}{\sqrt{w}} = \pm 0.6745 \sqrt{\frac{\Sigma wv^2}{w(n-1)}} \quad \ldots(9.8)$$

(b) Probable error of weighted arithmetic mean

$$= \pm 0.6745 \sqrt{\frac{\Sigma wv^2}{\Sigma w \times (n-1)}} \quad \ldots(9.9)$$

Case 3. Probable Error of Computed Quantities

The probable error of computed quantities follow the following *laws* depending upon the relation between the computed quantity and the observed quantity.

1. If a computed quantity is equal to sum or difference of the observed quantity plus or minus a constant, the probable error of the computed quantity is the same as that of the observed quantity.

Let x = observed quantity; y = computed quantity ; k = a constant

Such that $y = \pm x \pm k$

Then $e_y = e_x$ ...(9.10)

where e_x = probable error of the observed quantity

e_y = corresponding probable error of the computed quantity

For example, Let $\angle A + \angle B = 90°$

$\angle B = 46° 30' 20''$

p.e. in observation of $\angle B = \pm 0''.4$

Then, the p.e. in observation of $\angle A = \pm 0''.4$

Now $\angle A = 90° - \angle B = \angle 90° - 46° 30' 20'' = 43° 29' 40''$

and probable value of $\angle A = 43° 29' 40'' \pm 0''.4$.

2. If a computed quantity is equal to an observed quantity multiplied by a constant, the p.e. of computed quantity is equal to the p.e. of observed quantity multiplied by the constant.

Let x = observed quantity ; y = computed quantity ; k = a constant

Such that $y = kx$

Then $e_y = ke_x$ ...(9.11)

For example, let $A = 4.6\ B$

$B = 2.2$ (observed) ; p.e. in $B = \pm 0.02$

Then $A = 4.6\ B = 4.6 \times 2.2 = 10.12$

and p.e. in observation of $A = k \times$ (p.e. of B)

$= 4.6 \times (\pm 0.02) = \pm 0.092$

Hence probable value of $A = 10.12 \pm 0.092$.

3. If a computed quantity is equal to the sum of two or more observed quantities, the p.e. of the computed quantity is equal to the square root of sum of the square of p.e.'s of observed quantities.

Let x_1, x_2, x_3 ....be the observed quantities

y = computed quantity

Such that $y = x_1 + x_2 + x_3$ ...

Then $e_y = \sqrt{e_{x1}^2 + e_{x2}^2 + e_{x3}^2 \ldots\ldots}$ ...(9.12)

where e_y = p.e. of the computed quantity

e_{x1}, e_{x2}, e_{x3} ....etc = p.e. of the observed quantities.

For example, let $A + B + C = 180°$

$A = 30° 30' 12'' \pm 0''.2$

$B = 68° 45' 48'' \pm 0''.6$

$C = 80° 44' 00'' \pm 0''.4$

To determine the probable error of the summation.

Now $y = A + B + C = 180°$

$\therefore$ $e_y = \sqrt{e_a^2 + e_b^2 + e_c^2} = \sqrt{(0.2)^2 + (0.6)^2 + (0.4)^2}$

$= \sqrt{0.56} = \pm 0''.75$ = p.e. of the summation.

4. If a computed quantity is a function of an observed quantity, its probable error is obtained by multiplying the p.e. of the observed quantity with its differentiation with respect to that quantity.

Let x = observed quantity ; y = computed quantity

Such that $y = f(x)$.

Then $e_y = \frac{dy}{dx} e_x$ ...(9.13)

For example let $A = 4.6\ B$

$B = 2.2$ (observed)

p.e. of $B = \pm 0.02$

Now $A = 4.6\ B$

$\therefore$ $\frac{dA}{dB} = 4.6$

$\therefore$ $e_a = 4.6\ e_b = 4.6\ (\pm 0.02) = \pm 0.092$

which is the same as found by rule 2.

5. If a computed quantity is a function of two more observed quantities, its probable error is equal to the square root of summation of the squares of the p.e. of the observed quantity multiplied by its differentiation with respect to that quantity.

Let y = computed quantity

x_1, x_2, x_3 etc = observed quantities

Such that $y = f(x_1, x_2, x_3$ etc.)

Then $$e_y = \sqrt{\left(e_{x1}\frac{dy}{dx_1}\right)^2+\left(e_{x2}\frac{dy}{dx_2}\right)^2+\left(e_{x3}\frac{dy}{dx_3}\right)^2} \quad ...(9.14)$$

where e_y = probable error of the computed quantity

e_{x1}, e_{x2}, e_{x3} = probable errors of observed quantities.

For example, let $A = 4B \times C$; $B = 22 \pm 0.02$; $C = 10 \pm 0.01$

Now $A = 4\,BC$

$\therefore \quad \frac{dA}{dB} = 4C = 4 \times 10 = 40$

$\therefore \quad \frac{dA}{dC} = 4B = 4 \times 22 = 88$

$$\therefore \quad e_a = \sqrt{\left(e_b\frac{dA}{dB}\right)^2+\left(e_c\frac{dA}{dC}\right)^2}$$

$$= \sqrt{(0.02 \times 40)^2 + (0.01 \times 88)^2} = \sqrt{1.415} = \pm \mathbf{1.19}$$

Example 9.1. *The following are the observed values of an angle :*

Angle	*Weight*
40° 20′ 20″	*2*
40° 20′ 18″	*2*
40° 20′ 19″	*3*

Find : *(a) p.e. of single observation of unit weight*

(b) p.e. of weighted arithmetic mean

(c) p.e. of single observation of wegiht 3.

Solution.

The computations are arranged in the tabular form below. Since the error is in seconds only, the degrees and minutes of the quantities have not been included in the tabulation:

Value	*Weight*	*Value × Weight*	*v*	v^2	wv^2
20″	2	40″	+ 1	1	2
18″	2	36″	− 1	1	2
19″	3	57″	0	0	0
	$\Sigma w = 7$	Weighted mean = 19″			$\therefore \Sigma wv^2 = 4$

In the above table,

Weighted arithmetic mean of the seconds readings of the observed angles

$$= \frac{(20'' \times 2) + (18'' \times 2) + (19'' \times 3)}{2+2+3} = \frac{40'' + 36'' + 57''}{7} = \frac{133''}{7} = 19''$$

$\therefore \quad v_1 = 20'' - 19'' = 1''$; $v_2 = 18'' - 19'' = -1''$; $v_3 = 19'' - 19'' = 0$

(*a*) p.e. of single observation of unit weight = E_s

$$= \pm 0.6745\sqrt{\frac{\Sigma wv^2}{n-1}} = \pm 0.6745\sqrt{\frac{4}{3-1}} = \pm \mathbf{0.95}.$$

(*b*) p.e. of weighted arithmetic mean

$$= \pm 0.6745\sqrt{\frac{\Sigma wv^2}{\Sigma w\,(n-1)}} = \pm 0.6745\sqrt{\frac{4}{7 \times 2}} = \pm \mathbf{0.36}$$

(*c*) p.e. of single observation of weight 3

$$= \frac{E_s}{\sqrt{w}} = -\frac{0.95}{\sqrt{3}} = \pm \mathbf{0.55.}$$

9.7. DISTRIBUTION OF ERROR OF THE FIELD MEASUREMENTS

Whenever observations are made in the field, it is always necessary to check for the closing error, if any. The closing error should be distributed to the observed quantities. For example, the sum of the angles measured at a central angle should be 360° ; if the sum is not equal to 360°, the error should be distributed to the observed angles after giving proper weightage to the observations. The following rules should be applied for the distribution of errors :

(1) The correction to be applied to an observation is inversely proportional to the weight of the observation.

(2) The correction to be applied to an observation is directly proportional to the square of the probable error.

(3) In case of line of levels, the correction to be applied is proportional to the length.

Example 9.3. *The following are the three angles α, β and γ observed at a station P closing the horizon, along with their probable errors of measurement. Determine their corrected values.*

$\alpha = 78° \, 12' 12'' \pm 2''$; $\beta = 136° \, 48' 30'' \pm 4''$; $\gamma = 144° \, 59' 08'' \pm 5''$

Solution.

Sum of the three angles = 359° 59′ 50″

Discrepancy = 10″

Hence each angle is to be increased, and the error of 10″ is to be distributed in proportion to the square of the probable error.

Let c_1, c_2 and c_3 be the correction to be applied to the angles α, β and γ respectively.

$c_1 : c_2 : c_3 = (2)^2 : (4)^2 : (5)^2 = 4 : 16 : 25 \quad ...(1)$

Also, $c_1 + c_2 + c_3 = 10'' \quad ...(2)$

From (1), $c_2 = \frac{16}{4} c_1 = 4c_1$ and $c_3 = \frac{25}{4} c_1$

Substituting these values of c_2 and c_3 in (2), we get

$$c_1 + 4c_1 + \frac{25}{4} c_1 = 10''$$

or $$c_1\left(1 + 4 + \frac{25}{4}\right) = 10''$$

$$\therefore \quad c_1 = 10 \times \frac{4}{45} = 0''.89$$

$\therefore$ $c_2 = 4c_1 = 3''.56$ and $c_3 = \frac{25}{4} c_1 = 5''.55$

Check : $c_1 + c_2 + c_3 = 0''.89 + 3''.56 + 5''.55 = 10''$

Hence the corrected angles are

$$\alpha = 78° 12' 12'' + 0''.89 = 78° 12' 12''.89$$
$$\beta = 136° 48' 30'' + 3''.56 = 136° 48' 33''.56$$
and
$$\gamma = 144° 59' 08'' + 5''.55 = 144° 59' 13''.55$$

Sum $= 360° 00' 00''.00$

Example 9.4. *An angle A was measured by different persons and the following are the values :*

Angle		*Number of measurements*
65° 30′ 10″	...	*2*
65° 29′ 50″	...	*3*
65° 30′ 00″	...	*3*
65° 30′ 20″	...	*4*
65° 30′ 10″	...	*3*

Find the most probable value of the angle.

Solution. As stated earlier, the most probable value of an angle is equal to its weighted arithmetic mean.

$$65° 30' 10'' \times 2 = 131° 00' 20''$$
$$65° 29' 50'' \times 3 = 196° 29' 30''$$
$$65° 30' 00'' \times 3 = 196° 30' 00''$$
$$65° 30' 20'' \times 4 = 262° 01' 20''$$
$$65° 30' 10'' \times 3 = 196° 30' 30''$$

Sum $= 982° 31' 40''$

Σ weight $= 2 + 3 + 3 + 4 + 3 = 15$

$\therefore$ Weighted arithmetic mean $= \frac{982° 31' 40''}{15} = 65° 30' 6''.67$

Hence most probable value of the angle = **65° 30′ 6″.67**

Example 9.5. *Adjust the following angles closing the horizon :*

A = 110° 20′ 48″	*wt. 4*
B = 92° 30′ 12″	*wt. 1*
C = 56° 12′ 00″	*wt. 2*
D = 100° 57′ 04″	*wt. 3*

Solution. Sum of the observed angles = 360° 00′ 04″

$\therefore$ Error = + 4″

Total correction = – 4″

This error of 4″ will be distributed to the angles in an inverse proportion to their weights.

Let c_1, c_2, c_3 and c_4 be the corrections to the observed angles A, B, C and D respectively.

$$\therefore \quad c_1 : c_2 : c_3 : c_4 = \frac{1}{4} : \frac{1}{1} : \frac{1}{2} : \frac{1}{3}$$

or
$$c_1 : c_2 : c_3 : c_4 = 1 : 4 : 2 : \frac{4}{3} \quad ...(1)$$

Also $c_1 + c_2 + c_3 + c_4 = 4''$ ...(2)

From (1), we have

$$c_2 = 4c_1 \; ; \quad c_3 = 2c_1 \quad \text{and} \quad c_4 = \frac{4}{3} c_1$$

Substituting these values of c_2, c_3 and c_4 in (2), we get

$$c_1 + 4c_1 + 2c_1 + \frac{4}{3} c_1 = 4$$

or
$$c_1 \left(1 + 4 + 2 + \frac{4}{3} \right) = 4$$

$$\therefore \quad c_1 = \frac{4 \times 3}{25} = \frac{12}{25} = 0''.48$$

Hence
$$c_2 = 4c_1 = 1''.92$$
$$c_3 = 2c_1 = 0''.96$$
$$c_4 = \frac{4}{3} c_1 = 0''.64$$

Hence the corrected angles are

$$A = 110° 20' 48'' - 0''.48 = 110° 20' 47''.52$$
$$B = 92° 30' 12'' - 1''.92 = 92° 30' 10''.08$$
$$C = 56° 12' 00'' - 0''.96 = 56° 11' 59''.04$$
and
$$D = 100° 57' 04'' - 0''.64 = 100° 57' 03''.36$$

Sum $= 360° 00' 00''.00$

9.8. NORMAL EQUATIONS

A normal equation is the one which is formed by multiplying each equation by the coefficient of the unknown whose normal equation is to be found and by adding the equation thus formed. As the number of normal equations is the same as the number of unknowns, the most probable values of the unknowns can be found from the equations.

Consider a round of angles observed at a central station, the horizon closing with three angles x, y and z, which are geometrically fixed by the condition equation

$$x + y + z = 360° = -d \text{ (say)}$$

If all the angles are of equal weight, the error e in the round will be $(x + y + z + d)$. The most probable value of each angle can then be obtained by applying a correction of $\frac{1}{3} e$ to each observed angle.

If, however, one angle is measured directly and the others indirectly, the error equation takes the form

$$e = (ax + by + cz + d) \quad \ldots(1)$$

If the measurements are repeated, giving different values (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) etc., then we have

$$e_1 = ax_1 + by_1 + cz_1 + d \quad ; \quad e_2 = ax_2 + by_2 + cz_2 + d$$
$$e_3 = ax_3 + by_3 + cz_3 + d \quad \text{etc. etc.}$$

By the theory of least squares,

$$\Sigma e^2 = \Sigma(ax + by + cz + d)^2 \text{ should be minimum.}$$

Differentiating this, in order, with respect to x, y and z, and equating each expression so obtained to zero, we get

$$\Sigma a(ax + by + cz + d) = 0 \quad \text{(Normal equation for } x) \quad \ldots(2)$$
$$\Sigma b(ax + by + cz + d) = 0 \quad \text{(Normal equation for } y) \quad \ldots(3)$$
$$\Sigma c(ax + by + cz + d) = 0 \quad \text{(Normal equation for } z) \quad \ldots(4)$$

Equations (2), (3) and (4) are nothing but the fundamental equation (1) multiplied by the coefficient of x, y and z respectively.

These equations are known as the *normal equations* the solution of which will lead to the most probable value of x, y and z.

Thus, equation (2) is the normal equation in x, equation (3) is the normal equation for y, and equation (4) is the normal equation for z.

Now, $\Sigma a(ax + by + cz + d) = a[(ax_1 + by_1 + cz_1 + d) + (ax_2 + by_2 + cz_2 + d) + (ax_3 + by_3 + cz_3 + d) + \ldots]$

Similarly, $\Sigma b(ax + by + cz + d) = b\,[(ax_1 + by_1 + cz_1 + d) + (ax_2 + by_2 + cz_2 + d) + (ax_3 + by_3 + cz_3 + d) + \ldots]$

and $\Sigma c(ax + by + cz + d) = c\,[(ax_1 + by_1 + cz_1 + d) + (ax_2 + by_2 + cz_2 + d) + (ax_3 + by_3 + cz_3 + d) + \ldots]$

Hence if the observations are of equal weight, we derive the following rule for forming the normal equations :

Rule 1. ***To form a normal equation for each of the unknown quantities, multiply each observation equation by the algebraic co-efficient of that unknown quantity in that equation, and add the results.***

If, however, each set of the observations (x_1, y_1, z_1), (x_2, y_2, z_2) (x_n, y_n, z_n) have different weights $w_1, w_2, \ldots\ldots, w_n$ respectively, the error equations will take the following form :

$$e_1 = ax_1 + by_1 + cz_1 + d\,, \quad \text{(weight } w_1)$$
$$e_2 = ax_2 + by_2 + cz_2 + d\,, \quad \text{(weight } w_2)$$
$$\vdots$$
$$e_n = ax_n + by_n + cz_n + d\,, \quad \text{(weight } w_n) \quad \ldots(I)$$

By the theory of least squares,

$$\Sigma we^2 = \Sigma w(ax + by + cz + d)^2 \text{ should be a minimum.}$$

Differentiating this in order with respect to x, y, z and equating each expression so obtained to zero, we get

$$\Sigma wa\,(ax + by + cz + d) = 0 \quad \text{(Normal equation for } x) \quad \ldots(II)$$
$$\Sigma wb\,(ax + by + cz + d) = 0 \quad \text{(Normal equation for } y) \quad \ldots(III)$$
$$\Sigma wc\,(ax + by + cz + d) = 0 \quad \text{(Normal equation for } z) \quad \ldots(IV)$$

Equations (II), (III) and (IV) are nothing but the fundamental equations (I) multiplied by coefficients of x, y and z respectively, and the weight of each equation. These are therefore normal equations in x, y and z respectively.

Now $\Sigma wa\,(ax + by + cz + d) = a[w_1(ax_1 + by_1 + cz_1 + d) + w_2(ax_2 + by_2 + cz_2 + d) \ldots + w_n(ax_n + by_n + cz_n + d)]$

$\Sigma wb(ax + by + cz + d) = b[w_1(ax_1 + by_1 + cz_1 + d) + w_2(ax_2 + by_2 + cz_2 + d) \ldots + w_n\,(ax_n + by_n + cz_n + d)]$

$\Sigma wc(ax + by + cz + d) = c[w_1(ax_1 + by_1 + cz_1 + d) + w_2(ax_2 + by_2 + cz_2 + d) \ldots + w_n(ax_n + by_n + cz_n + d)]$

Hence if the observation equations are of different weights, we derive the following rule for forming the normal equations :

Rule 2. ***To form the normal equation for each of the unknown quantities, multiply each observation equation by the product of the algebraic coefficient of that unknown quantity in that equation and the weight of that observation and add the results.***

Example 9.6 (*a*) *Form the normal equations for x, y and z in the following equations of equal weight* :

$$3x + 3y + z - 4 = 0 \quad \ldots(1)$$
$$x + 2y + 2z - 6 = 0 \quad \ldots(2)$$
$$5x + y + 4z - 21 = 0 \quad \ldots(3)$$

(*b*) *If the weights of the above equations are 2, 3 and 1 respectively, form the normal equations for x, y and z.*

Solution.

(*a*) The normal equation of an unknown quantity is formed by multiplying each equation by the algebraic coefficient of that unknown quantity in that equation and adding the result.

Thus, in equations (1), (2) and (3) the coefficients of x are 3, 1 and 5 respectively. Hence

$$9x + 9y + 3z - 12 = 0$$
$$x + 2y + 2z - 6 = 0$$
$$25x + 5y + 20z - 105 = 0$$

$\therefore$ Normal equation for x is $35x + 16y + 25z - 123 = 0$ ...(I)

Similarly, the coefficents for y are 3, 2 and 1. Hence

$$9x + 9y + 3z - 12 = 0$$
$$2x + 4y + 4z - 12 = 0$$
$$5x + y + 4z - 21 = 0$$

$\therefore$ Normal equation for y is $16x + 14y + 11z - 45 = 0$ ...(II)

Similarly, the coefficients of z are 1, 2 and 4. Hence

$$3x + 3y + z - 4 = 0$$
$$2x + 4y + 4z - 12 = 0$$
$$20x + 4y + 16z - 84 = 0$$

$\therefore$ Normal equation for z is $25x + 11y + 21z - 100 = 0$ ...(III)

Hence the normal equations for x, y and z are

$$35x + 16y + 25z - 123 = 0 \quad ...(I)$$
$$16x + 14y + 11z - 45 = 0 \quad ...(II)$$
$$25x + 11y + 21z - 100 = 0 \quad ...(III)$$

(*b*) The normal equation of an unknown quantity is formed by multiplying each equation by the algebraic co-efficient of that quantity in that equation and the weight of that equation, and adding the result.

Thus, in equations (1), (2) and (3) the product of coefficients of x and weight of respective equations are : (3×2), (1×3) and (5×1). Hence

$$18x + 18y + 6z - 24 = 0 \text{ (from 1)}$$
$$3x + 6y + 6z - 18 = 0 \text{ (from 2)}$$
$$25x + 5y + 20z - 105 = 0 \text{ (from 3)}$$

Normal equation for x is $46x + 29y + 32z - 147 = 0$ ...(I a)

Similarly, the product of coefficient of y and weight of each equation, in the original equations are (3×2), (2×3) and (1×1) respectively. Hence

$$18x + 18y + 6z - 24 = 0 \text{ (from 1)}$$
$$6x + 12y + 12z - 36 = 0 \text{ (from 2)}$$
$$5x + y + 4z - 21 = 0 \text{ (from 3)}$$

$\therefore$ Normal equation for y is $29x + 31y + 22z - 81 = 0$ ...(II a)

And, the product of coefficient of z and weight of each equation, in the origional equations are (1×2), (2×3) and (4×1) reespectively. Hence

$$6x + 6y + 2z - 8 = 0 \text{ (from 1)}$$
$$6x + 12y + 12z - 36 = 0 \text{ (from 2)}$$
$$20x + 4y + 16z - 84 = 0 \text{ (from 3)}$$

$\therefore$ Normal equation for z is $32x + 22y + 30z - 128 = 0$ ...(III a)

Hence the normal equations for x, y and z are as follows :

$$46x + 29y + 32z - 147 = 0 \quad ...(I\ a)$$
$$29x + 31y + 22z - 81 = 0 \quad ...(II\ a)$$
$$32x + 22y + 30z - 128 = 0 \quad ...(III\ a)$$

9.9. DETERMINATION OF THE MOST PROBABLE VALUES

As defined earlier, the most probable value of a quantity is the one which has more chances of being true than has any other. It is deduced from the several measurements on which it is based. In practice, the following cases may arise of which the most probable value may be required to be determined :

1. Direct observations of equal weights.
2. Direct observations of unequal weights.
3. Indirectly observed quantities involving unknowns of equal weights.
4. Indirectly observed quantities involving unknowns of unequal weights.
5. Observation equations accompanied by condition equation.

Case 1. Direct Observations of Equal Weights

As stated earlier, *the most probable value of the directly observed quantity of equal weights is equal to the arithmetic mean of the observed values.*

Thus, if $V_1, V_2, V_3, ... V_n$ is the observed value of a quantity of equal weight, and M is the arithmetic mean, then

$$M = \frac{V_1 + V_2 + V_3 + ... V_n}{n} = \text{most probable value} \quad ...(9.16)$$

Case 2. Direct Observations of Unequal Weights

As proved earlier, *the most probable value of an observed quantity of unequal weights is equal to the weighted arithmetic mean of the observed quantities.*

Thus, if V_1, V_2, V_3 etc. are the observed quantities with weights w_1, w_2, w_3 etc. and N is the most probable value of the quantity, we have

$$N = \frac{w_1 V_1 + w_2 V_2 + w_3 V_3 + ... w_n V_n}{w_1 + w_2 + w_3 + ... w_n} \quad ...(9.16)$$

Case 3 and 4. Indirectly Observed Quantities Involving Unknowns of Equal Weights or Unequal Weights.

When the unknowns are independent of each other, their most probable values can be found by forming the normal equations for each of the unknown quantities, and treating them as simultaneous equations to get the values of the unknowns. The rules for forming the normal equations have already been discussed. See examples 9.7, 9.8 9.9, 9.10 and 9.11 for illustration.

Case 5. Observation Equations Accompained by Condition Equation

When the observation equations are accompanied by one or more condition equations, the latter may be reduced to an observation equation which will eliminate one of the unknowns. The normal equation can then be formed for the remaining unknowns. There is also another method, known as the *method of correlates* by which the observation equations are eliminated. However, the former method (*i.e.*, eliminating the condition equation) is suitable for simple cases while the latter method is used for more complicated problems.

Example 9.7 *Find the most probable value of the angle A from the following observation equation* :

$$A = 30°\ 28'\ 40'' \ ;\ 3A = 91°\ 25'\ 55''\ ;\ 4A = 121°\ 54'\ 30''$$

Solution.

There is only one unknown, and all the observations are of equal weight. The coefficients of A in the three equations are 1, 3 and 4. Hence multiply these equations by 1, 3 and 4 respectively and add the resulting equations to get the normal equation for A.

Thus,

$$A = 30°\ 28'\ 40''$$
$$9A = 274°\ 17'\ 45''$$
$$16A = 487°\ 38'\ 00''$$

$\therefore$ $26A = 792°\ 24'\ 25''$ (Normal equation in A)

$\therefore$ $A = \mathbf{30°\ 28'\ 37''.9}$.

Alternative Solution

From first equation $A = 30°\ 28'\ 40''$, weight 1

From second equation, $A = \dfrac{91°\ 25'\ 55''}{3} = 30°\ 28'\ 38''.33$

$$\text{Weight} = \frac{1}{(\frac{1}{3})^2} = 9$$

From third equation, $A = \dfrac{121°\ 54'\ 30''}{4} = 30°\ 28'\ 37''.5$

$$\text{Weight} = \frac{1}{(\frac{1}{4})^2} = 16$$

Sum of weights $= 1 + 9 + 16 = 26$

Weighted mean (A) $= 30°\ 28' + \dfrac{1}{26}[(40 \times 1) + (38.33 \times 9) + (37.5 \times 16)] = 30°\ 28' + 37''.9$

$= \mathbf{30°\ 28'\ 37''.9}$.

Example 9.8. *Find the most probable value of the angle A from the following observation equations* :

$$A = 30°\ 28'\ 40'' \quad \textit{weight } 2.$$
$$3A = 91°\ 25'\ 55'' \quad \textit{weight } 3.$$

Solution.

There is only one unknown. However, the observations are of unequal weight. The normal equation can be formed by multiplying each of the two observation equations by the corresponding weight and coefficient of A, and adding them.

Thus, in the first equation, coefficient of A is 1 and weight of observation is 2. Hence multiply it by $2 (= 2 \times 1)$. Similarly, in the second equation, the co-efficient of A is 3 and the weight of the observation is 3. Hence multiply it by $9 (= 3 \times 3)$. Thus, we have

$$2A = 60°\ 57'\ 20''$$
$$27A = 822°\ 53'\ 15''$$

$\therefore$ $29A = 883°\ 50'\ 35''$ (Normal equation in A)

or $A = \mathbf{30°\ 28'\ 38''.5}$.

Alternative Solution

From Eq. (1) $A = 30°\ 28'\ 40''$, weight 2

From Eq. (2), $A = \dfrac{90°\ 25'\ 55''}{3} = 30°\ 28'\ 38''.33$

$$\text{Weight} = \frac{3}{(\frac{1}{3})^2} = 27$$

Sum of weights $= 2 + 27 = 29$

$\therefore$ $A = 30°\ 28' + \dfrac{1}{29}[(40 \times 2) + (38.33 \times 27)] = \mathbf{30°\ 28'\ 38''.45}$.

Example 9.9. *Find the most probable values of the angles A and B from the following observations at a station O* :

$$A = 9°\ 48'\ 36''.6 \quad \textit{weight } 2 \quad (1)$$
$$B = 54°\ 37'\ 48''.3 \quad \textit{weight } 3 \quad (2)$$
$$A + B = 104°\ 26'\ 28''.5 \quad \textit{weight } 4. \quad (3)$$

(*B.U.*)

Solution.

There are two unknowns A and B and both are independent of each other, and there will be two normal equations.

Two find the normal equation for A multiply equation (1) by 2 (since co-efficient of $A \times$ weight $= 1 \times 2 = 2$), equation (2) by zero, (since co-efficient of A is zero) and equation (3) by 4 (since the co-efficient of $A \times$ weight $= 1 \times 4 = 4$). Thus, we have

$$2A = 99°\ 37'\ 13''.2$$

and $4A + 4B = 417°\ 45'\ 54''.0$

$\therefore$ $6A + 4B = 517°\ 23'\ 07''.2$ ....I (Normal Eq. for A)

Similarly, to find the normal equation for B, multiply equation (1) by zero (since the coefficient of B is zero), equation (2) by 3 and equation (3) by 4. Thus,

$$3B = 163°\ 53'\ 24''.9$$
$$4A + 4B = 417°\ 45'\ 54''.0$$

$\therefore$ $4A + 7B = 581°\ 39'\ 18''.9$ ...II (Normal Eq. for B)

Hence the normal equations are

$$6A + 4B = 517° \, 23' \, 7''.2 \quad \text{...(I)}$$

$$4A + 7B = 581° \, 39' \, 18''.9 \quad \text{...(II)}$$

To solve these for A and B, multiply I by 2 and II by 3. Thus,

$$12A + 8B = 1034° \, 46' \, 14''.4 \quad \text{...(1)}$$

$$12A + 21B = 1744° \, 57' \, 56''.7 \quad \text{...(2)}$$

Subtracting (1) from (2), we get

$$13B = 710° \, 11' \, 42''.3$$

$$B = \mathbf{54° \, 37' \, 49''.4}$$

Substituting value of B in (1), we get

$$A = \mathbf{49° \, 48' \, 38''.3}.$$

Example 9.10. *The following are mean values observed in the measurement of three angles* α, β *and* γ *at one station* ;

$$\alpha = 76° \, 42' \, 46''.2 \text{ with weight } 4$$

$$\alpha + \beta = 134° \, 36' \, 32''.6 \text{ with weight } 3$$

$$\beta + \gamma = 185° \, 35' \, 24''.8 \text{ with weight } 2$$

$$\alpha + \beta + \gamma = 262° \, 18' \, 10''.4 \text{ with weight } 1$$

Calculate the most probable value of each angle. (U.L.)

Solution.

To form the normal equation for unknown, multiply each equation by the coefficient of that unknown and also by the weight of the equation, and take the sum of the resulting equations.

Thus, forming normal equation for α for we have

$$4\alpha = 306° \, 51' \, 04''.8$$

$$3\alpha + 3\beta = 403° \, 49' \, 37''.8$$

$$\alpha + \beta + \gamma = 262° \, 18' \, 10''.4$$

$$\therefore \quad 8\alpha + 4\beta + \gamma = 972° \, 58' \, 53''.0 \quad \text{...(Normal equation for } \alpha)$$

Forming normal equation for β, we have

$$3\alpha + 3\beta = 403° \, 49' \, 37''.8$$

$$2\beta + 2\gamma = 371° \, 10' \, 49''.6$$

$$\alpha + \beta + \gamma = 262° \, 18' \, 10''.4$$

$$\therefore \quad 4\alpha + 6\beta + 3\gamma = 1037° \, 18' \, 37''.8 \quad \text{...(Normal equation for } \beta)$$

Forming normal equation for γ we have

$$2\beta + 2\gamma = 371° \, 10' \, 49''.6$$

$$\alpha + \beta + \gamma = 262° \, 18' \, 10''.4$$

$$\therefore \quad \alpha + 3\beta + 3\gamma = 633° \, 29' \, 00''.0 \quad \text{...(Normal equation for } \gamma)$$

Hence the three normal equations are

$$8\alpha + 4\beta + \gamma = 972° \, 58' \, 53''.0 \quad \text{...(1)}$$

$$4\alpha + 6\beta + 3\gamma = 1037° \, 18' \, 37''.8 \quad \text{...(2)}$$

$$\alpha + 3\beta + 3\gamma = 633° \, 29' \, 00''.0 \quad \text{...(3)}$$

Solving the above three equations simultaneously for α, β and γ we get

$$\alpha = \mathbf{76° \, 42' \, 46''.17}$$

$$\beta = \mathbf{57° \, 53' \, 46''.13}$$

$$\gamma = \mathbf{127° \, 41' \, 38''.26}$$

9.10. ALTERNATIVE METHOD OF DIFFERENCES

The above *direct method* of solving the normal equations is very laborious since it involves large numbers. In order to make them as small as possible, we can solve the equation by *method of differences.* A set of values is assumed for the most probable values of the unknown quantities and the most probable series of errors are determined by normal equations. The errors so found are then added algebraically to the observed values to get the most probable values of the measurements. The procedure for the solution of the problem is as follows :

(1) Let k_1, k_2, k_3 etc. be the corrections (or the residual errors) to the observed values.

(2) Replace the observation equations by equations in terms of k_1, k_2, k_3 etc., to express the discrepancy between the observed results and those given by the assumed values, *always subtracting the latter from the former.*

(3) Form the normal equations in terms of k_1, k_2, k_3, etc. and solve them to get k_1, k_2, k_3 etc.

(4) Add these algebraically to the quantities to get their most probable values.

Example 9.11. *The following observations of three angles A, B and C were taken at one station* :

$$A = 75° \, 32' \, 46''.3 \text{ with weight } 3$$

$$B = 55° \, 09' \, 53''.2 \text{ with weight } 2$$

$$C = 108° \, 09' \, 28''.8 \text{ with weight } 2$$

$$A + B = 130° \, 42' \, 41''.6 \text{ with weight } 2$$

$$B + C = 163° \, 19' \, 22''.5 \text{ with weight } 1$$

$$A + B + C = 238° \, 52' \, 9''.8 \text{ with weight } 1$$

Determine the most probable value of each angle.

Solution.

Let k_1, k_2, k_3 be the most probable correction to A, B and C. Then the most probable values of A, B and C are :

$$A = 75° \, 32' \, 46''.3 + k_1 \quad \text{...(1)}$$

$$B = 55° \, 09' \, 53''.2 + k_2 \quad \text{...(2)}$$

$$C = 108° \, 09' \, 28''.8 + k_3 \quad \text{...(3)}$$

$$A + B = 130° \, 42' \, 39''.5 + k_1 + k_2 \text{ by adding (1) and (2).} \quad \text{...(4)}$$

$B + C = 163° 19' 22''.0 + k_2 + k_3$ by adding (2) and (3). ...(5)

and $A + B + C = 238° 52' 08''.3 + k_1 + k_2 + k_3$ by adding (1), (2) and (3). ...(6)

Subtracting these from the corresponding observation equations, we get the following *reduced observation equations* :

$k_1 = 0$ weight 3

$k_2 = 0$ weight 2

$k_3 = 0$ weight 2

$k_1 + k_2 = +2''.1$ weight 2

$k_2 + k_3 = +0''.5$ weight 1

$k_1 + k_2 + k_3 = +1''.5$ weight 1

Normal equation of k_1 :

$3k_1 = 0$

$2k_1 + 2k_2 = +4.2$

$k_1 + k_2 + k_3 = +1.5$

$\therefore \quad 6k_1 + 3k_2 + k_3 = +5.7$

Normal equation for k_2 :

$2k_2 = 0$

$2k_1 + 2k_2 = +4.2$

$k_2 + k_3 = +0.5$

$k_1 + k_2 + k_3 = +1.5$

$\therefore \quad 3k_1 + 6k_2 + 2k_3 = +6.2$

Normal equation for k_3 :

$2k_3 = 0$

$k_2 + k_3 = +0.5$

$k_1 + k_2 + k_3 = +1.5$

$\therefore \quad k_1 + 2k_2 + 4k_3 = +2.0$

Hence the three normal equations are :

$6k_1 + 3k_2 + k_3 = +5.7$

$3k_1 + 6k_2 + 2k_3 = +6.2$

$k_1 + 2k_2 + 4k_3 = +2.0$

Solving the simultaneously for k_1, k_2 and k_3 , we get

$k_1 = +0''.58$

$k_2 = +0''.75$

$k_3 = -0''.02$

Hence the most probable values of A, B and C are

$A = 75° 32' 46''.3 + 0''.58 =$ **$75° 32' 46''.88$**

$B = 55° 09' 53''.2 + 0''.75 =$ **$55° 09' 53''.95$**

$C = 108° 09' 28''.8 - 0''.02 =$ **$108° 09' 28''.78$**.

Example 9.12. *The following are the observed values of A, B and C at a station, the angles being subject to the condition that $A + B = C$:*

$A = 30° 12' 28''.2$

$B = 35° 48' 12''.6$

$C = 66° 0' 44''.4$

Find the most probable values of A, B and C.

Solution.

To avoid the condition equation $A + B = C$, we can write the third observation equation as

$A + B = 66° 0' 44''.4$

Hence the three observation equations are :

$A = 30° 12' 28''.2$ ...(1)

$B = 35° 48' 12''.6$ ...(2)

$A + B = 66° 00' 44''.4$ ...(3)

Normal equation for A :

$A = 30° 12' 28''.2$

$A + B = 66° 00' 44''.4$

$2A + B = 96° 13' 12''.6$ (Normal equation for A)

Normal equation for B

$B = 35° 48' 12''.6$

$A + B = 66° 00' 44''.4$

$A + 2B = 101° 48' 57''.0$ (Normal equation for B)

Hence the two normal equations are :

$2A + B = 96° 13' 12''.6$ ...(1

$A + 2B = 101° 48' 57''.0$ ...(2)

Solving these, we get

$A = 30° 12' 29''.4$

$B = 35° 48' 13''.8$

$\therefore$ **$C = A + B = 66° 00' 43''.2$**

Example 9.13. *Find the most probable values of angles A, B and C of triangle ABC from the following observation equations:*

$A = 68° 12' 36''$

$B = 53° 46' 12''$

$C = 58° 01' 16''$

Solution.

The condition equation is

$$A + B + C = 180°$$

From which $C = 180° - (A + B)$

Thus, the third unknown C can be eliminated writing one more observation equation;

$$C = 180° - (A + B) = 58°\ 01'\ 16''$$

or $$A + B = 180° - 58°\ 01'\ 16'' = 121°\ 58'\ 44''$$

Hence, the new observation equations are :

$$A = 68°\ 12'\ 36''$$
$$B = 53°\ 46'\ 12''$$

and $$A + B = 121°\ 58'\ 44''$$

Normal equation for A :

$$A = 68°\ 12'\ 36''$$
$$A + B = 121°\ 58'\ 44''$$

$$2A + B = 190°\ 11'\ 20''$$

Normal equation for B :

$$B = 53°\ 46'\ 12''$$
$$A + B = 121°\ 58'\ 44''$$

$$A + 2B = 175°\ 44'\ 56''$$

Hence, the normal equations are :

$$2A + B = 190°\ 11'\ 20'' \quad ...(1)$$
$$A + 2B = 175°\ 44'\ 56'' \quad ...(2)$$

Solving these, we get

$$A = \mathbf{68°\ 12'\ 34''.7}$$
$$B = \mathbf{53°\ 46'\ 10''.6}$$

$\therefore$ $C = 180° - (A + B) = 180° - (68°\ 12'\ 34''.7 + 53°\ 46'\ 10''.6) = \mathbf{58°\ 1'\ 14''.7}$

Alternative Solution

$$A = 68°\ 12'\ 36''$$
$$B = 53°\ 46'\ 12''$$
$$C = 58°\ 01'\ 16''$$

$\therefore$ $A + B + C = 180°\ 0'\ 04''$

$\therefore$ Total correction $= -4''$

Since the weight of each of the observations is equal, the corrections will be equally divided.

Hence corrected A (most probable values of A)

$= 68°\ 12'\ 36'' - 1''.33 = \mathbf{68°\ 12'\ 34''.67}$

$B = 53°\ 46'\ 12'' - 1''.33 = \mathbf{53°\ 46'\ 10''.67}$

and $C = 58°\ 01'\ 16'' - 1''.33 = \mathbf{58°\ 01'\ 14''.67}$

Example 9.14. *The angles of a triangle ABC were recorded as follows :*

$A = 77°\ 14'20''$ weight 4

$B = 49°\ 40'35''$ weight 3

$C = 53°\ 04'52''$ weight 2

Give the corrected values of the angles. (*K.U.*)

Solution.

The condition equation is

$$A + B + C = 180° \quad \text{or} \quad C = 180° - (A + B)$$

Thus, the unknown C can be eliminated by forming one more observation equation in terms of the two unknowns A and B :

$$C = 180° - (A + B) = 53°\ 4'\ 52''$$

or $$A + B = 180° - 53°\ 4'\ 52'' = 126°\ 55'\ 8''$$

Hence the observation equations are :

$$A = 77°\ 14'\ 20'' \quad \text{(weight 4)}$$
$$B = 49°\ 40'\ 35'' \quad \text{(weight 3)}$$
$$A + B = 126°\ 55'\ 08'' \quad \text{(weight 2)}$$

Normal equation for A :

$$4A = 308°\ 57'\ 20''$$
$$2A + 2B = 253°\ 50'\ 16''$$

$$6A + 2B = 562°\ 47'\ 36''$$

or $$3A + B = 281°\ 23'\ 48''$$

Normal equation for B :

$$3B = 149°\ 01'\ 45''$$
$$2A + 2B = 253°\ 50'\ 16''$$

$$2A + 5B = 402°\ 52'\ 01''$$

Hence, the normal equations are :

$$3A + B = 281°\ 23'\ 48''$$

and $$2A + 5B = 402°\ 52'\ 01''$$

Solving the above simultaneously for A and B, we get

$$A = \mathbf{77°\ 14'\ 23''}$$
$$B = \mathbf{49°\ 40'\ 39''}$$
$$C = 180° - (A + B) = 180° - (77°\ 14'\ 23'' + 49°\ 40'\ 39'') = \mathbf{53°\ 4'\ 58''}$$

Alternative Solution

The problem can be solved by the *method of differences*, thus simplifying the calculation work. The reduced observation equations are :

$$A = 77° 14' 20'' \quad ...(1)$$
$$B = 49° 40' 35'' \quad ...(2)$$
$$A + B = 126° 55' 08'' \quad ...(3)$$

Let k_1 and k_2 be the corrections to A and B so that the most probable values of A, B and $(A+B)$ are :

$$A = 77° 14' 20'' + k_1 \quad ...(1a)$$
$$B = 49° 40' 35'' + k_2 \quad ...(2a)$$

$$A + B = 126° 54' 55'' + k_1 + k_2 \quad ...(3a)$$

Subtracting these from the corresponding observation equations, we get

$$k_1 = 0 \quad ...(\text{wt. } 4) \quad ...(1b)$$
$$k_2 = 0 \quad ...(\text{wt. } 3) \quad ...(2b)$$
$$k_1 + k_2 = +13'' \quad ...(\text{wt. } 2) \quad ...(3b)$$

Normal equation for k_1 :

$$4k_1 = 0$$
$$2k_1 + 2k_2 = +26''$$

$$6k_1 + 2k_2 = +26''$$

Normal equation for k_2 :

$$3k_2 = 0$$
$$2k_1 + 2k_2 = +26''$$

$$2k_1 + 5k_2 = +26''$$

Hence the normal equations are :

$$6k_1 + 2k_2 = +26''$$

and

$$2k_1 + 5k_2 = +26''$$

Solving which, we get $k_2 = +4''$; $k_1 = +3''$

Applying these corrections to the observed angles, we get the most probable values as follows :

$$A = 77° 14' 23''$$
$$B = 49° 40' 39''$$
$$C = 53° \; 4' 58''.$$

Example 9.15. *The following angles were measured at a station O so as to close the horizon* :

$$\angle AOB = 83° 42' 28''.75 \quad \textit{weight } 3$$
$$\angle BOC = 102° 15' 43''.26 \quad \textit{weight } 2$$

$$\angle COD = 94° 38' 27''.22 \quad \textit{weight } 4$$
$$\angle DOA = 79° 23' 23''.77 \quad \textit{weight } 2.$$ *Adjust the angles. (K.U.)*

Solution. The condition equation is

$$\angle AOB + \angle BOC + \angle COD + \angle DOA = 360°$$

$\therefore$ $$\angle DOA = 360° - (\angle AOB + \angle BOC + \angle COD)$$

Hence $$79° 23' 23''.77 = 360° - (\angle AOB + \angle BOC + \angle COD)$$

or $$\angle AOB + \angle BOC + \angle COD = 360° - 79° 23' 23''.77 = 280° 36' 36''.23$$

Hence the observation equation are :

$$\angle AOB = 83° 42' 28''.75 \quad \text{wt. } 3 \quad ...(1)$$
$$\angle BOC = 102° 15' 43''.26 \quad \text{wt. } 2 \quad ...(2)$$
$$\angle COD = 94° 38' 27''.22 \quad \text{wt. } 4 \quad ...(3)$$
$$\angle AOB + \angle BOC + \angle COD = 280° 36' 36''.23 \quad \text{wt. } 2 \quad ...(4)$$

Let k_1, k_2, k_3 be the corrections to the assumed values of $\angle AOB$, $\angle BOC$ and $\angle COD$, so that their most probable values are:

$$\angle AOB = 83° 42' 28''.75 + k_1 \quad ...(1a)$$
$$\angle BOC = 102° 15' 43''.26 + k_2 \quad ...(2a)$$
$$\angle COD = 94° 38' 27''.22 + k_3 \quad ...(3a)$$

$$\angle AOB + \angle BOC + \angle COD = 280° 36' 39''.23 + k_1 + k_2 + k_3 \quad ...(4a)$$

Subtracting these from the corresponding observation equations, we get

$$k_1 = 0 \quad \text{wt. } 3 \quad ...(1b)$$
$$k_2 = 0 \quad \text{wt. } 2 \quad ...(2b)$$
$$k_3 = 0 \quad \text{wt. } 4 \quad ...(3b)$$
$$k_1 + k_2 + k_3 = -3' \quad \text{wt. } 2 \quad ...(4b)$$

Normal equation for k_1 :

$$3k_1 = 0$$
$$2k_1 + 2k_2 + 2k_3 = -6$$

$$5k_1 + 2k_2 + 2k_3 = -6$$

Normal equation for k_2 :

$$2k_2 = 0$$
$$2k_1 + 2k_2 + 2k_3 = -6$$

$$2k_1 + 4k_2 + 2k_3 = -6$$

Normal equation for k_3 :

$$4k_3 = 0$$
$$2k_1 + 2k_2 + 2k_3 = -6$$

$$2k_1 + 2k_2 + 6k_3 = -6$$

Hence the three normal equations for k_1, k_2, k_3 are :

$$5k_1 + 2k_2 + 2k_3 = -6$$
$$2k_1 + 4k_2 + 2k_3 = -6$$
$$2k_1 + 2k_2 + 6k_3 = -6$$

Solving these simultaneously for k_1, k_2 and k_3 we get

$$k_1 = -0''.63 \quad ; \quad k_2 = -0''.95 \quad ; \quad k_3 = -0''.47$$

Hence the most probable values of the angles are

$AOB = 83° 42' 28''.75 - 0''.63 =$ **83° 42' 28''.12**

$BOC = 102° 15' 43''.26 - 0''.95 =$ **102° 15' 42''.31**

$COD = 94° 38' 27''.22 - 0''.47 =$ **94° 38' 26''.75**

$DOA =$ **79° 23' 22''.82**

9.11. METHOD OF CORRELATES

Correlates or correlatives are the unknown multiples or independent constants used for finding most probable values of unknowns. We have already studied the method of normal equations for finding the most probable values of quantities from observations involving condition equations. The direct method of normal equations was used for simple cases while the 'method of differences' or 'corrections' was used for reducing the arithmetical work. The condition equation was used to eliminate one of the unknown thus giving one more observation equation. However, the method of normal equations become more tedious when the number of conditions are more. In that case, the method of correlates may be used.

In the method of correlates, all the condition equations are collected. To this is added one more equation of condition imposed by the theory of least squares, *i.e.*, the sum of the squares of the residual errors should be minimum.

Suppose, for example, the angles A, B, C, D are measured at a station closing the horizon, the observed values of angles, A, B, C, D may be of weights w_1, w_2, w_3 and w_4 respectively.

Let E be the total residual error in the summation of the four angles such that

$$A + B + C + D - 360° = E$$

Let e_1, e_2, e_3 and e_4 be the corrections to be applied to the observed angles. Then, we have one *equation of condition* :

$$\Sigma e = e_1 + e_2 + e_3 + e_4 = E \quad \ldots(1)$$

Further the *least square condition* requires that

$$\Sigma(we^2) = w_1 e_1^2 + w_2 e_2^2 + w_3 e_3^2 + w_4 e_4^2 = \text{a minimum} \quad \ldots(2)$$

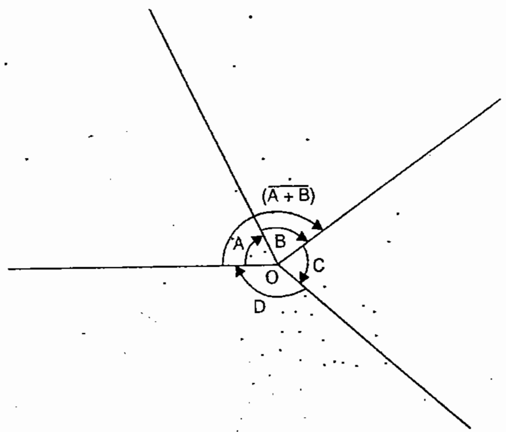


FIG. 9.2. METHOD OF CORRELATES.

Thus, we get two condition equations. Differentiating these two equations, we get

$$\Sigma(\delta e) = \delta e_1 + \delta e_2 + \delta e_3 + \delta e_4 = 0 \quad \ldots(3)$$

and
$$\Sigma(w e\, \delta e) = w_1 e_1 \delta e_1 + w_2 e_2 \delta e_2 + w_3 e_3 \delta e_3 + w_4 e_4 \delta e_4 = 0 \quad \ldots(4)$$

Multiply equation (3) by a correlative $-\lambda_1$ and add the result to equation (4). Thus,

$$-\lambda_1 \delta e_1 - \lambda_1 \delta e_2 - \lambda_1 \delta e_3 - \lambda_1 \delta e_4 = 0$$
$$w_1 e_1 \delta e_1 + w_2 e_2 \delta e_2 + w_3 e_3 \delta e_3 + w_4 e_4 \delta e_4 = 0$$

$$\therefore \quad \delta e_1(w_1 e_1 - \lambda_1) + \delta e_2(w_2 e_2 - \lambda_1) + \delta e_3(w_3 e_3 - \lambda_1) + \delta e_4(w_4 e_4 - \lambda_1) = 0 \quad \ldots(5)$$

Since $\delta e_1, \delta e_2, \delta e_3$ are definite quantities and are independent of each other, their coefficient must vanish independently, or

$$\lambda_1 = w_1 e_1 = w_2 e_2 = w_3 e_3 = w_4 e_4$$

From which
$$e_1 = \frac{\lambda_1}{w_1}, \quad e_2 = \frac{\lambda_1}{w_2}, \quad e_3 = \frac{\lambda_1}{w_3}, \quad e_4 = \frac{\lambda_1}{w_4} \quad \ldots(6)$$

Equation (6) shows that the corrections to be applied are inversely proportional to the weights.

To find the value of the correlative λ_1, substitute these values of e_1, e_2, e_3 and e_4 in equation (1). Thus,

$$\frac{\lambda_1}{w_1} + \frac{\lambda_1}{w_2} + \frac{\lambda_1}{w_3} + \frac{\lambda_1}{w_4} = E$$

or
$$\lambda_1 \left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3} + \frac{1}{w_4} \right) = E \quad \ldots(7) \quad \ldots(9.17)$$

From equation (7), the value of λ_1 can be calculated since w_1, w_2, w_3, w_4 and E are known. Knowing the value of λ_1, the corrections e_1, e_2, e_3, e_4 can be calculated from equation (6). These corrections, when applied to the observed angles, will give the most probable values of the angles.

In the above treatment, only one condition equation [*i.e.* Σ (angle) = 360°] was imposed and, therefore, there was only one correlative λ_1. However, if there are more than one condition equations, the first equation (in the form of equation 3 obtained after differentiation) is multiplied by $-\lambda_1$, second by $-\lambda_2$, third by $-\lambda_3$ and so on, and these are added to equation 4 (obtained from the least square principles) to get pairs of equations such as equation (5) and $\lambda_1, \lambda_2, \lambda_3$ etc. can be calculated.

For example, in addition to the individual angles A, B, C and D, angle $(\overline{A+B})$ was also measured with weight w_5. Let e_5 be the correction to be applied to $(\overline{A+B})$. Let E' be the error of closure between the combined angle $(\overline{A+B})$ and the summation of angles A and B, such that

$$(\overline{A+B}) - A - B = E'$$

Hence we get total two condition equations in terms of corrections :

$$e_1 + e_2 + e_3 + e_4 = E \quad \text{...(1a)}$$

and $$e_5 - (e_1 + e_2) = E' \quad \text{...(1b)}$$

In addition to this, the *least square conditions* requires that

$$w_1 e_1^2 + w_2 e_2^2 + w_3 e_3^2 + w_4 e_4^2 + w_5 e_5^2 = \text{a minimum} \quad \text{..(2)}$$

Differentiating equations (1*a*, 1*b*) and (2) partially, we get

$$\delta e_1 + \delta e_2 + \delta e_3 + \delta e_4 = 0 \quad \text{...(3a)}$$

$$\delta e_5 - \delta e_1 - \delta e_2 = 0 \quad \text{...(3b)}$$

and $$w_1 e_1 \delta e_1 + w_2 e_2 \delta e_2 + w_3 e_3 \delta e_3 + w_4 e_4 \delta e_4 + w_5 e_5 \delta e_5 = 0 \quad \text{...(4)}$$

Multiply equation (3*a*) by $-\lambda_1$, (3*b*) by $-\lambda_2$ and add these to equation (4). Thus,

$$-\lambda_1 \delta e_1 - \lambda_1 \delta e_2 - \lambda_1 \delta e_3 - \lambda_1 \delta e_4 = 0$$

$$\lambda_2 \delta e_1 + \lambda_2 \delta e_2 - \lambda_2 \delta e_5 = 0$$

$$w_1 e_1 \delta e_1 + w_2 e_2 \delta e_2 + w_3 e_3 \delta e_3 + w_4 e_4 \delta e_4 + w_5 e_5 \delta e_5 = 0$$

$$\delta e_1(w_1 e_1 - \lambda_1 + \lambda_2) + \delta e_2(w_2 e_2 - \lambda_1 + \lambda_2) + \delta e_3(w_3 e_3 - \lambda_1) + \delta e_4(w_4 e_4 - \lambda_1) + \delta e_5 (w_5 e_5 - \lambda_2) = 0 \quad \text{...(5)}$$

Since the coefficients of δe_1, δe_2, δe_3, δe_4 and δe_5 must vanish independently, we have

$$\left.\begin{aligned} w_1 e_1 - \lambda_1 + \lambda_2 &= 0 \quad \text{or} \quad e_1 = \frac{\lambda_1 - \lambda_2}{w_1} \\ w_2 e_2 - \lambda_1 + \lambda_2 &= 0 \quad \text{or} \quad e_2 = \frac{\lambda_1 - \lambda_2}{w_2} \\ w_3 e_3 - \lambda_1 &= 0 \quad \text{or} \quad e_3 = \frac{\lambda_1}{w_3} \\ w_4 e_4 - \lambda_1 &= 0 \quad \text{or} \quad e_4 = \frac{\lambda_1}{w_4} \\ \text{and} \quad w_5 e_5 - \lambda_1 &= 0 \quad \text{or} \quad e_5 = \frac{\lambda_1}{w_5} \end{aligned}\right\} \quad \text{...(6)}$$

Substituting these values in (1*a*) and (1*b*), we get

$$\frac{\lambda_1}{w_1} - \frac{\lambda_2}{w_1} + \frac{\lambda_1}{w_2} - \frac{\lambda_2}{w_2} + \frac{\lambda_1}{w_3} + \frac{\lambda_1}{w_4} = E$$

or $$\lambda_1\left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3} + \frac{1}{w_4}\right) - \lambda_2\left(\frac{1}{w_1} + \frac{1}{w_2}\right) = E \quad \text{...(7a)}$$

and $$\frac{\lambda_2}{w_5} - \frac{\lambda_1 - \lambda_2}{w_1} - \frac{\lambda_1 - \lambda_2}{w_2} = E'$$

or $$-\lambda_1\left(\frac{1}{w_1} + \frac{1}{w_2}\right) + \lambda_2\left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_5}\right) = E' \quad \text{...(7b)}$$

Since w_1, w_2, w_3, w_4, w_5, E' and E are all known, λ_1 and λ_2 can be calculated. These values can then be substituted in Eq. (6) to get the corrections e_1, e_2, e_3, e_4 etc.

Example 9.16. *Solve example 9.13 by method of correlates.*

Solution.

The observed equations are

$$A = 68° 12' 36'' \ ; \quad B = 53° 46' 12'' \ ; \quad C = 58° 01' 16''$$

The conditions equation is $A + B + C = 180°$

Now $A + B + C = 180° 00' 04''$

Hence $E = 180° - (A + B + C) = 180° - (180° 0' 4'') = -4'' =$ total correction

Let e_1, e_2 and e_3 be corrections to the angles A, B and C.

Hence, we have the condition equation

$$e_1 + e_2 + e_3 = -4'' \quad \text{...(1)}$$

Also, from least squares condition, $\Sigma we^2 = 0$

Since all the observations are of equal weight, we have

$$e_1^2 + e_2^2 + e_3^2 = 0 \quad \text{...(2)}$$

Differentiating (1) and (2), we get

$$\delta e_1 + \delta e_2 + \delta e_3 = 0 \quad \text{...(3)}$$

and $$e_1 \delta e_1 + e_2 \delta e_2 + e_3 \delta e_3 = 0 \quad \text{...(4)}$$

Multiplying Eq. (3) by $-\lambda$ and adding to (4), we get

$$-\lambda \delta e_1 - \lambda \delta e_2 - \lambda \delta e_3 = 0$$

$$e_1 \delta e_1 + e_2 \delta e_2 + e_3 \delta e_3 = 0$$

$$\therefore \quad \delta e_1 (-\lambda + e_2) + \delta e_2(-\lambda + e_2) + \delta e_3(-\lambda + e_3) = 0 \quad \text{...(5)}$$

Since the coefficient of δe_1, δe_2 and δe_3 must vanish independently, we have

$$\lambda = e_1 = e_2 = e_3 \quad \text{...(6)}$$

Substituting these values of e_1, e_2, e_3 in (1), we get

$$\lambda + \lambda + \lambda = -4 \quad \text{or} \quad \lambda = -\frac{4}{3} = -1''.33 = e_1 = e_2 = e_3$$

This shows that for the observations of equal weight, the error is distributed equally to all the angles.

Knowing e_1, e_2 and e_3 the corrected values can be found.

Example 9.17. *Solve example 9.14 by method of correlates.*

Solution.

The observed angles are :

$A = 77° 14' 20''$ wt. 4
$B = 49° 40' 35''$ wt. 3
$C = 53° 04' 52''$ wt. 2

Sum = 179° 59′ 47″

Hence total correction to be applied = 180° − (179° 59′ 47″) = + 13″

Let e_1, e_2 and e_3 be the corrections

$$\therefore \quad e_1 + e_2 + e_3 = +13'' \quad \text{...(1)}$$

From the least square condition, Σwe^2 = a minimum

$\therefore \quad 4e_1^2 + 3e_2^2 + 2e_3^2$ = a minimum ...(2)

Differentiating (1) and (2), we get

$$\delta e_1 + \delta e_2 + \delta e_3 = 0 \quad ...(3)$$

and $\quad 4\,e_1\,\delta e_1 + 3\,e_2\,\delta e_2 + 2\,e_3\,\delta e_3 = 0$ ...(4)

Multiplying (3) by $-\lambda$ and adding it to (4), we get

$$\delta e_1(4e_1 - \lambda) + \delta e_2(3e_2 - \lambda) + \delta e_3(2e_3 - \lambda) = 0 \quad ...(5)$$

Since the coefficients of $\delta e_1, \delta e_2, \delta e_3$ must vanish independently, we have

$$4\,e_1 - \lambda = 0 \quad \text{or} \quad e_1 = \frac{\lambda}{4}$$
$$3\,e_2 - \lambda = 0 \quad \text{or} \quad e_2 = \frac{\lambda}{3}$$
$$2\,e_3 - \lambda = 0 \quad \text{or} \quad e_3 = \frac{\lambda}{2}$$

Substituting these values of e_1, e_2 and e_3 in (1), we get

$$\frac{\lambda}{4} + \frac{\lambda}{3} + \frac{\lambda}{2} = 13'' \quad \text{or} \quad \lambda\left(\frac{13}{12}\right) = 13''$$

or $\quad \lambda = +12''$ and $e_1 = \frac{\lambda}{4} = \frac{12}{4} = +3''$

$$e_2 = \frac{\lambda}{3} = \frac{12}{3} = +4'' \quad \text{and} \quad e_3 = \frac{\lambda}{2} = \frac{12}{2} = +6''$$

Hence the corrected angles are

$A = 77°\ 14'\ 20'' + 3'' =$ **77° 14′ 23″**

$B = 49°\ 40'\ 35'' + 4'' =$ **49° 40′ 39″**

$C = 53°\ \ 4'\ 52'' + 6'' =$ **53° 4′ 58″**.

Note. This example was solved by the two methods of normal equations. It can be seen that the method of correlates applied above to solution of the same problem is very much easier since the computations are very much reduced.

Example 9.18. *Solve example 9.15 by method of correlates.*

Solution.

AOB =	85° 42′ 28″.75	wt. 3
BOC =	102° 15′ 43″.26	wt. 2
COD =	94° 38′ 27″.22	wt. 4
DOA =	79° 23′ 24″.77	wt. 2
Sum =	360° 00′ 03″.00	

Hence, the total correction $E = 360° - (360°\ 0'\ 3'') = -3''$.

Let e_1, e_2, e_3 and e_4 be the individual corrections to the four angles respectively. Then, by the condition equation, we get

$$e_1 + e_2 + e_3 + e_4 = -3'' \quad ...(1)$$

Also, from the least square principle, $\Sigma(we^2)$ = a minimum

Hence $3e_1^2 + 2e_2^2 + 4e_3^2 + 2e_4^2$ = a minimum ...(2)

Differentiating (1) and (2), we get

$$\delta e_1 + \delta e_2 + \delta e_3 + \delta e_4 = 0 \quad ...(3)$$

and $\quad 3\,e_1\,\delta e_1 + 2\,e_2\,\delta e_2 + 4\,e_3\,\delta e_3 + 2\,e_4\,\delta e_4 = 0$ ...(4)

Multiplying equation (3) by $-\lambda$ and adding it to (4), we get

$$\delta e_1(3e_1 - \lambda) + \delta e_2(2e_2 - \lambda) + \delta e_3(4e_3 - \lambda) + \delta e_4(2e_4 - \lambda) = 0 \quad ...(5)$$

Since the coefficients of $\delta e_1, \delta e_2, \delta e_3$ and δe_4 must vanish independently, we have

$$3\,e_1 - \lambda = 0 \quad \text{or} \quad e_1 = \frac{\lambda}{3}$$
$$2\,e_2 - \lambda = 0 \quad \text{or} \quad e_2 = \frac{\lambda}{2}$$
$$4\,e_3 - \lambda = 0 \quad \text{or} \quad e_3 = \frac{\lambda}{4}$$
$$2\,e_4 - \lambda = 0 \quad \text{or} \quad e_4 = \frac{\lambda}{2} \quad ...(6)$$

Substituting these values in (1), we get

$$\frac{\lambda}{3} + \frac{\lambda}{2} + \frac{\lambda}{4} + \frac{\lambda}{2} = -3'' \quad \text{or} \quad \lambda\left(\frac{19}{12}\right) = -3'' \quad \text{or} \quad \lambda = -\frac{3 \times 12}{19}$$

Hence

$$e_1 = -\frac{1}{3}\cdot\frac{3 \times 12}{19} = -\frac{12}{19} = -0.63''$$
$$e_2 = -\frac{1}{2}\cdot\frac{3 \times 12}{19} = -\frac{18}{19} = -0.95''$$
$$e_3 = -\frac{1}{4}\cdot\frac{3 \times 12}{19} = -\frac{9}{19} = -0.47''$$
$$e_4 = -\frac{1}{2}\cdot\frac{3 \times 12}{19} = -\frac{18}{19} = -0.95''$$

Sum = −3.0″

Hence the corrected angles

AOB =	83° 42′ 28″.75 − 0″.63 =	83° 42′ 28″.12
BOC =	102° 15′ 43″.26 − 0″.95 =	102° 15′ 42″.31
COD =	94° 38′ 27″.22 − 0″.47 =	94° 38′ 26″.75
DOA =	79° 23′ 23″.77 − 0″.95 =	79° 23′ 22″.82
Sum =		360° 00′ 00″.00

This example was also solved by the method of differences of normal equations. The method of correlates is however, much more easier.

Example 9.19. *The following round of angles was observed from central station to the surrounding stations of a triangulation survey :*

$A = 93° 43' 22''$ *weight 3*

$B = 74° 32' 39''$ *weight 2*

$C = 101° 13' 44''$ *weight 2*

$D = 90° 29' 50''$ *weight 3*

In addition, one angle $\overline{(A+B)}$ *was measured separately as combined angle with a mean value of* $168° 16' 06''$ *(wt. 2).*

Determine the most probable values of the angles A, B, C and D.

Solution.

$A + B + C + D = 359° \; 59' \; 35''$

Total correction $E = 360° - (359° \; 59' \; 35'') = +25''$

Similarly, $\overline{(A+B)} = (A+B)$

Hence correction $E' = A + B - \overline{(A+B)} = 168° \; 16' \; 01'' - 168° \; 16' \; 06'' = -5''$

Let e_1, e_2, e_3, e_4 and e_5 be the correction to A, B, C, D and $\overline{(A+B)}$

We have, then, the condition equations

$$e_1 + e_2 + e_3 + e_4 = +25'' \quad \ldots(1a)$$

$$e_5 - e_1 - e_2 = -5'' \quad \ldots(1b)$$

Also, from the least square condition, $\Sigma(we^2) =$ a minimum

Hence $3e_1^2 + 2e_2^2 + 2e_3^2 + 3e_4^2 + 2e_5^2 =$ a minimum ...(2)

Differentiating (1*a*), (1*b*) and (2), we get

$$\delta e_1 + \delta e_2 + \delta e_3 + \delta e_4 = 0 \quad \ldots(3a)$$

$$\delta e_5 - \delta e_1 - \delta e_2 = 0 \quad \ldots(3b)$$

$$3e_1\,\delta e_1 + 2e_2\,\delta e_2 + 2\,e_3\,\delta e_3 + 3e_4\,\delta e_4 + 2e_5\,\delta e_5 = 0 \quad \ldots(4)$$

Multiplying (3*a*) by $-\lambda_1$, (3*b*) by $-\lambda_2$ and adding to (3), we get

$$\delta e_1(-\lambda_1 + \lambda_2 + 3\,e_1) + \delta e_2(-\lambda_1 + \lambda_2 + 2\,e_2) + \delta e_3(-\lambda_1 + 2\,e_3) + \delta e_4(-\lambda_1 + 3\,e_4) + \delta e_5(-\lambda_2 + 2\,e_5) = 0 \quad \ldots(5)$$

Since the co-efficients of δe_1, δe_2, δe_3 etc. must vanish independently, we get

$$\left.\begin{aligned} -\lambda_1 + \lambda_2 + 3\,e_1 &= 0 \quad \text{or} \quad e_1 = \frac{\lambda_1}{3} - \frac{\lambda_2}{3} \\ -\lambda_1 + \lambda_2 + 2\,e_2 &= 0 \quad \text{or} \quad e_2 = \frac{\lambda_1}{2} - \frac{\lambda_2}{2} \\ -\lambda_2 + 2e_3 &= 0 \quad \text{or} \quad e_3 = \frac{\lambda_1}{2} \\ -\lambda_1 + 3\,e_4 &= 0 \quad \text{or} \quad e_4 = \frac{\lambda_1}{3} \\ -\lambda_2 + 2\,e_5 &= 0 \quad \text{or} \quad e_5 = \frac{\lambda_2}{2} \end{aligned}\right\} \quad \ldots(6)$$

Substituting these values of e_1, e_2, e_3, e_4 and e_5 in Eqs. (1*a*) and (1*b*) ; we get

$$\frac{\lambda_1}{3} - \frac{\lambda_2}{3} + \frac{\lambda_1}{2} - \frac{\lambda_2}{2} + \frac{\lambda_1}{2} + \frac{\lambda_1}{3} = 25 \quad \text{from (1}a\text{)}$$

or

$$\frac{5}{3}\lambda_1 - \frac{5}{6}\lambda_2 = 25$$

or

$$\frac{\lambda_1}{3} - \frac{\lambda_2}{6} = 5 \quad \ldots(\text{I})$$

and

$$\frac{\lambda_2}{2} - \frac{\lambda_1}{3} + \frac{\lambda_2}{3} - \frac{\lambda_1}{2} + \frac{\lambda_2}{2} = -5 \quad \text{from (1 } b\text{)}$$

or

$$\frac{4}{3}\lambda_2 - \frac{5}{6}\lambda_1 = -5 \quad \ldots(\text{II})$$

Solving (I) and (II) simultaneously, we get

$$\lambda_1 = +\frac{210}{11}$$

and

$$\lambda_2 = +\frac{90}{11}$$

Hence

$$e_1 = \frac{1}{3}\cdot\frac{210}{11} - \frac{1}{3}\cdot\frac{90}{11} = +\frac{40''}{11} = +3''.64$$

$$e_2 = \frac{1}{2}\cdot\frac{210}{11} - \frac{1}{2}\cdot\frac{90}{11} = +\frac{60}{11} = +5''.45$$

$$e_3 = \frac{1}{2}\cdot\frac{210}{11} = +\frac{105}{11} = +9''.55$$

$$e_4 = \frac{1}{3}\cdot\frac{210}{11} = +\frac{70}{11} = +6''.36$$

Total $= +25''.00$

Also

$$e_5 = \frac{1}{2}\cdot\frac{90}{11} = 4''.09$$

Hence the corrected angles are

$A = 93° \; 43' \; 22'' + 3''.64 = 93° \; 43' \; 25''.64$

$B = 74° \; 32' \; 39'' + 5''.45 = 74° \; 32' \; 44''.45$

$C = 101° \; 13' \; 44'' + 9''.55 = 101° \; 13' \; 53''.55$

$D = 90° \; 29' \; 50'' + 6''.36 = 90° \; 29' \; 56''.36$

Sum $= 360° \; 00' \; 00''.00$

Example 9.20. *A surveyor carried out levelling operations of a closed circuit ABCDA starting from A and made the following observations :*

B was 8.164 m above A, weight 2

C was 6.284 m above B, weight 2

D was 5.626 m above C, weight 3

and *D was 19.964 m above A, weight 3*

Determine the probable heights of B, C and D above A by method of correlates.

Solution.

Error of closure = (8.164 + 6.284 + 5.626) − 19.964 = 20.074 − 19.964 = 0.11 m

Total correction = − 0.11 m

Let e_1, e_2, e_3 and e_4 be the corrections to the observed quantities taken in order. Hence we have condition equation :

$$e_1 + e_2 + e_3 + e_4 = -0.11 \text{ m} \qquad \ldots(1)$$

Also, from least square condition, $\Sigma(we^2)$ = a minimum

$$2e_1^2 + 2e_2^2 + 3e_3^2 + 3e_4^2 = \text{a minimum} \qquad \ldots(2)$$

Differentiating (1) and (2), we get

$$\delta e_1 + \delta e_2 + \delta e_3 + \delta e_4 = 0 \qquad \ldots(3)$$

and $$2e_1\,\delta e_1 + 2e_2\,\delta e_2 + 3e_3\,\delta e_3 + 3e_4\,\delta e_4 = 0 \qquad \ldots(4)$$

Multiplying equation (3) by $-\lambda$ and adding it to (4), we get

$$\delta e_1(2e_1 - \lambda) + \delta e_2(2e_2 - \lambda) + \delta e_3(3e_3 - \lambda) + \delta e_4(3e_4 - \lambda) = 0$$

Since the co-efficients of δe_1, δe_2, δe_3 and δe_4 must vanish independently, we get

$$2e_1 - \lambda = 0 \quad \text{or} \quad e_1 = \frac{\lambda}{2}$$

$$2e_2 - \lambda = 0 \quad \text{or} \quad e_2 = \frac{\lambda}{2}$$

$$3e_3 - \lambda = 0 \quad \text{or} \quad e_3 = \frac{\lambda}{3}$$

$$3e_4 - \lambda = 0 \quad \text{or} \quad e_4 = \frac{\lambda}{3}$$

Substituting the values of e_1, e_2, e_3 and e_4 in (1), we get

$$\frac{\lambda}{2} + \frac{\lambda}{2} + \frac{\lambda}{3} + \frac{\lambda}{3} = -0.11$$

or $$\lambda = -0.11 \times \frac{3}{5} = -0.066 \text{ m}$$

Hence $$e_1 = \frac{\lambda}{2} = -0.033 \text{ m}$$

$$e_2 = \frac{\lambda}{2} = -0.033 \text{ m}$$

$$e_3 = \frac{\lambda}{3} = -0.022 \text{ m}$$

$$e_4 = \frac{\lambda}{3} = -0.022 \text{ m}$$

Total = − 0.110 m

Hence the corrected levels are

B = 8.164 − 0.033 = 8.131 above A

C = 6.284 − 0.033 = 6.251 above B = 14.382 above A

D = 5.626 − 0.022 = 5.604 above C = 19.986 above A

Check : Level of A above D = − 19.964 − 0.022 = − 19.986

9.12. TRIANGULATION ADJUSTMENTS

In a triangulation system, all the measured angles should be corrected so that they satisfy :

(*i*) Conditions imposed by the station of observation, known as the *station adjustment;* and

(*ii*) Conditions imposed by the figure, known as the *figure adjustment*.

The most accurate method is that of least squares, and the most rigid application follows when the entire system is adjusted in one mass, all the angles being simultaneously involved. The process is exceedingly laborious, even in nets comprising few figures. As such, it is always convenient to break it into three parts which are each adjusted separately.

(*i*) Single angle adjustment. (*ii*) Station adjustment.

and (*iii*) Figure adjustment.

(1) Single Angle Adjustment

Generally, several observations are taken for a single angle. The corrections to be applied are inversely proportional to the weight and directly proportional to the square of probable errors. In the case of the measurement of the angle with equal weights, the most probable value is equal to the arithmetic mean of the observations. In the case of the weighted observations, the most probable value of the angle is equal to the weighted arithmetic mean of the observed angles. See examples 9.2, 9.3, 9.4 and 9.5.

(2) Station Adjustment

Station adjustment is the determination of the most probable values of two or more angles measured at a station so as to satisfy the condition of being geometrically consistent. There are three cases of station adjustment :

(*i*) when the horizon is closed with angles of equal weights

(*ii*) when the horizon is closed with angles from unequal weights

(*iii*) when several angles are measured at a station individually, and in combination.

Case 1. When the horizon is closed with angles of equal weights.

In Fig. 9.3, angles A, B and C have been measured and the horizon is closed. Hence $A + B + C$ should be equal to 360°. If this condition is not satisfied, the error is distributed *equally* to all the three angles.

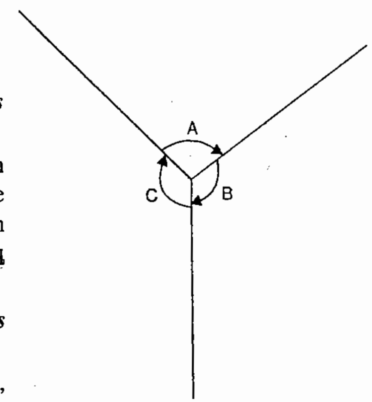


FIG. 9.3

Case 2. When the horizon is closed with angles of unequal weights.

If the angles observed are of unequal weight, discrepancy is distributed among the angles inversely as the respective weights.

Case 3. When the several angles are measured at a station individually and also in combination.

In Fig. 9.4, the three angles A, B and C are measured individually. Also the summation angles $A + B$ and $A + B + C$ have been measured. As discused earlier, the most probable value of the angles can be found by forming the normal equations for the unknowns and solving them simultaneously. See example 9.9, 9.10, 9.11, 9.21 and 9.22.

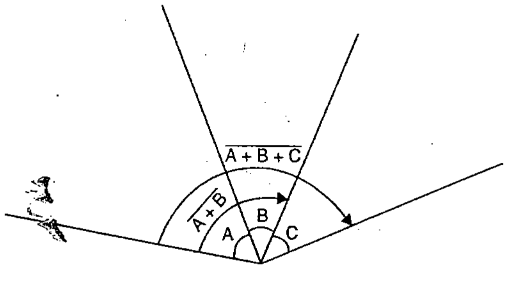


FIG. 9.4.

Example 9.21. *Given the following equations*

$A = 42°\,36'\,28''$	*wt.* 2
$B = 28°\,12'\,42''$	*wt.* 2
$C = 65°\,25'\,16''$	*wt.* 1
$A + B = 70°\,49'\,14''$	*wt.* 2
$B + C = 93°\,37'\,55''$	*wt.* 1

Find the most probable values of A, B and C.

Solution.

Let k_1, k_2, k_3 be the most probable corrections to A, B and C. Then the most probable values of A, B and C are

$$A = 42°\,36'\,28'' + k_1 \quad ...(1)$$

$$B = 28°\,12'\,42'' + k_2 \quad ...(2)$$

$$C = 65°\,25'\,16'' + k_3 \quad ...(3)$$

$$A + B = 70°\,49'\,10'' + k_1 + k_2 \text{ by adding (1) and (2)} \quad ...(4)$$

$$B + C = 93°\,37'\,58'' + k_2 + k_3 \text{ by adding (2) and (3)} \quad ...(5)$$

Substituting these in the corresponding observation equations, we get the following *reduced observation equations* :

$k_1 = 0$	weight 2
$k_2 = 0$	weight 2
$k_3 = 0$	weight 1
$k_1 + k_2 = +4''$	weight 2
$k_2 + k_3 = -3''$	weight 1

Normal equation for k_1 :

$$2k_1 = 0$$
$$2k_1 + 2k_2 = +8$$
$$\overline{4k_1 + 2k_2 = +8}$$

Normal equation for k_2 :

$$2k_2 = 0$$
$$2k_1 + 2k_2 = +8$$
$$k_2 + k_3 = -3$$

$$\therefore \quad 2k_1 + 5k_2 + k_3 = +5$$

Normal equation for k_3

$$k_3 = 0$$
$$k_2 + k_3 = -3$$
$$\overline{k_2 + 2k_3 = -3}$$

Hence the three normal equations are

$$4k_1 + 2k_2 = +8 \quad ...(1a)$$

$$2k_1 + 5k_2 + k_3 = +5 \quad ...(2a)$$

$$k_2 + 2k_3 = -3 \quad ...(3a)$$

Solving these simultaneously for k_1, k_2 and k_3, we get

$$k_1 = +1''.93$$

$$k_2 = +0''.14$$

$$k_3 = -1''.57$$

Hence the most probable values of the angles are

$$A = 42°\,36'\,28'' + 1''.93 = \mathbf{42°\,36'\,29''.93}$$

$$B = 28°\,12'\,42'' + 0''.14 = \mathbf{28°\,12'\,42''.14}$$

$$C = 65°\,25'\,16'' - 1''.57 = \mathbf{65°\,25'\,14''.43}$$

Example 9.22. *Find the most probable values of the angles A, B and C from the following observations at a station P* :

$A = 38°\,25'\,20''$	*wt.* 1
$B = 32°\,36'\,12''$	*wt.* 1
$A + B = 71°\,01'\,29''$	*wt.* 2
$A + B + C = 119°\,10'\,43''$	*wt.* 1
$B + C = 80°\,45'\,28''$	*wt.* 2

Solution.

Let k_1, k_2 and k_3 be the corrections to the angle A, B and C. Here angle C has not been observed directly.

Assume $\quad C = (B + C) - B$

$$= 80°\,45'\,28'' - 32°\,36'\,12''$$
$$= 48°\,9'\,16''$$

Hence the most probable values of the angles are

$$A = 38° 25' 20'' + k_1 \quad ...(1)$$
$$B = 32° 36' 12'' + k_2 \quad ...(2)$$
$$C = 48° \ 9' 16'' + k_3 \quad ...(3)$$
$$A + B = 71° 01' 32'' + k_1 + k_2 \text{ by adding (1) and (2).} \quad ...(4)$$
$$A + B + C = 119° 10' 48'' + k_1 + k_2 + k_3 \text{ by adding (1), (2) and (3).} \quad ...(5)$$
$$B + C = 80° 45' 28'' + k_2 + k_3 \text{ by adding (2) and (3).} \quad (6)$$

Substituting these values in the observation equations, we get

k_1	$= 0$	weight 1
k_2	$= 0$	weight 1
$k_1 + k_2$	$= -3$	weight 2
$k_1 + k_2 + k_3$	$= -5$	weight 1
$k_2 + k_3$	$= 0$	weight 2

Normal equation for k_1 :

$$k_1 = 0$$
$$2k_1 + 2k_2 = -6$$
$$k_1 + k_2 + k_3 = -5$$

$$\therefore \quad 4k_1 + 3k_2 + k_3 = -11$$

Normal equation for k_2 :

$$k_2 = 0$$
$$2k_1 + 2k_2 = -6$$
$$k_1 + k_2 + k_3 = -5$$
$$2k_2 + 2k_3 = 0$$

$$\therefore \quad 3k_1 + 6k_2 + 3k_3 = -11$$

Normal equation for k_3 :

$$k_1 + k_2 + k_3 = -5$$
$$2k_2 + 2k_3 = 0$$

$$k_1 + 3k_2 + 3k_3 = -5$$

Hence the three normal equations are

$$4k_1 + 3k_2 + k_3 = -11 \quad ...(I)$$
$$3k_1 + 6k_2 + 3k_3 = -11 \quad ...(II)$$
$$k_1 + 3k_2 + 3k_3 = -5 \quad ...(III)$$

Solving these simultaneously for k_1, k_2 and k_3 we get

$$k_1 = -2''.29$$
$$k_2 = -1''.24$$
$$k_3 = +1''.88$$

Hence the most probable values of the angles are

$$A = 38° 25' 20'' - 2''.29 = \mathbf{38° 25'' 17''.71}$$
$$B = 32° 36' 12'' - 1''.24 = \mathbf{32° 36' 10''.76}$$
$$C = 48° \ 9' 16'' + 1''.88 = \mathbf{48° 9' 17''.88}$$

9.13. FIGURE ADJUSTMENT

The determination of the most probable values of the angles involved in any geometrical figure so as to fulfil geometrical conditions is called the figure adjustment. The figure adjustment, therefore, involves one or more condition equations. We have already discussed the simple cases of condition equations by the method of normal equations and also by the method of correlates. When the condition equations are more, the method of correlates is much simpler.

The triangulation system mainly consists of the following geometrical figures :

(*i*) triangles

(*ii*) quadrilaterals

(*iii*) polygons with central figure.

We shall discuss the adjustments of all the three figures separately in detail.

9.14. ADJUSTMENT OF A GEODETIC TRIANGLE

A triangle is the basic figure of any triangulation system. All the three angles of a triangle are to corrected. The following are the *general rules* for applying the corrections to the observed angles.

Let A, B and C be the observed angles

e_1, e_2 and e_3 be the corresponding corrections

e = the total correction (equal to the discrepancy)

n_1, n_2 and n_3 = number of observations for angles A, B and C respectively

w_1, w_2 and w_3 = relative weights for A, B and C

E_1, E_2 and E_3 = probable error of A, B and C.

Rule 1. Equal corrections. ***If all the angles are of equal weight, the discrepancy is distributed equally to all the three angles.***

i.e.,
$$e_1 = e_2 = e_3 = \frac{1}{3} e$$

Rule 2. Inverse weight corrections. ***If all the angles are of unequal weight, the discrepancy is distributed to all the angles in inverse proportion to the weights.***

i.e.,
$$e_1 : e_2 : e_3 = \frac{1}{w_1} : \frac{1}{w_2} : \frac{1}{w_3}$$

Hence
$$e_1 = \frac{\frac{1}{w_1}}{\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3}} e$$

$$e_2 = \frac{\frac{1}{w_2}}{\frac{1}{w_1}+\frac{1}{w_2}+\frac{1}{w_3}}\, e \quad \text{and} \quad e_3 = \frac{\frac{1}{w_3}}{\frac{1}{w_1}+\frac{1}{w_2}+\frac{1}{w_3}}\, e$$

Rule 3. Inverse corrections. *If the weights of observations are not given, the discrepancy is distributed to all the three angles in inverse proportion to their number of observations.*

i.e., $e_1 : e_2 : e_3 = \frac{1}{n_1} : \frac{1}{n_2} : \frac{1}{n_3}$

Hence $e_1 = \frac{\frac{1}{n_1}}{\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}}\, e$; $e_2 = \frac{\frac{1}{n_2}}{\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}}\, e$ and $e_3 = \frac{\frac{1}{n_3}}{\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}}\, e$

Rule 4. Inverse square correction. *The discrepancy is distributed to all the angles in inverse proportion to the square of the number of observations.*

i.e., $e_1 : e_2 : e_3 = \left(\frac{1}{n_1}\right)^2 : \left(\frac{1}{n_2}\right)^2 : \left(\frac{1}{n_3}\right)^2$

Hence $$e_1 = \frac{(1/n_1)^2}{\left(\frac{1}{n_1}\right)^2+\left(\frac{1}{n_2}\right)^2+\left(\frac{1}{n_3}\right)^2}\cdot e$$

$$e_2 = \frac{(1/n_2)^2}{\left(\frac{1}{n_1}\right)^2+\left(\frac{1}{n_2}\right)^2+\left(\frac{1}{n_3}\right)^2}\cdot e$$

and $$e_3 = \frac{(1/n_3)^2}{\left(\frac{1}{n_1}\right)^2+\left(\frac{1}{n_2}\right)^2+\left(\frac{1}{n_3}\right)^2}\cdot e$$

There is little mathematical justification for this rule.

Rule 5. Probable error square corrections. *If the probable errors of each angle are known, the discrepancy is distributed to all the angles in direct proportion to the squares of the probable errors.*

i.e. $e_1 : e_2 : e_3 = E_1^2 : E_2^2 : E_3^2$

Hence $e_1 = \frac{E_1^2}{E_1^2+E_2^2+E_3^2}\cdot e$; $e_2 = \frac{E_2^2}{E_1^2+E_2^2+E_3^2}\cdot e$

and $e_3 = \frac{E_3^2}{E_1^2+E_2^2+E_3^2}\cdot e$

Rule 6. Gauss's Rule. This rule is applied when the weights of the observations are not directly known. *If the residual error of each observation is known, the weights can be calculated by the Gauss's rule given by the expression :*

$$w = \frac{\frac{1}{2}n^2}{\Sigma v^2}$$

where w is the weight to the assigned to a quantity.

n = total number of observations made for the quantity.

Σv^2 = sum of the squares of the residuals.

Hence $w_1 = \frac{\frac{1}{2}n_1^2}{\Sigma v_1^2}$ or $\frac{1}{w_1} = \frac{\Sigma v_1^2}{\frac{1}{2}n_1^2} = K_1$ (say) ;

$w_2 = \frac{\frac{1}{2}n_2^2}{\Sigma v_2^2}$ or $\frac{1}{w_2} = \frac{\Sigma v_2^2}{\frac{1}{2}n_2^2} = K_2$ (say)

and $w_3 = \frac{\frac{1}{2}n_3^2}{\Sigma v_3^2}$ or $\frac{1}{w_3} = \frac{\Sigma v_3^2}{\frac{1}{2}n_3^2} = K_3$ (say)

Knowing the values of the weights, the corrections are applied by rule (2).

$$e_1 : e_2 : e_3 = \frac{1}{w_1} : \frac{1}{w_2} : \frac{1}{w_3} = K_1 : K_2 : K_3$$

Hence $e_1 = \frac{K_1}{K_1+K_2+K_3}\cdot e$; $e_2 = \frac{K_2}{K_1+K_2+K_3}\cdot e$

and $e_3 = \frac{K_3}{K_1+K_2+K_3}\cdot e$

Generally, rules 1, 2 and 6 are the most commonly used for the adjustments of angles.

FIGURE ADJUSTMENT OF A TRIANGLE

A tiangle is a simple figure having three interior angles. If all the three angles are measured independently (as is generally the case), their sum must be equal to 180°in the case of a *plane triangle* or should be equal to (180° + spherical excess) in the case of a spherical triangle. If the sum is not equal to 180° in a plane triangle or equal to (180° + spherical excess) in a spherical triangle, the discrepancy is distributed to all the three angles according to *any one* of the rules stated above. *The corrected angles so found are then used to calculate the other two sides of the triangle if length of one side is known.*

CALCULATION OF SPHERICAL EXCESS

The spherical excess (∈) is the amount by which the sum of the three angles exceeds 180°. Its value depends upon the area of the geodetic triangle, and may be ignored if the length of the sides ae less than 3 km. However, for large triangles, it must be calculated. The value of spherical excess is approximately 1" for every 200 square km.

The spherical excess (∈) can be calculated from the formula:

$$\in \text{ (in seconds)} = \frac{A}{R^2 \sin 1''} = \frac{648000\, A}{\pi R^2} \quad \text{...(9.19)}$$

where A = area of the spherical triangle

R = radius of the earth

Both A and R^2 should be substituted in the same units.

If A = area in sq. ft.

$$R = 20889000 \text{ ft.}$$

$$\sin 1'' = \frac{\pi}{180 \times 60 \times 60} = \frac{\pi}{648000}$$

$$\in = \frac{A}{(20889000)^2} \cdot \frac{648000}{\pi} \text{ seconds}$$

If Δ is the area in sq. miles,

$$\in = \frac{\Delta\,(5280)^2}{(20889000)^2} \cdot \frac{648000}{\pi} \text{ seconds}$$

or $$\in = \frac{\Delta}{76} \text{ seconds (approximately)} \quad ...(1)$$

If S = area in square km, we have

$$\in = \frac{S \times 0.386}{76} \quad \text{(since 1 sq. km.} = 0.386 \text{ sq. mile)}$$

or $$\in = \frac{S}{197} \text{ seconds} \quad ...(2)$$

The above expression can also be obtained independently by substituting $R = 6370$ km in Eq. 9.19.

Thus, $$\in = \frac{648000\,S}{\pi\,(6370)^2} = \frac{S}{179} \text{ seconds}$$

Hence, generalising the expression for $\in$ we get

$$\in = \frac{A}{76} \text{ seconds, when } A \text{ is in sq. miles}$$
and $$\in = \frac{A}{197} \text{ seconds, when } A \text{ is in sq. km} \quad ...(9.20)$$

Knowing the spherical excess, the discrepancy in the observed angles is given by

$$e = 180° + \in - (A + B + C) \quad ...(9.21)$$

This discrepancy is to be distributed to the angle (A, B and C) as per rules already discussed.

In the calculation of the spherical excess ($\in$) , the area (S) of the spherical triangle is involved. This area cannot be accurately determined unless the angles are accurately known which, in turn, can be known only if the spherical excess is known. Hence, in the first approximation, the area S is calculated by treating the triangle as a plane triangle, and using the observed angles.

Thus $$S = \frac{1}{2}\,ab.\sin C \quad ...(9.22)$$

or $$= \frac{1}{2}\,c^2 \frac{\sin A \sin B}{\sin C} \quad ...(9.23)$$

where c is the known side and A, B and C are the *observed angles*.

Knowing the area, the spherical excess can be calculated and the corrected angles can be computed.

COMPUTATION OF THE SIDES OF A SPHERICAL TRIANGLE

The sides of the spherical triangle can be calculated by one of the following three methods :

1. By spherical trigonometry
2. By Delambre's method
3. By Legendre's method

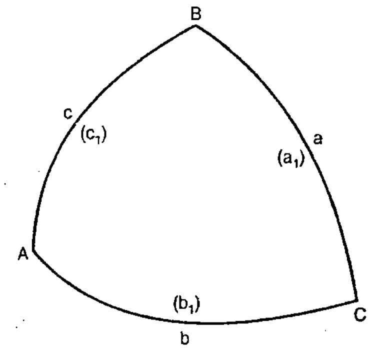


FIG. 9.5.

1. By Spherical Trigonometry

Knowing the length of one side and the three adjusted angles, the lengths of the other two sides can be calculated by the formulae of spherical trigonometry.

Let A, B and C = adjusted angles of the spherical triangle

$a = BC$, $b = AC$ and $c = AB$,

$BC = a$ = known side.

a_1 = angle subtended by side BC at the centre of the sphere

b_1 = angle subtended by side CA at the centre of the sphere

c_1 = angle subtended by side AB at the centre of the sphere

The computations are done in the following steps :

Step 1. Calculate the central angle a_1 of the side $BC\,(= a)$

$$\text{arc} = R \times \text{central angle}$$

or $$\text{central angle} = \frac{\text{arc}}{R}$$

or $$a_1 = \frac{180°\,a}{\pi R}, \text{ where } a_1 \text{ is in degrees}$$

and R is the radius of the earth.

Step 2. Knowing a_1 calculate the central angles b_1 and c_1 by the sine rule.

$$\sin b_1 = \sin a_1 \cdot \frac{\sin B}{\sin A}$$

and $$\sin c_1 = \sin a_1 \cdot \frac{\sin C}{\sin A}$$

Step 3. Knowing the central angles b_1 and c_1, calculate the corresponding lengths of the arcs $CA\,(=b)$ and $AB(=c)$ by the relations

$$b = \frac{\pi R b_1}{180°}$$

and $$c = \frac{\pi R c_1}{180°}$$

2. By Delambre's Method

The method is used when the corrected plane angles are known. The points A, B and C are assumed to be joined by the chord, and the plane angles are the angles between the chords.

In Fig. 9.6, A_0, B_0 and C_0 are the plane angles. The computations are done in the following steps :

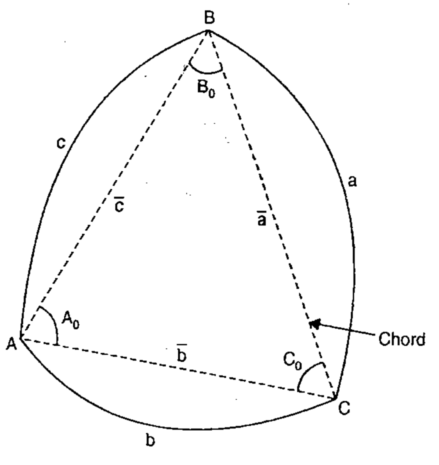


FIG. 9.6

Step 1. Knowing the length BC (= a), calculate the central angle a_1.

Thus, $$a_1 = \frac{180° \, a}{\pi R}.$$

Step 2. Knowing the central angle a_1, the corresponding chord length $\bar{a}$ is calculated from the relation

$$\bar{a} = 2R \sin \frac{a_1}{2}$$

Step 3. From the known chord length $\bar{a}$ and the three corrected plane angles A_0, B_0 and C_0, the other two chord lengths $\bar{b}$ and $\bar{c}$ are computed by the sine rule. Thus

$$\bar{b} = \bar{a}\,\frac{\sin B_0}{\sin A_0} \quad \text{and} \quad \bar{c} = \bar{a}\,\frac{\sin C_0}{\sin A_0}$$

Step 4. Knowing the chord lengths $\bar{b}$ and $\bar{c}$, calculate the corresponding central angles b_1 and c_1 from the relations :

$$\sin \frac{b_1}{2} = \frac{\bar{b}}{2R} \quad \text{and} \quad \sin \frac{c_1}{2} = \frac{\bar{c}}{2R}$$

Step 5. Knowing the central angles b_1 and c_1 the corresponding lengths of the arcs are known by the relations :

$$b = \frac{\pi R b_1}{180°} \quad \text{and} \quad c = \frac{\pi R c_1}{180°}$$

3. By Legendre's Method

Legendre's theorem : The following is the statement of the Legendre's theorem : ***" In any spherical triangle, the sides of which are small compared with the radius of the sphere, if each of the angles be diminished by one-third of the spherical excess, the sines of these angles will be proportional to the lengths of the opposite sides and the triangle may be calculated as if it were plane."***

Let A_0, B_0 and C_0 be the corrected plane angles.

Thus, $$b = a\,\frac{\sin B_0}{\sin A_0}$$

and $$c = a\,\frac{\sin C_0}{\sin A_0}$$

Example 9.23. *Adjust the following angles of the triangle ABC:*

$A = 56° \, 12' \, 36''$	$B = 68° \, 36' \, 12''$	$C = 55° \, 11' \, 14''$
$32''$	$14''$	$18''$
$34''$	$16''$	$12''$
$32''$	$14''$	$15''$
$38''$	$16''$	$16''$
$35''$	$18''$	
	$12''$	
	$14''$	

Solution.

Mean value of $A = 56° \, 12' \, 34''.5$; number of observations = 6

Mean value of $B = 68° \, 36' \, 14''.5$; number of observations = 8

Mean value of $C = 55° \, 11' \, 15''$; number of observations = 5

Sum = $180° \, 00' \, 04''$

$\therefore$ Discrepancy $e = +4''$

Total correction = $-4''$

Weight of any angle $= \dfrac{\frac{1}{2} n^2}{\Sigma v^2}$ where v = (mean − observed)

For A, $\Sigma v^2 = \{(-1.5)^2 + (2.5)^2 + (0.5)^2 + (2.5)^2 + (-3.5)^2 + (-0.5)^2\} = 27.5$; $n = 6$

$\therefore$ $w_1 = $ weight of $A = \dfrac{\frac{1}{2} n_1^2}{\Sigma v_1^2} = \dfrac{\frac{1}{2}(6)^2}{27.5} = \dfrac{18}{27.5}$

$\therefore$ $K_1 = \dfrac{1}{w_1} = \dfrac{27.5}{18} = 1.528$

For B, $\Sigma v^2 = \{(2.5)^2 + (0.5)^2 + (-1.5)^2 + (0.5)^2 + (-1.5)^2 + (-3.5)^2 + (2.5)^2 + (0.5)^2\} = 30$

$n = 8$

$\therefore$ $w_2 = $ weight of $B = \dfrac{\frac{1}{2} n_2^2}{\Sigma v_2^2} = \dfrac{\frac{1}{2}(8)^2}{30} = \dfrac{32}{30}$

$\therefore$ $K_2 = \dfrac{1}{w_2} = \dfrac{30}{32} = 0.937$

For C, $\Sigma v^2 = \{(1)^2 + (-3)^2 + (3)^2 + (0)^2 + (-1)^2\} = 20$; $n = 5$

$\therefore$ $w_3 = $ weight of $C = \dfrac{\frac{1}{2} n_3^2}{\Sigma v_3^2} = \dfrac{\frac{1}{2}(5)^2}{20} = \dfrac{5}{8}$

$\therefore$ $K_3 = \dfrac{1}{w_3} = \dfrac{8}{5} = 1.6$

Hence correction to $$A = \frac{K_1}{K_1 + K_2 + K_3} \cdot e$$

$$= \frac{1.528 \times 4}{1.528 + 0.937 + 1.6} = \frac{1.528 \times 4}{4.065} = 1''.51 \text{ (– ve)}$$

correction to $B = \frac{K_2}{K_1 + K_2 + K_3} \cdot e = \frac{0.937 \times 4}{4.065} = 0''.92$ (–ve)

correction to $C = \frac{K_3}{K_1 + K_2 + K_3} \cdot e = \frac{1.6 \times 4}{4.065} = 1''.57$ (–ve)

Check : Total correction = 1".51 + 0".92 + 1".57 = 4".00

Hence the corrected values of the angles are

$A = 56° 12' 34''.5 - 1''.51 = 56° 12' 32''.99$

$B = 68° 36' 14''.5 - 0''.92 = 68° 36' 13''.58$

$C = 55° 11' 15'' \quad - 1''.57 = 55° 11' 13''.43$

Sum = 180° 00' 00".00

9.15. ADJUSTMENT OF CHAIN OF TRIANGLES

Let us consider a chain of triangles *ABC, ACD, DCE* etc. as shown in Fig. 9.7. The numbers 1, 2, 3 etc. represent the angle numbers and not their values. All the angles have been measured with equal precision. The adjustment is done into two steps :

(*i*) Station adjustment.

(*ii*) Figure adjustment.

(*i*) **Station Adjustment**

Since all the angles at a station have been measured, their sum must be equal to 360°. Hence we get the following condition equations :

$\angle 1 + \angle 2 + \angle 3 = 360°$ ...(1)

$\angle 4 + \angle 5 = 360°$ ...(2)

$\angle 6 + \angle 7 + \angle 8 + \angle 9 = 360°$ ...(3)

$\angle 10 + \angle 11 + \angle 12 = 360°$ ...(4)

$\angle 13 + \angle 14 = 360°$ ...(5)

The discrepancy denoted by each of the angles should be distributed equally to the component angles since all the angles have been measured with equal precision.

(*ii*) **Figure Adjustment**

After having adjusted the individual angles, each triangle is taken separately for figure adjustment. The sum of the three angles in each triangle should be equal to

Thus in triangle *ABC*,

$\angle 1 + \angle 4 + \angle 6 = 180°$ ...(6)

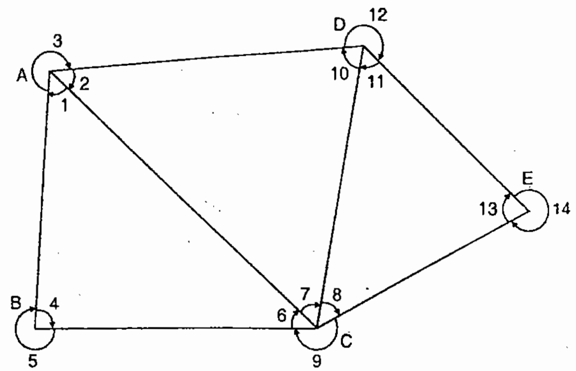


FIG. 9.7.

Thus in triangle *ACD*, $\angle 2 + \angle 7 + \angle 10 = 180°$ ...(7)

Thus in triangle *CDE*, $\angle 8 + \angle 11 + \angle 13 = 180°$ ...(8)

If the angles are of equal weight, the discrepancy is distribued equally to all the three angles. If the angles are weighted, the discrepancy is distributed in inverse proportion to their weights.

9.16. ADJUSTMENT OF TWO CONNECTED TRIANGLES

Fig 9.8 shows two connected triangles *ACD* and *BCD*. The angles A, B, C_1, C_2, D_1 and D_2 have been measured. The summation angles C (= ACB) and D(= ADB) have also been measured. Thus, there are eight angles. There are four independent condition equations that must be satisfied by the adjusted values of the angles. These equations are called the *angle equations* and are as follows :

$\angle A + \angle C_1 + \angle D_1 = 180°$ ...(1)

$\angle B + \angle C_2 + \angle D_2 = 180°$ ...(2)

$\angle C_1 + \angle C_2 = C$ ...(3)

$\angle D_1 + \angle D_2 = D$ ...(4)

There are total eight unknowns, out of which C_1, C_2, D_1 and D_2 must be regarded as the independent unknowns, and the remaining four as the dependent ones since they can be easily obtained from the condition equations.

The solution can be obtained either by means of normal equations or by correlates.

If the method of the normal equation is adopted, the four unknowns A, B, C and D can be expressed in terms of the independent unknowns, C_1, C_2, D_1 and D_2. Thus,

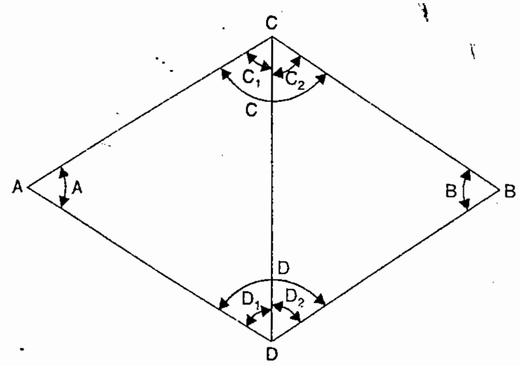


FIG. 9.8. TWO CONNECTED TRIANGLES

$\angle A = 180° - (\angle C_1 + \angle D_1)$ ...(*a*)

$\angle B = 180° - (\angle C_2 + \angle D_2)$ ...(*b*)

$\angle C = \angle C_1 + \angle C_2$ ...(*c*)

and $\angle D = \angle D_1 + \angle D_2$ ...(*d*)

From the new observation equations so formed in terms of C_1, C_2, D_1 and D_2 the normal equations can be formed in terms of the differences (or corrections) and their values can be known.

Example 9.24 illustrates the procedure for the adjustment.

Example 9.24. *The following are the measured values of equal weight for two connected triangles ACD and BCD (Fig. 9.8) :*

A 68° 12' 24"

B 52° 28' 46"

C	$128° 16' 30''$
D	$110° 02' 25''$
C_1	$62° 18' 40''$
C_2	$65° 57' 51''$
D_1	$49° 28' 59''$
D_2	$16° 33' 28''$

Adjust the values of the angles.

Solution.

(a) *By Method of Normal Equations*

The condition equations are

$$A + C_1 + D_1 = 180°$$
$$B + C_2 + D_2 = 180°$$
$$C_1 + C_2 = C$$
$$D_1 + D_2 = D$$

Let A, B, C and D be the dependent quantities. Expressing them in terms of the independent quantities, we get

$$A = 68° 12' 24'' = 180° - (C_1 + D_1)$$

or $$C_1 + D_1 = 180° - 68° 12' 24' = 111° 47' 36''$$

$$B = 52° 28' 46'' = 180° - (C_2 + D_2)$$

or $$C_2 + D_2 = 180° - 52° 28' 46'' = 127° 31' 14''$$

$$C = 128° 16' 30'' = C_1 + C_2$$

and $$D = 111° 02' 25'' = D_1 + D_2$$

Hence the new observation equations are

$$C_1 = 62° 18' 40'' \quad ...(1)$$
$$C_2 = 65° 57' 51'' \quad ...(2)$$
$$D_1 = 49° 28' 59'' \quad ...(3)$$
$$D_2 = 61° 33' 28'' \quad ...(4)$$
$$C_1 + D_1 = 111° 47' 36'' \quad ...(5)$$
$$C_2 + D_2 = 127° 31' 14'' \quad ...(6)$$
$$C_1 + C_2 = 128° 16' 30'' \quad ...(7)$$

and $$D_1 + D_2 = 111° 02' 25'' \quad ...(8)$$

Let k_1, k_2, k_3 and k_4 be the corrections (in seconds) to the angles C_1, C_2, D_1 and D_2 respectively, so that their most probable values are :

$$C_1 = 62° 18' 40'' + k_1 \quad ...(1a)$$
$$C_2 = 65° 57' 51'' + k_2 \quad ...(2a)$$
$$D_1 = 49° 28' 59'' + k_3 \quad ...(3a)$$
$$D_2 = 61° 33' 28'' + k_4 \quad ...(4a)$$
$$C_1 + D_1 = 111° 47' 39'' + k_1 + k_3 \text{ by adding (1a) and (3a)} \quad ...(5a)$$

$$C_2 + D_2 = 127° 31' 19'' + k_2 + k_4 \text{ by adding (2a) and (4a)} \quad ...(6a)$$
$$C_1 + C_2 = 128° 16' 31'' + k_1 + k_2 \text{ by adding (1a) and (2a)} \quad ...(7a)$$
$$D_1 + D_2 = 111° 02' 27'' + k_3 + k_4 \text{ by adding (3a) and (4a)} \quad ...(8a)$$

Substituting these values of Eqs. (1a), (2a), (3a) etc. in the corresponding observations equations (1), (2), (3) etc., we get the following reduced equations :

$$k_1 = 0$$
$$k_2 = 0$$
$$k_3 = 0$$
$$k_4 = 0$$
$$k_1 + k_3 = -3$$
$$k_2 + k_4 = -5$$
$$k_1 + k_2 = -1$$
$$k_3 + k_4 = -2$$

From the above reduced equations, the normal equations for k_1, k_2, k_3 and k_4 can be formulated.

Normal equation for k_1 :

$$k_1 = 0$$
$$k_1 + k_3 = -3$$
$$k_1 + k_2 = -1$$

$$3k_1 + k_2 + k_3 = -4$$

Normal equation for k_2 :

$$k_2 = 0$$
$$k_2 + k_4 = -5$$
$$k_1 + k_2 = -1$$

$$k_1 + 3k_2 + k_4 = -6$$

Normal equation for k_3 :

$$k_3 = 0$$
$$k_1 + k_3 = -3$$
$$k_3 + k_4 = -2$$

$$k_1 + 3k_3 + k_4 = -5$$

Normal equation for k_4 :

$$k_4 = 0$$
$$k_2 + k_4 = -5$$
$$k_3 + k_4 = -2$$

$$k_2 + k_3 + 3k_4 = -7$$

Hence, the normal equations are :

$$3k_1 + k_2 + k_3 = -4 \quad \ldots(i)$$
$$k_1 + 3k_2 + k_4 = -6 \quad \ldots(ii)$$
$$k_1 + 3k_3 + k_4 = -5 \quad \ldots(iii)$$
$$k_2 + k_3 + 3k_4 = -7 \quad \ldots(iv)$$

Solving these equations, we get

$$k_1 = -0''.46$$
$$k_2 = -1''.48$$
$$k_3 = -1''.15$$
$$k_4 = -1''.08$$

Hence, the corrected values of the angles are :

$$C_1 = 62° \ 18' \ 40'' - 0''.46 = 62° \ 18' \ 39''.54$$
$$C_2 = 65° \ 57' \ 51'' - 1''.48 = 65° \ 57' \ 49''.52$$
$$D_1 = 49° \ 28' \ 59'' - 1''.15 = 49° \ 28' \ 57''.85$$
$$D_2 = 61° \ 33' \ 28'' - 1''.08 = 61° \ 33' \ 26''.92$$

Also $A = 180° - (C_1 + D_1) = 180° - 111° \ 47' \ 37''.39 = 68° \ 12' \ 22''.61$

$$B = 180° - (C_2 + D_2) = 180° - 127° \ 31' \ 16''.44 = 52° \ 28' \ 43''.56$$
$$C = C_1 + C_2 = 128° \ 16' \ 29''.06$$
$$D = D_1 + D_2 = 111° \ 02' \ 24''.77$$

Check : Sum $A + B + C + D = 360° \ 00' \ 00''.00$

9.17. ADJUSTMENT OF A GEODETIC QUADRILATERAL

In a godetic quadrilateral, all the eight angles ($\theta_1, \theta_2, \ldots. \theta_8$) shown in Fig. 9.9 are measured *independently*. If the size of the quadrilateral is small, it may be taken to be a plane quadrilateral. However, if the size is large, the spherical excess of each triangle can be calculated separately and a correction of $\frac{1}{3}$ spherical excess may be applied to each angle of the triangles, thus giving the plane angles. There are three methods of adjusting a geodetic quadrilateral

1. Rigorous method of least squares.
2. Approximate method.
3. Method of equal shifts. (See § 9.20)

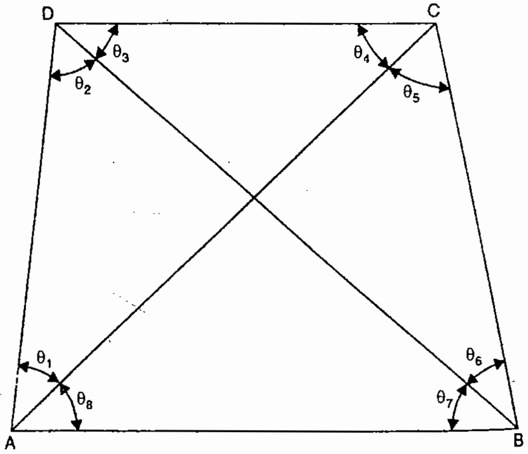


FIG. 9.9. GEODETIC QUADRILATERAL

1. ADJUSTMENT OF QUADRILATERAL BY METHODS OF LEAST SQUARES

In Fig. 9.9, $\theta_1, \theta_2, \theta_3, \ldots. \theta_8$ are the eight corner angles. The theodolite is set up only at the four stations *A, B, C* and *D* and *not* at the intersection of the diagonals. If we imagine to stand at the intersection of the diagonals and see the sides *AD, DC, CB,* and *BA* in turn, then the angles to the left are known as *left angles* and angles to the right are known as *right angles*. Thus $\theta_1, \theta_3, \theta_5$ and θ_7 are the left angles while $\theta_2, \theta_4, \theta_6$ and θ_8 are the right angles.

The conditions to be fulfilled by the adjusted values of the angles are :

(*i*) **Angle equations**

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5 + \theta_6 + \theta_7 + \theta_8 = 360° \quad \ldots(1)$$
$$\theta_1 + \theta_2 = \theta_5 + \theta_6 \quad \ldots(2)$$
$$\theta_3 + \theta_4 = \theta_7 + \theta_8 \quad \ldots(3)$$

(*ii*) **Side equations**

In addition to the three angle equations, a geodetic quadrilateral (or any other figure) must also fulfil the side equation so that the figure is closed. Even if the angle equations are satisfied, the quadrilateral may not be closed as shown in Fig. 9.10, by drawing all the lines parallel to those of Fig. 9.9.

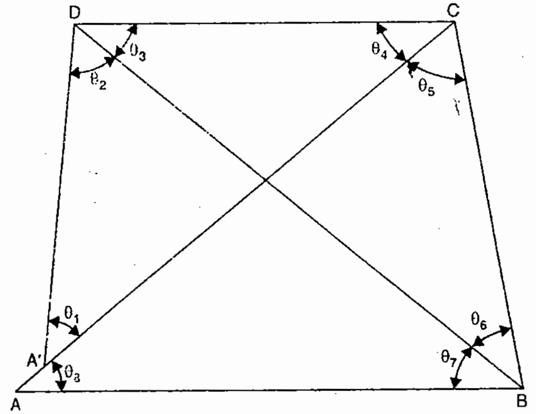


FIG. 9.10. GEODETIC QUADRILATERAL NOT FULFILLING THE SIDE EQUATION.

Let us consider Fig. 9.9 again, which is a closed figure.

From triangle *ADC*,

$$DC = AD \frac{\sin \theta_1}{\sin \theta_4}$$

$$= AB \frac{\sin \theta_7}{\sin \theta_2} \cdot \frac{\sin \theta_1}{\sin \theta_4} \qquad \text{since } AD = AB \frac{\sin \theta_7}{\sin \theta_2}$$

$$= BC \frac{\sin \theta_5}{\sin \theta_8} \cdot \frac{\sin \theta_7}{\sin \theta_2} \cdot \frac{\sin \theta_1}{\sin \theta_4} \qquad \text{since } AB = BC \frac{\sin \theta_5}{\sin \theta_8}$$

$$= DC \frac{\sin \theta_3}{\sin \theta_6} \frac{\sin \theta_5}{\sin \theta_8} \frac{\sin \theta_7}{\sin \theta_2} \frac{\sin \theta_1}{\sin \theta_4}; \qquad \text{since } BC = DC \frac{\sin \theta_3}{\sin \theta_6}$$

Hence $\sin \theta_1 \cdot \sin \theta_3 \cdot \sin \theta_5 \sin \theta_7 = \sin \theta_2 \cdot \sin \theta_4 \cdot \sin \theta_6 \cdot \sin \theta_8$

Taking log of both the sides, and denoting the left angles by θ_L and right angles by θ_R, we get

$$\Sigma \log \sin \theta_L = \Sigma \log \sin \theta_R \ ; \quad \text{or simply} \quad \Sigma \log \sin L = \Sigma \log \sin R \quad \ldots(4) \ \ldots(9.24)$$

where *L* denotes left angles and *R* denotes the right angles.

Thus, we have four condition equations.

Let e_1, e_2, e_3, e_4, e_5, e_6, e_7 and e_8 be the corrections to θ_1, θ_2, θ_3.... , θ_8 respectively.

Also, let E_1, E_2, E_3, E_4 be the discrepancies, *i.e.*, the amount by which the condition equations are not fulfilled. That is

$$E_1 = 360° - (\theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5 + \theta_6 + \theta_7 + \theta_8)$$
$$E_2 = (\theta_5 + \theta_6) - (\theta_1 + \theta_2)$$
$$E_3 = (\theta_7 + \theta_8) - (\theta_3 + \theta_4)$$

and
$$E_4 = \Sigma \log \sin R - \Sigma \log L$$

Then, we have

$$e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8 = E_1 \quad ...(1a)$$
$$e_1 + e_2 - e_5 - e_6 = E_2 \quad ...(1b)$$
$$e_3 + e_4 - e_7 - e_8 = E_3 \quad ...(1c)$$

and
$$e_1 f_1 - e_2 f_2 + e_3 f_3 - e_4 f_4 + e_5 f_5 - e_6 f_6 + e_7 f_7 - e_8 f_8 = E_4 \quad ...(1d)$$

where $f_1, f_2, ..., f_8$ are log sine difference for 1" in the values of θ_1, θ_2 ....., θ_8, to be obtained from log tables.

Also, $e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_5^2 + e_6^2 + e_7^2 + e_8^2$ = a minimum ...(1e)

Differentiating equations (1a), (1b), (1c), (1d), and (1e), we get

$$\delta e_1 + \delta e_2 + \delta e_3 + \delta e_4 + \delta e_5 + \delta e_6 + \delta e_7 + \delta e_8 = 0 \quad ...(2a)$$
$$\delta e_1 + \delta e_2 - \delta e_5 - \delta e_6 = 0 \quad ...(2b)$$
$$\delta e_3 + \delta e_4 - \delta e_7 - \delta e_8 = 0 \quad ...(2c)$$
$$f_1\,\delta e_1 - f_2\,\delta e_2 + f_3\,\delta e_3 - f_4\,\delta e_4 + f_5\,\delta e_5 - f_6\,\delta e_6 + f_7\,\delta e_7 - f_8\,\delta e_8 = 0 \quad ...(2d)$$
$$e_1\,\delta e_1 + e_2\,\delta e_2 + e_3\,\delta e_3 + e_4\,\delta e_4 + e_5\,\delta e_5 + e_6\,\delta e_6 + e_7\delta e_7 + e_8\,\delta e_8 = 0 \quad ...(2e)$$

Multiplying equations (2a), (2b, (2c) and (2d) by $-\lambda_1$, $-\lambda_2$, $-\lambda_3$ and $-\lambda_4$ respectively and adding to Eq. (2e), and equating the co-efficient of δe_1, δ_2, ...δe_8 to zero we get

$$e_1 = \lambda_1 + \lambda_2 + f_1\,\lambda_4 \quad ...(3a)$$
$$e_2 = \lambda_1 + \lambda_2 - f_2\,\lambda_4 \quad ...(3b)$$
$$e_3 = \lambda_1 + \lambda_3 + f_3\,\lambda_4 \quad ...(3c)$$
$$e_4 = \lambda_1 + \lambda_3 - f_4\,\lambda_4 \quad ...(3d)$$
$$e_5 = \lambda_1 - \lambda_2 + f_5\,\lambda_4 \quad ...(3e)$$
$$e_6 = \lambda_1 - \lambda_2 - f_6\,\lambda_4 \quad ...(3f)$$
$$e_7 = \lambda_1 - \lambda_3 + f_7\,\lambda_4 \quad ...(3g)$$

and
$$e_8 = \lambda_1 - \lambda_3 - f_8\,\lambda_4 \quad ...(3h)$$

Substituting the values of $e_1, e_2, e_3,....e_8$ in equations (1a), (1b), (1c) and (1d), we get

$$8\,\lambda_1 + (f_1 - f_2 + f_3 - f_4 + f_5 - f_6 + f_7 - f_8)\,\lambda_4 = E_1 \quad ...(4a)$$
$$4\lambda_2 + \{(f_1 - f_2) - (f_5 - f_6)\}\,\lambda_4 = E_2 \quad ...(4b)$$
$$4\lambda_3 + \{(f_3 - f_4) - (f_7 - f_8)\}\,\lambda_4 = E_3 \quad ...(4c)$$

and
$$\{(f_1 - f_2) + (f_3 - f_4) + (f_5 - f_6) + (f_7 - f_8)\}\,\lambda_1 + \{(f_1 - f_2) - (f_5 - f_6)\}\,\lambda_2 + \{(f_3 - f_4) - (f_7 - f_8)\}\,\lambda_3 + \{f_1^2 + f_2^2 + f_3^2 + f_4^2 + f_5^2 + f_6^2 + f_7^2 + f_8^2\}\,\lambda_4 = E_4 \quad ...(4d)$$

Solving equation (4a), (4b), (4c) and (4d), the values of λ_1, λ_2, λ_3 and λ_4 can be known. Substituting the values of λ_1, λ_2, λ_3 and λ_4 in equations (3a), (3b)...(3h), the correction $e_1, e_2, e_3,..., e_8$ can be calculated, and hence the corrected angles can be found.

The method is illustrated by example 9.25.

Alternative Method of Formulating Equations (3a), (3b) ...(3h)

The equations (3a), (3b)...(3h) above were formed by differentiating each of the five condition equations, multiplying the first four by $-\lambda_1$, $-\lambda_2$, $-\lambda_3$ and $-\lambda_4$, and equating the coefficients of $\delta e_1, \delta e_2, ... \delta e_8$ to zero. However, a simplified and direct method of formulating these equations is given below :

The original condition equations are

$$e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8 = E_1 \quad ....(1a)$$
$$e_1 + e_2 - e_5 - e_6 = E_2 \quad ...(1b)$$
$$e_3 + e_4 - e_7 - e_8 = E_3 \quad ...(1c)$$
$$e_1 f_1 - e_2 f_2 + e_3 f_3 - e_4 f_4 + e_5 f_5 - e_6 f_6 + e_7 f_7 - e_8 f_8 = E_4 \quad ...(1d)$$

Prepare the following table :

(1)	(2)	(3)	(4)	(5)
e	λ_1	λ_2	λ_3	λ_4
e_1	+1	+1	0	$+f_1$
e_2	+1	+1	0	$-f_2$
e_3	+1	0	+1	$+f_3$
e_4	+1	0	+1	$-f_4$
e_5	+1	−1	0	$+f_5$
e_6	+1	−1	0	$-f_6$
e_7	+1	0	−1	$+f_7$
e_8	+1	0	−1	$-f_8$

In the above table, there are five columns with the heading e, $\lambda_1, \lambda_2, \lambda_3$ and λ_4. In the first column under e, the corrections e_1, e_2, e_3,......, e_8, are entered. In the second column under λ_1, the coefficients of e_1, e_2, e_3, ..., e_8 in equation (1a) are entered. In third column under λ_2, the coefficients of $e_1, e_2, e_3,....e_8$ in equation (1b) are entered. In the fourth column under λ_3, coefficients of $e_1, e_2, e_3,......, e_8$ in equation (1c) are entered. Finally, in the last column under λ_4, the coefficients of e_1, e_2, e_3, ..., e_8 in equation (1d) are entered.

The equations for $e_1, e_2, e_3, ...e_8$ are then formed by multiplying $\lambda_1, \lambda_2, \lambda_3$ and λ_4 *by the corresponding coefficients under them and in horizontal line with* e_1, e_2, e_3, ..., e_8 *respectively.*

Thus, the equation for e_1 is obtained by multiplying λ_1, λ_2, λ_3 and λ_4 by the coefficients +1, +1, 0 and $+f_1$ respectively, under them and in horizontal line with e_1. The required equations are therefore :

$$e_1 = \lambda_1 + \lambda_2 + f_1 \lambda_4 \quad ...(3a)$$
$$e_2 = \lambda_1 + \lambda_2 - f_2 \lambda_4 \quad ...(3b)$$
$$e_3 = \lambda_1 + \lambda_3 + f_3 \lambda_4 \quad ...(3c)$$
$$e_4 = \lambda_1 + \lambda_3 - f_4 \lambda_4 \quad ...(3d)$$
$$e_5 = \lambda_1 - \lambda_2 + f_5 \lambda_4 \quad ...(3e)$$
$$e_6 = \lambda_1 - \lambda_2 - f_6 \lambda_4 \quad ...(3f)$$
$$e_7 = \lambda_1 - \lambda_3 + f_7 \lambda_4 \quad ...(3g)$$
$$e_8 = \lambda_1 - \lambda_3 - f_8 \lambda_4 \quad ...(3h)$$

which are the same as found earlier.

2. ADJUSTMENT OF QUADRILATERAL BY APPROXIMATE METHOD

This method is generally used for adjusting a quadriateral of moderate size or of minor importance. The method gives fairly accurate results. However, the least square condition is not satisfied by this method.

Refer to Fig. 9.9 ; the condition equations are

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5 + \theta_6 + \theta_7 + \theta_8 = 360° \quad ...(1)$$
$$\theta_1 + \theta_2 = \theta_5 + \theta_6 \quad ...(2)$$
$$\theta_3 + \theta_4 = \theta_7 + \theta_8 \quad ...(3)$$
and
$$\Sigma \log \sin L = \Sigma \log \sin R \quad ...(4)$$

In the above, $\theta_1, \theta_2, ...\theta_8$ are the angles adjusted for spherical excess if necessary.

All the four equations are satisfied in the following steps :

(1) Find the sum of $\theta_1, \theta_2, ..., \theta_8$ and subtract it from 360°. Correct each angle by distributing one-eighth of the discrepancy so that equation 1 is satisfied.

(2) From the values of the angles so obtained, find the difference between $(\theta_1 + \theta_2)$ and $(\theta_5 + \theta_6)$. Correct each angle by one fourth of the discrepancy. If $(\theta_1 + \theta_2)$ is less than $(\theta_5 + \theta_6)$, the sign of corrections to θ_1 and θ_2 is positive and that for θ_5 and θ_6 is negative. Similarly, if $(\theta_1 + \theta_2)$ is more than $(\theta_5 + \theta_6)$ the sign of correction to θ_1 and θ_2 is negative and that for θ_5 and θ_6 is positive. Thus equation (2) is satisfied.

(3) Similarly, find the difference between $(\theta_3 + \theta_4)$ and $(\theta_7 + \theta_8)$. Correct each angle by one-fourth of the discrepancy. If $(\theta_3 + \theta_4)$ is less than $(\theta_7 + \theta_8)$, the sign of corrections to θ_3 and θ_4 is positive and that for θ_7 and θ_8 is negative and *vice versa*.

Thus equation (3) is satisfied.

(4) The adjusted values of $\theta_1, \theta_2, \theta_3, ..., \theta_8$ are then tested for side equation. Find the log sine of angles $\theta_1, \theta_2, ..., \theta_8$. Take the sum of log sine of left angles, and also of log sine of right angles. Find the difference between Σ log sin L and Σ log sin R. and find the discrepancy.

Let m be the discrepancy

i.e. $$m = \Sigma \log \sin L - \Sigma \log \sin R$$

Let $f_1, f_2, f_3, ..., f_8$ be the tabular differences 1″ for log sin θ_1, log sin θ_2,...log sin θ_8.

Find Σf^2.

i.e. $$\Sigma f^2 = f_1^2 + f_2^2 + f_3^2 + ... f_8^2$$

Then the correction to angle $\theta_1 = \frac{f_1}{\Sigma f^2} m$

Then the correction to angle $\theta_2 = \frac{f_2}{\Sigma f^2} m$

... ...

Correction to angle $\theta_8 = \frac{f_8}{\Sigma f^2} m$

If Σ log sin L is greater than Σ log sin R, the corrections to left angles are negative and corrections to right angles are positive and *vice versa* so that Eq. (4) is satisfied.

Due to the fulfilment of the side equation, the values of the angles will be changed, thus disturbing the condition equations (1), (2) and (3). In case more accuracy is required, both the adjustments are repeated.

Example 9.25. *The following are the observed values of eight angles of a Geodetic quadrilateral after they have been corrected for spherical excess. Adjust the quadrilateral. The observations may be assumed to be of equal weight.*

$\theta_1 = 71° 26' 03''.59$
$\theta_2 = 53° 39' 54''.60$
$\theta_3 = 31° 18' 10''.53$
$\theta_4 = 23° 35' 52''.03$
$\theta_5 = 89° 40' 10''.42$
$\theta_6 = 35° 25' 47''.08$
$\theta_7 = 14° 18' 02''.87$
$\theta_8 = 40° 36' 00''.15$

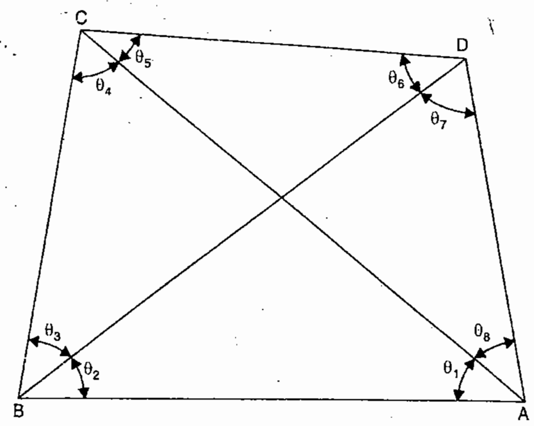


FIG. 9.11.

(a) Solution by method of least squares

The angle equations are :

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5 + \theta_6 + \theta_7 + \theta_8 = 360°$$
$$\theta_1 + \theta_2 = \theta_5 + \theta_6$$
$$\theta_3 + \theta_4 = \theta_7 + \theta_8$$

Sum of the observed angles = 360° 00′ 01″.27

$\therefore$ $$E_1 = 360° - \Sigma\theta = -1''.27$$

$$\theta_1 + \theta_2 = 125° 05' 58''.19$$
$$\theta_5 + \theta_6 = 125° 05' 57''.50$$

$\therefore \quad E_2 = (\theta_5 + \theta_6) - (\theta_1 + \theta_2) = -0''.69$

$\theta_3 + \theta_4 = 54° \, 54' \, 02''.56$

$\theta_7 + \theta_8 = 54° \, 54' \, 03''.02$

$E_3 = (\theta_7 + \theta_8) - (\theta_3 + \theta_4) = +0''.46$

To find Σ log sin *L* and Σ log sin *R*, the following table is prepared:

Left				Right			
	Angle ° ′ ″	Log sine	*f*		Angle ° ′ ″	Log sine	*f*
θ_1	71 26 03.59	9.9767897	7.0	θ_2	53 39 54.60	9.9061023	15.5
θ_3	31 18 10.53	9.7156380	34.6	θ_4	23 35 52.03	9.6024004	48.2
θ_5	89 40 10.42	9.9999928	0.1	θ_6	35 25 47.08	9.7632065	29.6
θ_7	14 18 02.87	9.3927189	82.6	θ_8	40 36 00.15	9.8134304	24.6
	Sum	39.0851394			Sum	39.0851399	

$\therefore \quad E_4 = \Sigma \log \sin R - \Sigma \log \sin L = +5$

Thus, if $e_1, e_2, e_3 \ldots, e_8$ are the corrections to the angles $\theta_1, \theta_2, \theta_3, \ldots \theta_8$, we have the following conditions :

$e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8 = -1.27$ ...(1)

$e_1 + e_2 - e_5 - e_6 = -0.69$ ...(2)

$e_3 + e_4 - e_7 - e_8 = +0.46$ ...(3)

$e_1 f_1 - e_2 f_2 + e_3 f_3 - e_4 f_4 + e_5 f_5 - e_6 f_6 + e_7 f_7 - e_8 f_8 = E_4$

or $7.0\, e_1 - 15.5\, e_2 + 34.6\, e_3 - 48.2\, e_4 + 0.1\, e_5 - 29.6\, e_6 + 82.6\, e_7 - 24.6\, e_8 = +5$ ...(4)

Also, from least square condition,

$e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_5^2 + e_6^2 + e_7^2 + e_8^2 = \text{a minimum}$ ...(5)

To form the equations for e_1, e_2, e_3 etc., either differentiate all the five equations, multiply Eqs. (1) to (4) by $-\lambda_1, -\lambda_2, -\lambda_3, -\lambda_4$, respectively, add to eq. (5) and equate the coefficients $\delta e_1, \delta e_2, \ldots, \delta e_8$ of zero or use the following table as discussed earlier :

(1)	(2)	(3)	(4)	(5)
e	λ_1	λ_2	λ_3	λ_4
e_1	+1	+1	0	+ 7.0
e_2	+1	+1	0	− 15.5
e_3	+1	0	+1	+ 34.6
e_4	+1	0	+1	− 48.2
e_5	+1	−1	0	+ 0.1
e_6	+1	−1	0	− 29.6
e_7	+1	0	−1	+ 82.6
e_8	+1	0	−1	− 24.6

Hence the equations are

$e_1 = \lambda_1 + \lambda_2 + 7\,\lambda_4$ ...(2a)

$e_2 = \lambda_1 + \lambda_2 - 15.5\,\lambda_4$ ...(2b)

$e_3 = \lambda_1 + \lambda_3 + 34.6\,\lambda_4$ ...(2c)

$e_4 = \lambda_1 + \lambda_3 - 48.2\,\lambda_4$ ...(2d)

$e_5 = \lambda_1 - \lambda_2 + 0.1\,\lambda_4$ ...(2e)

$e_6 = \lambda_1 - \lambda_2 - 29.6\,\lambda_4$ ...(2f)

$e_7 = \lambda_1 - \lambda_3 + 82.6\,\lambda_4$ ...(2g)

$e_8 = \lambda_1 - \lambda_3 - 24.6\,\lambda_4$ ...(2h)

Substituting these values in Eqs. (1) to (4), we get

$8\,\lambda_1 + 69.4\,\lambda_4 = -1.27$ ...(i)

$4\,\lambda_2 + 21.0\,\lambda_4 = -0.69$ ...(ii)

$4\,\lambda_3 - 71.6\,\lambda_4 = +0.46$ ...(iii)

$6.4\,\lambda_1 + 21\,\lambda_2 - 71.6\,\lambda_3 + 1211\,\lambda_4 = +5$ ...(iv)

Solving the above four equations and substituting the values of $\lambda_1, \lambda_2, \lambda_3$ and λ_4 in Eqs. (2a) to (2h), we get

$e_1 = -0''.330$

$e_2 = -0''.367$

$e_3 = +0''.042$

$e_4 = -0''.095$

$e_5 = +0''.022$

$e_6 = -0''.029$

$e_7 = -0''.166$

$e_8 = -0''.347$

The corrected angles along with their log sines are given in the table below :

	Observed Angles ° ′ ″	Corrections ″	Corrected angles ° ′ ″	Log sine
θ_1	71 26 03.59	− 0.330	71 26 03.260	9.9767895
θ_3	31 18 10.53	+ 0.042	31 18 10.572	9.7156382
θ_5	89 40 10.42	+ 0.022	89 40 10.442	9.9999928
θ_7	14 18 02.87	− 0.166	14 18 02.704	9.3927175
			Sum	39.0851380
θ_2	53 39 54.60	− 0.367	53 39 54.233	9.9061018
θ_4	23 35 52.03	− 0.095	23 35 51.935	9.6023999
θ_6	35 25 47.08	− 0.029	35 25 47.051	9.7632064
θ_8	40 36 00.15	− 0.347	40 35 59.803	9.8134299
		Sum	360° 00′ 00″.000	39.0851380

(*b*) **Solution by Approximate Method**

The quadrilateral is adjusted by approximate method in the following steps :

Step 1. $\Sigma\theta = 360° \ 00' \ 01''.27$

$\therefore$ $E_1 = 360° - \Sigma\theta = -01''.27$

Distributing this equally to all the eight angles, the correction to each angle $= -\frac{1}{8} \times 1''.27 = -0''.16$ (approximately). Hence the corrected angles are

$\theta_1 = 71° \ 26' \ 03''.43$
$\theta_2 = 53° \ 39' \ 54''.44$
$\theta_3 = 31° \ 18' \ 10''.37$
$\theta_4 = 23° \ 35' \ 51''.87$
$\theta_5 = 89° \ 40' \ 10''.26$
$\theta_6 = 35° \ 25' \ 46''.92$
$\theta_7 = 14° \ 18' \ 02''.71$
$\theta_8 = 40° \ 36' \ 00''.00$ (approx.)

Sum $= 360° \ 00' \ 00''.00$

Step 2. $\theta_1 + \theta_2 = 125° \ 05' \ 57''.87$

$\theta_5 + \theta_6 = 125° \ 05' \ 57''.18$

$\therefore$ $E_2 = 125° \ 05' \ 57''.87 - 125° \ 05' \ 57''.18 = -0''.69$

Distributing this equally to all the four angles, correction to each angle $= \frac{1}{4} \times 0''.69 = 0''.17$ approximately. The correction will be negative to θ_1 and θ_2 and positive to θ_5 and θ_6

Hence the corrected angles are

$\theta_1 = 71° \ 26' \ 03''.43 - 0''.17 = 71° \ 26' \ 03''.26$
$\theta_2 = 53° \ 39' \ 54''.44 - 0''.17 = 53° \ 39' \ 54''.27$

Sum $= 125° \ 05' \ 57''.53$

$\theta_5 = 89° \ 40' \ 10''.26 + 0''.17 = 89° \ 40' \ 10''.43$
$\theta_6 = 35° \ 25' \ 46''.92 + 0''.17 = 35° \ 25' \ 47''.09$

Sum $= 125° \ 05' \ 57''.52$

Step 3.

$\theta_3 + \theta_4 = 54° \ 54' \ 02''.24$

$\theta_7 + \theta_8 = 54' \ 54' \ 02''.71$

$\therefore$ $E_3 = 54° \ 54' \ 02''.71 - 54° \ 54' \ 02''.24 = 0''.47$

Distributing this equally to all the four angles, the correction to each angle $= \frac{1}{4}(0''.47) = 0''.12$ approximately. The correction will be positive to θ_3 and θ_4 and negative to θ_7 and θ_8. Hence the corrected angles are

$\theta_3 = 31° \ 18' \ 10''.37 + 0''.12 = 31° \ 18' \ 10''.49$
$\theta_4 = 23° \ 35' \ 51''.87 + 0''.12 = 23° \ 35' \ 51''.99$

Sum $= 54° \ 54' \ 02''.48$

$\theta_7 = 14° \ 18' \ 02''.71 - 0''.12 = 14° \ 18' \ 02''.59$
$\theta_8 = 40° \ 36' \ 00''.00 - 0''.12 = 40° \ 35' \ 59''.88$

Sum $= 54° \ 54' \ 02''.47$

Step 4.

The log sines of the corrected angles are tabulated below:

Left				Right			
	Angle	*Log sine*	*f*		*Angle*	*Log sine*	*f*
θ_1	71° 26′ 03″.26	9.9767895	7.0	θ_2	53° 39′ 54″.27	9.9061018	15.5
θ_3	31° 18′ 10″.49	9.7156380	34.6	θ_4	23° 35′ 51″.99	9.6024004	48.2
θ_5	89° 40′ 10″.43	9.9999928	0.1	θ_6	35° 25′ 47″.09	9.7632065	29.6
θ_7	14° 18′ 02″.59	9.3927157	82.6	θ_8	40° 35′ 59″.88	9.8134300	24.0
	Sum	39.0851360			Sum	39.0851387	

Hence $m = \Sigma \log \sin L - \Sigma \log \sin R = -27$

This correction will be distributed to each angle in proportion to $\frac{f}{\Sigma f^2} m$. The correction will be positive for left angles and negative for right angles.

From the above table, $\Sigma f^2 = 12114$

Correction for $\theta_1 = +7 \times \frac{27}{12114} = +0''.016$

Similarly Correction for $\theta_3 = +0''.077$
Correction for $\theta_5 =$ zero
Correction for $\theta_7 = +0''.184$
Correction for $\theta_2 = -0''.035$
Correction for $\theta_4 = -0''.107$
Correction for $\theta_6 = -0''.066$
Correction for $\theta_8 = -0''.053$

Hence the corrected angles are

$\theta_1 = 71° 26' 03''.26 + 0''.016 = 71° 26' 03''.276$

$\theta_2 = 53° 39' 54''.27 - 0''.035 = 53° 39' 54''.235$

$\theta_3 = 31° 18' 10''.49 + 0''.077 = 31° 18' 10''.567$

$\theta_4 = 23° 35' 51''.99 - 0''.107 = 23° 35' 51''.883$

$\theta_5 = 89° 40' 10''.43 + \text{zero} = 89° 40' 10''.430$

$\theta_6 = 35° 25' 47''.09 - 0''.066 = 35° 25' 47''.024$

$\theta_7 = 14° 18' 02''.59 + 0''.184 = 14° 18' 02''.774$

$\theta_8 = 40° 35' 59''.88 - 0''.053 = 40° 35' 59''.827$

Sum = 360° 00' 00''.016

∴ Residual Discrepancy = 0''.016, which can be eliminated by applying the second round of corrections.

9.18. ADJUSTMENT OF A QUADRILATERAL WITH A CENTRAL STATION BY METHOD OF LEAST SQUARES

Fig 9.12 shows a geodetic quadrilateral *ABCD* with a central station *O*. All the twelve angles ($\theta_1, \theta_2 \ldots\ldots \theta_{12}$) have been measured independently. The equations of conditions are :

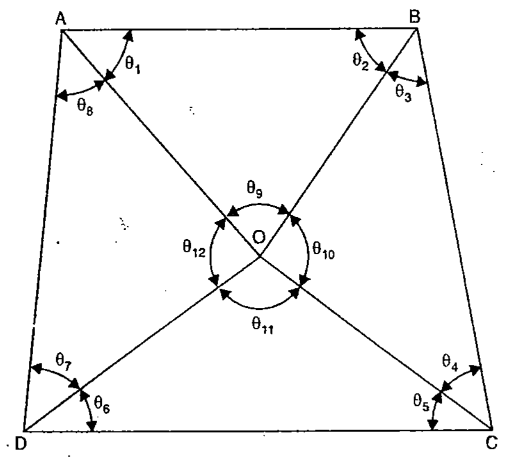


FIG. 9.12. QUADRILATERAL WITH CENTRAL STATION.

$\theta_1 + \theta_2 + \theta_9 = 180°$

$\theta_3 + \theta_4 + \theta_{10} = 180°$

$\theta_5 + \theta_6 + \theta_{11} = 180°$

$\theta_7 + \theta_8 + \theta_{12} = 180°$

$\theta_9 + \theta_{10} + \theta_{11} + \theta_{12} = 360°$

$\Sigma(\log \sin \theta_L) = \Sigma (\log \sin \theta_R)$

Let $E_1 = 180° - (\theta_1 + \theta_2 + \theta_9)$

$E_2 = 180° - (\theta_3 + \theta_4 + \theta_{10})$

$E_3 = 180° - (\theta_5 + \theta_6 + \theta_{11})$

$E_4 = 180° - (\theta_7 + \theta_8 + \theta_{12})$

$E_5 = 360° - (\theta_9 + \theta_{10} + \theta_{11} + \theta_{12})$

and $E_6 = \Sigma \log \sin R - \Sigma \log \sin L$

Hence, if $e_1, e_2, e_3 \ldots\ldots e_{12}$ are the corrections to $\theta_1, \theta_2, \theta_3 \ldots\ldots \theta_{12}$, we have

$$e_1 + e_2 + e_9 = E_1 \quad \ldots(1)$$

$$e_3 + e_4 + e_{10} = E_2 \quad \ldots(2)$$

$$e_5 + e_6 + e_{11} = E_3 \quad \ldots(3)$$

$$e_7 + e_8 + e_{12} = E_4 \quad \ldots(4)$$

$$e_9 + e_{10} + e_{11} + e_{12} = E_5 \quad \ldots(5)$$

$$e_1 f_1 - e_2 f_2 + e_3 f_3 - e_4 f_4 + e_5 f_5 - e_6 f_6 + e_7 f_7 - e_8 f_8 = E_6 \quad \ldots(6)$$

Also, from least square condition, Σe^2 = a minimum

$$\therefore \; e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_5^2 + e_6^2 + e_7^2 + e_8^2 + e_9^2 + e_{10}^2 + e_{11}^2 + e_{12}^2 = \text{a minimum} \quad \ldots(7)$$

Differentiating equations (1) to (7), we get

$$\delta e_1 + \delta e_2 + \delta e_9 = 0 \quad \ldots(1a)$$

$$\delta e_3 + \delta e_4 + \delta e_{10} = 0 \quad \ldots(2a)$$

$$\delta_5 + \delta e_6 + \delta e_{11} = 0 \quad \ldots(3a)$$

$$\delta e_7 + \delta e_8 + \delta e_{12} = 0 \quad \ldots(4a)$$

$$\delta e_9 + \delta e_{10} + \delta e_{11} + \delta e_{12} = 0 \quad \ldots(5d)$$

$$f_1 \,\delta e_1 - f_2 \,\delta e_2 + f_3 \,\delta e_3 - f_4 \,\delta e_4 + f_5 \,\delta e_5 - f_6 \,\delta e_6 + f_7 \,\delta e_7 - f_8 \,\delta e_8 = 0 \quad \ldots(6a)$$

and $e_1 \,\delta e_1 + e_2 \,\delta e_2 + e_3 \,\delta e_3 + e_4 \,\delta e_4 + e_5 \,\delta e_5 + e_6 \,\delta e_6 + e_7 \,\delta e_7 + e_8 \,\delta e_8$

$$+ e_9 \,\delta e_9 + e_{10} \,\delta e_{10} + e_{11} \,\delta e_{11} + e_{12} \,\delta e_{12} = 0 \quad \ldots(7a)$$

Multiplying equation (1*a*), (2*a*), (3*a*), (4*a*), (5*a*) and (6*a*) by $-\lambda_1, -\lambda_2, -\lambda_3, -\lambda_4, -\lambda_5, -\lambda_6$ respectively, adding to equation (7*a*) and equating the coefficients of $\delta e_1, \delta e_2, \delta e_3 \ldots\ldots, \delta e_{12}$ to zero, we get the following equations for $e_1, e_2, e_3, \ldots.. e_{12}$:

$$e_1 = \lambda_1 + f_1 \lambda_6 \; ; \; e_2 = \lambda_1 - f_2 \lambda_6$$

$$e_3 = \lambda_2 + f_3 \lambda_8 \; ; \; e_4 = \lambda_2 - f_4 \lambda_6$$

$$e_5 = \lambda_3 + f_5 \lambda_6 \; ; \; e_6 = \lambda_3 - f_6 \lambda_6$$

$$e_7 = \lambda_4 + f_7 \lambda_6 \; ; \; e_8 = \lambda_4 - f_8 \lambda_6 \quad \ldots(9.27)$$

$$e_9 = \lambda_1 + \lambda_5 \; ; \; e_{10} = \lambda_2 + \lambda_5$$

$$e_{11} = \lambda_3 + \lambda_5 \; ; \; e_{12} = \lambda_4 + \lambda_5$$

Alternatively, the above equations can also be formed from the table below, as explained in the previous article.

(1) e	*(2)* λ_1	*(3)* λ_2	*(4)* λ_3	*(5)* λ_4	*(6)* λ_5	*(7)* λ_6
e_1	+ 1	0	0	0	0	$+f_1$
e_2	+ 1	0	0	0	0	$-f_2$
e_3	0	+ 1	0	0	0	$+f_3$
e_4	0	+ 1	0	0	0	$-f_4$
e_5	0	0	+ 1	0	0	$+f_5$
e_6	0	0	+ 1	0	0	$-f_6$
e_7	0	0	0	+ 1	0	$+f_7$
e_8	0	0	0	+ 1	0	$-f_8$
e_9	+ 1	0	0	0	+ 1	0
e_{10}	0	+ 1	0	0	+ 1	0
e_{11}	0	0	+ 1	0	+ 1	0
e_{12}	0	0	0	+ 1	+ 1	0

In this first column under e, the correction $e_1, e_2, e_3, ... e_{12}$ are entered. In the second column under λ_1, the coefficients of $e_1, e_2, e_3, e_4, ... e_{12}$ in equation (1) are entered. In third column under λ_2, the coefficients of $e_1, e_2, e_3, ..., e_{12}$ in equation (2) are entered. In the fourth column under λ_3, the coefficients of $e_1, e_2, ... e_{12}$ in equation (3) are entered. Similarly, in the fifth, sixth and seventh columns under λ_4, λ_5 and λ_6, the coefficients $e_1, e_2, ... e_{12}$ of in equation (4), (5) and (6) respectively are entered.

The table is thus completed.

The equations for $e_1, e_2, ..., e_{12}$ are then obtained by multiplying by $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ by the coefficients under them and in horizontal line with, $e_1, e_2, ... e_{12}$ respectively. Thus,

$$\begin{aligned} e_1 &= \lambda_1 + f_1 \lambda_6 & e_7 &= \lambda_4 + f_7 \lambda_6 \\ e_2 &= \lambda_1 - f_2 \lambda_6 & e_8 &= \lambda_4 - f_8 \lambda_6 \\ e_3 &= \lambda_2 + f_3 \lambda_6 & e_9 &= \lambda_1 + \lambda_5 \\ e_4 &= \lambda_2 - f_4 \lambda_6 & e_{10} &= \lambda_2 + \lambda_5 \\ e_5 &= \lambda_3 + f_5 \lambda_6 & e_{11} &= \lambda_3 + \lambda_5 \\ e_6 &= \lambda_3 - f_6 \lambda_6 & e_{12} &= \lambda_4 + \lambda_5 \end{aligned} \quad ...(9.27)$$

The equations are the same as derived earlier.

Substituting the values of $e_1, e_2, e_3, ... e_{12}$, in equation (1), (2), (3), (4), (5) and (6), we get the following equations for $\lambda_4, \lambda_5\ \lambda_1, \lambda_2, \lambda_3$, and λ_6 :

$$3\lambda_1 + \lambda_5 + \lambda_6 (f_1 - f_2) = E_1 \quad ...(i)$$
$$3\lambda_2 + \lambda_5 + \lambda_6 (f_3 - f_4) = E_2 \quad ...(ii)$$
$$3\lambda_3 + \lambda_5 + \lambda_6 (f_5 - f_6) = E_3 \quad ...(iii) \quad ...(9.28)$$
$$3\lambda_4 + \lambda_5 + \lambda_6 (f_7 - f_8) = E_4 \quad ...(iv)$$
$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 4\lambda_5 = E_5 \quad ...(v)$$
$$\lambda_1 (f_1 - f_2) + \lambda_2 (f_3 - f_4) + \lambda_3 (f_5 - f_6) + \lambda_4 (f_7 - f_8) + \lambda_6 \Sigma(f^2) = E_6 \quad ...(vi)$$

Solving theses equation simultaneously, we get $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ and λ_6. Substituting these values in equations for $e_1, e_2, ... e_{12}$ we get the values of the corrections and hence the corrected angles can be known.

Approximate Solution for $\lambda_1, \lambda_2, ... \lambda_6$ by Dale's Method

The equation (*i*) to (*vi*) for $\lambda_1, \lambda_2, ... \lambda_6$ given above can be solved by the *method of successive approximations* suggested by Prof. J.B. Dale. The solution is done in the following steps :

Step 1.

Neglecting the terms for λ_6, add equations (*i*), (*ii*), (*iii*) and (*iv*)

$$3\lambda_1 + 3\lambda_2 + 3\lambda_3 + 3\lambda_4 + 4\lambda_5 = E_1 + E_2 + E_3 + E_4 \quad ...(a)$$

Step 2.

Multiply equation (*v*) by (3). Thus

$$3\lambda_1 + 3\lambda_2 + 3\lambda_3 + 3\lambda_4 + 12\lambda_5 = 3E_5 \quad ...(b)$$

Step 3.

Subtracting Eq. (*a*) from Eq. (*b*), we get

$$\lambda_5 = \frac{3E_5 - (E_1 + E_2 + E_3 + E_4)}{8} \quad ...(c)...(9.29)$$

(**Note.** If there are n sides of the polygon, the denominator will be $2n$).

Step 4.

Substituting this value of λ_5 in equation (*i*), (*ii*), (*iii*) and (*iv*) and *still ignoring the terms for* λ_6, we get

$$\lambda_1 = \frac{E_1 - \lambda_5}{3} \qquad \lambda_3 = \frac{E_3 - \lambda_5}{3}$$
$$\lambda_2 = \frac{E_2 - \lambda_5}{3} \qquad \lambda_4 = \frac{E_4 - \lambda_5}{3} \quad ...(9.30)$$

Step 5.

Substituting these approximate values of $\lambda_1, \lambda_2, \lambda_3$ and λ_4 in Eq. (*vi*), we get

$$\lambda_6 = \frac{E_6 - \Sigma\lambda\,(f_L - f_R)}{\Sigma f^2} \quad ...(9.31)$$

Thus the approximate values of $\lambda_1, \lambda_2, ... \lambda_6$ are known.

Step 6.

For the second approximation, let

$$E_1' = E_1 - \lambda_6 (f_1 - f_2)$$
$$E_2' = E_2 - \lambda_6 (f_3 - f_4)$$
$$E_3' = E_3 - \lambda_6 (f_5 - f_6)$$
$$E_4' = E_4 - \lambda_6 (f_7 - f_8)$$

Then
$$\lambda_5 = \frac{3E_5 - (E_1' + E_2' + E_3' + E_4')}{8} \quad ...[9.29\ (a)]$$

Step 7. Finally :

$$\lambda_1 = \frac{E_1' - \lambda_5}{3} \qquad \lambda_3 = \frac{E_3' - \lambda_5}{3}$$
$$\lambda_2 = \frac{E_2' - \lambda_5}{3} \qquad \lambda_4 = \frac{E_4' - \lambda_5}{3} \quad ...[9.30\ (a)]$$

Step 8.

$$\lambda_6 = \frac{E_6' - \Sigma\lambda\,(f_L - f_R)}{\Sigma f^2} \quad ...[9.31\ (a)]$$

If this value of λ_6 differs appreciably from its previous value, repeat steps (6), (7) and (8) till no appreciable change is effected by repeating the process.

The method of solution has been explained fully in example 9.26.

9.19. ADJUSTMENTS OF GEODETIC TRIANGLES WITH CENTRAL STATION BY METHOD OF LEAST SQUARES

Let ABC be a geodetic triangle with P as the central station. $\theta_1, \theta_2, ... \theta_9$ are the observed angles corrected for the spherical excess, if any.

The condition equations are

$\theta_1 + \theta_2 + \theta_7 = 180°$

$\theta_3 + \theta_4 + \theta_8 = 180°$

$\theta_5 + \theta_6 + \theta_9 = 180°$

$\theta_7 + \theta_8 + \theta_9 = 360°$

$\Sigma \log \sin L = \Sigma \log \sin R$

Let $E_1 = 180° - (\theta_1 + \theta_2 + \theta_7)$

$E_2 = 180° - (\theta_3 + \theta_4 + \theta_8)$

$E_3 = 180° - (\theta_5 + \theta_6 + \theta_9)$

$E_4 = 360° - (\theta_7 + \theta_8 + \theta_9)$

and $E_5 = \Sigma \log \sin R - \Sigma \log \sin L$

Hence if $e_1, e_2, \ldots e_9$ are the corrections to $\theta_1, \theta_2, \ldots \theta_9$ we have the following equations :

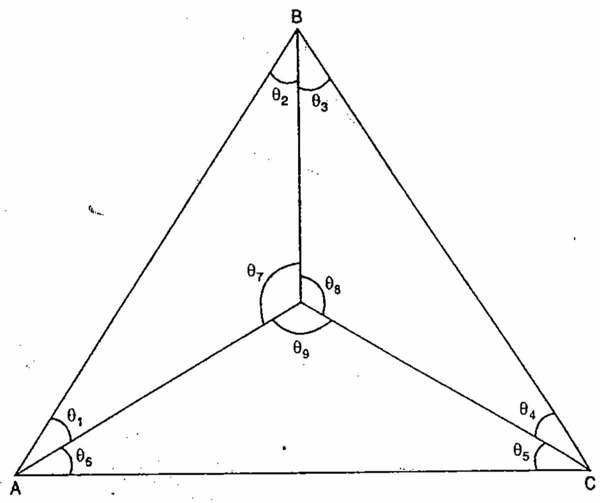


FIG. 9.13. TRIANGLE WITH CENTRAL STATION.

$$e_1 + e_2 + e_7 = E_1 \quad \ldots(1)$$

$$e_3 + e_4 + e_8 = E_2 \quad \ldots(2)$$

$$e_5 + e_6 + e_9 = E_3 \quad \ldots(3)$$

$$e_7 + e_8 + e_9 = E_4 \quad \ldots(4)$$

$$e_1 f_1 - e_2 f_2 + e_3 f_3 - e_4 f_4 + e_5 f_5 - e_6 f_6 = E_5 \quad \ldots(5)$$

Also, from least square condition,

$$e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_5^2 + e_6^2 + e_7^2 + e_8^2 + e_9^2 = \text{a minimum} \quad \ldots(6)$$

Differentiating equations (1) to (6), we get

$$\delta e_1 + \delta e_2 + \delta e_7 = 0 \quad \ldots(1a)$$

$$\delta e_3 + \delta e_4 + \delta e_8 = 0 \quad \ldots(2a)$$

$$\delta e_5 + \delta e_6 + \delta e_9 = 0 \quad \ldots(3a)$$

$$\delta e_7 + \delta e_8 + \delta e_9 = 0 \quad \ldots(4a)$$

$$f_1\, \delta e_1 - f_2\, \delta e_2 + f_3\, \delta e_3 - f_4\, \delta e_4 + f_5\, \delta e_5 - f_6\, \delta e_6 = 0 \quad \ldots(5a)$$

$$e_1\, \delta e_1 + e_2\, \delta e_2 + e_3\, \delta e_3 + e_4\, \delta e_4 + e_5\, \delta e_5 + e_6\, \delta e_6 + e_7\, \delta e_7 + e_8\, \delta e_8 + e_9\, \delta e_9 = 0 \quad \ldots(6a)$$

Multiplying equations (1*a*), (2*a*), (3*a*), (4*a*), (5*a*) by $-\lambda_1, -\lambda_2, -\lambda_3, -\lambda_4$ and $-\lambda_5$ respectively, adding it to equation (6*a*) and equating the coefficients of $\delta e_1, \delta e_2, \ldots\ldots, \delta e_9$ to zero, we get the following equations:

$e_1 = \lambda_1 + f_1 \lambda_5$

$e_2 = \lambda_1 - f_2 \lambda_5$

$e_3 = \lambda_2 + f_3 \lambda_5$

$e_4 = \lambda_2 - f_4 \lambda_5$

$e_5 = \lambda_3 + f_5 \lambda_5$

$e_6 = \lambda_3 - f_6 \lambda_5$

$e_7 = \lambda_1 + \lambda_4 \quad \ldots(9.32)$

$e_8 = \lambda_2 + \lambda_4$

$e_9 = \lambda_3 + \lambda_4$

Alternatively, the above equations can also be found from the table below :

(1)	(2) λ_1	(3) λ_2	(4) λ_3	(5) λ_4	(6) λ_5
e_1	+1	0	0	0	$+f_1$
e_2	+1	0	0	0	$-f_2$
e_3	0	+1	0	0	$+f_3$
e_4	0	+1	0	0	$-f_4$
e_5	0	0	+1	0	$+f_5$
e_6	0	0	+1	0	$-f_6$
e_7	+1	0	0	+1	0
e_8	0	+1	0	+1	0
e_9	0	0	+1	+1	0

In the first column under e, the corrections $e_1, e_2, \ldots\ldots e_9$ are entered. In the second column under λ_1, the coefficients of $e_1, e_2, e_3, e_4, \ldots\ldots e_9$ in equation (1) are entered. Similarly, in the third, fourth, fifth and sixth columns under $\lambda_2, \lambda_3, \lambda_4, \lambda_5$ respectively, the coefficients of $e_1, e_2, \ldots\ldots e_9$ in equations (2), (3), (4) and (5) respectively are entered. This table is thus completed.

The equations for $e_1, e_2, \ldots\ldots e_9$ are then obtained by multiplying $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ by the coefficients under them and in horizontal line with $e_1, e_2, \ldots\ldots e_9$ respectively. Thus

$e_1 = \lambda_1 + f_1 \lambda_5$

$e_2 = \lambda_1 - f_2 \lambda_5$

$e_3 = \lambda_2 + f_3 \lambda_5$

$e_4 = \lambda_2 - f_4 \lambda_5$

$e_5 = \lambda_3 + f_5 \lambda_5$

$e_6 = \lambda_3 - f_6 \lambda_5$

$e_7 = \lambda_1 + \lambda_4$

$e_8 = \lambda_2 + \lambda_4$

$e_9 = \lambda_3 + \lambda_4$

Substituting these values of $e_1, e_2, \ldots\ldots e_9$ in equations (1) to (5), we get

$$3\lambda_1 + \lambda_4 + \lambda_5 (f_1 - f_2) = E_1$$
$$3\lambda_2 + \lambda_4 + \lambda_5 (f_3 - f_4) = E_2$$
$$3\lambda_3 + \lambda_4 + \lambda_5 (f_5 - f_6) = E_3$$
$$\lambda_1 + \lambda_2 + \lambda_3 + 3\lambda_4 = E_4$$
$$\lambda_1 (f_1 - f_2) + \lambda_2 (f_3 - f_4) + \lambda_3 (f_5 - f_6) + \lambda_6 \Sigma (f)^2 = E_5$$

Solving these equations simultaneously, we get $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and λ_5. Substituting these values in equations for $e_1, e_2, \ldots, e_9$, we can get the values of the corrections, and hence the corrected angles can be known.

Example 9.26. *The following are the measured angles of a quadrilateral ABCD with the central point E:*

Triangle	*Central Angle*	*L.H. Angle*	*R.H. Angle*
AEB	*59° 03′ 10″*	*61° 00′ 54″*	*59° 56′ 06″*
BEC	*118° 23′ 50″*	*32° 03′ 54″*	*29° 32′ 06″*
CED	*60° 32′ 05″*	*56° 28′ 01″*	*62° 59′ 49″*
DEA	*122° 00′ 55″*	*28° 42′ 00″*	*29° 17′ 00″*

Adjust the quadrilateral. (U.L.)

Solution.

Fig. 9.14 shows the quadrilateral in which the L.H. angles have been denoted by odd numbers and R.H. angles by even numbers.

$\theta_1 = 61° 00' 54''$
$\theta_2 = 59° 56' 06''$
$\theta_3 = 32° 03' 54''$
$\theta_4 = 29° 32' 06''$
$\theta_5 = 56° 28' 01''$
$\theta_6 = 62° 59' 49''$
$\theta_7 = 28° 42' 00''$
$\theta_8 = 29° 17' 00''$
$\theta_9 = 59° 03' 10''$
$\theta_{10} = 118° 23' 50''$
$\theta_{11} = 60° 32' 05''$
$\theta_{12} = 122° 00' 55''$

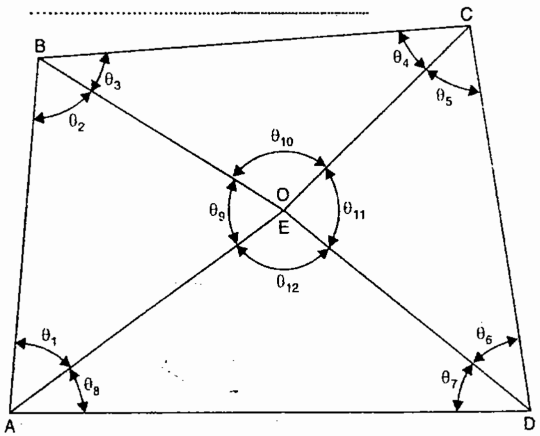


FIG. 9.14

The condition equations are

$$\theta_1 + \theta_2 + \theta_9 = 180°$$
$$\theta_3 + \theta_4 + \theta_{10} = 180°$$
$$\theta_5 + \theta_6 + \theta_{11} = 180°$$
$$\theta_7 + \theta_8 + \theta_{12} = 180°$$
$$\theta_9 + \theta_{10} + \theta_{11} + \theta_{12} = 360°$$
$$\Sigma \log \sin L = \Sigma \log \sin R$$

$$E_1 = 180° - (\theta_1 + \theta_2 + \theta_9) = 180° - 180° 00' 10'' = -10''$$
$$E_2 = 180° - (\theta_3 + \theta_4 + \theta_{10}) = 180° - 179° 59' 50'' = +10''$$
$$E_3 = 180° - (\theta_5 + \theta_6 + \theta_{11}) = 180° - 179° 59' 55'' = +5''$$
$$E_4 = 180° - (\theta_7 + \theta_8 + \theta_{12}) = 180° - 179° 59' 55'' = +5''$$
$$E_5 = 360° - (\theta_9 + \theta_{10} + \theta_{11} + \theta_{12}) = 360° - 360° = 0$$
$$E_6 = \Sigma \log \sin R - \Sigma \log \sin L$$

The values of log sine of the angles are tabulated below :

	Left				Right		
	Angle	Log sine	f		Angle	Log sine	f
θ_1	61° 00′ 54″	9.9418823	11.67	θ_2	59° 56′ 06″	9.9372458	12.18
θ_3	32° 03′ 54″	9.7249972	33.63	θ_4	29° 32′ 06″	9.6928074	37.15
θ_5	56° 28′ 01″	9.9209407	13.93	θ_6	62° 59′ 49″	9.9498691	10.73
θ_7	28° 42′ 00″	9.6814434	38.45	θ_8	29° 17′ 00″	9.6894232	37.57
	Sum	39.2692636			Sum	39.2693455	

$$\Sigma f^2 = 5995$$

$\therefore$ $E_6 = 39.2693455 - 39.2692636 = +819$

Hence if $e_1, e_2, \ldots e_{12}$ are the corrections to $\theta_1, \theta_2, \ldots \theta_{12}$ we get the following equations :

$$e_1 + e_2 + e_9 = -10 \quad \ldots(1)$$
$$e_3 + e_4 + e_{10} = +10 \quad \ldots(2)$$
$$e_5 + e_6 + e_{11} = +5 \quad \ldots(3)$$
$$e_7 + e_8 + e_{12} = +5 \quad \ldots(4)$$
$$e_9 + e_{10} + e_{11} + e_{12} = 0 \quad \ldots(5)$$
$$11.67\, e_1 - 12.18\, e_2 + 33.63\, e_3 - 37.15\, e_4 + 13.93\, e_5 - 10.73\, e_6 + 38.45 e_7 - 37.57 e_8 = +819 \quad \ldots(6)$$

To form the equations, we prepare a table such as given in § 9.16

The following equations are obtained :

$e_1 = \lambda_1 + 11.67\, \lambda_6$	$e_7 = \lambda_4 + 38.45\, \lambda_6$
$e_2 = \lambda_1 - 12.18\, \lambda_6$	$e_8 = \lambda_4 - 37.57\, \lambda_6$
$e_3 = \lambda_2 + 33.63\, \lambda_6$	$e_9 = \lambda_1 + \lambda_5$
$e_4 = \lambda_2 - 37.15\, \lambda_6$	$e_{10} = \lambda_2 + \lambda_5$
$e_5 = \lambda_3 + 13.93\, \lambda_6$	$e_{11} = \lambda_3 + \lambda_5$
$e_6 = \lambda_3 - 10.73\, \lambda_6$	$e_{12} = \lambda_4 + \lambda_5$

Substituting these values of $e_1, e_2, \ldots\ldots, e_{12}$ in equations (1), (2), (3), (4), (5) and (6), we get

$$3\lambda_1 + \lambda_5 - 0.51\lambda_6 = E_1 = -10 \quad \ldots(i)$$
$$3\lambda_2 + \lambda_5 - 3.52\lambda_6 = E_2 = +10 \quad \ldots(ii)$$
$$3\lambda_3 + \lambda_5 + 3.2\lambda_6 = E_3 = +5 \quad \ldots(iii)$$
$$3\lambda_4 + \lambda_5 + 0.88\lambda_6 = E_4 = +5 \quad \ldots(iv) \quad \ldots(9.28)$$
$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 4\lambda_5 = E_5 = 0 \quad \ldots(v)$$
$$-0.51\lambda_1 - 3.52\lambda_2 + 3.2\lambda_3 + 0.88\lambda_4 + 5995\lambda_6 = E_6 = +819 \quad \ldots(vi)$$

(where $\Sigma f^2 = 5995$).

The above six normal equations for $\lambda_1, \lambda_2, \ldots\ \lambda_6$ will now be solved by the *approximate method* discussed in § 9.16.

From equation 9.29

$$\lambda_5 = \frac{3E_5 - (E_1 + E_2 + E_3 + E_4)}{8} = \frac{0 - (-10 + 10 + 5 + 5)}{8} = -1.25$$

Substituting the value of λ_5 in equations 9.30, we get

$$\lambda_1 = \frac{E_1 - \lambda_5}{3} = \frac{-10 + 1.25}{3} = -2.92 \qquad \lambda_3 = \frac{E_3 - \lambda_5}{3} = \frac{5 + 1.25}{3} = 2.08$$

$$\lambda_2 = \frac{E_2 - \lambda_5}{3} = \frac{10 + 1.25}{3} = 3.75 \qquad \lambda_4 = \frac{E_4 - \lambda_5}{3} = \frac{5 + 1.25}{3} = 2.08$$

Substituting these approximate values of $\lambda_1, \lambda_2, \lambda_3$ and λ_4 in Eq. 9.31, we get

$$\lambda_6 = \frac{E_6 - \Sigma\lambda(f_L - f_R)}{\Sigma f^2} = \frac{819 - \{(-2.92)(-0.51) + 3.75(-3.52) + 2.08(3.20) + 2.08(0.88)\}}{5995}$$

$$= \frac{819 - (1.49 - 13.20 + 6.66 + 1.83)}{5995} = 0.1372.$$

Second Approximation

Using this value of λ_6, let us again calculate the values of λ_1, λ_2 etc. by *including* the terms containing λ_6. Thus

$$E_1' = E_1 - \lambda_6(f_1 - f_2) = -10 - 0.1372(-0.51) = -9.93$$
$$E_2' = E_2 - \lambda_6(f_3 - f_4) = +10 - 0.1372(-3.52) = +10.48$$
$$E_3' = E_3 - \lambda_5(f_5 - f_6) = +5 - 0.1372 \times 3.20 = +4.56$$
$$E_4' = E_4 - \lambda_6(f_7 - f_8) = +5 - 0.1372 \times (0.88) = +4.88$$

Hence from Eq. 9.29 (*a*),

$$\lambda_5 = \frac{3E_5 - (E_1' + E_2' + E_3' + E_4')}{8} = \frac{0 - (-9.93 + 10.48 + 4.56 + 4.88)}{8}$$
$$= -1.25, \text{ as before.}$$

From Eq. 9.30 (*a*), we get

$$\lambda_1 = \frac{E_1' - \lambda_5}{3} = \frac{-9.93 + 1.25}{3} = -2.89 \qquad \lambda_3 = \frac{E_3' - \lambda_5}{5} = \frac{4.56 + 1.25}{3} = 1.94$$

$$\lambda_2 = \frac{E_1' - \lambda_5}{3} = \frac{10.48 + 1.25}{3} = 3.91 \qquad \lambda_4 = \frac{E_4' - \lambda_5}{3} = \frac{4.88 + 1.25}{3} = 2.04$$

Substituting the values in Eq. 9.31 (*a*), we get

$$\lambda_6 = \frac{E_6' - \Sigma(f_L - f_R)}{\Sigma f^2} = \frac{819 - \{(-2.89)(-0.51) + 3.91(-3.52) + 1.94(3.20) + 2.04(0.88)\}}{5995}$$

$$= \frac{819 - (1.47 - 13.76 + 6.21 + 1.80)}{5995} = 0.1373, \text{ as against } 0.1372 \text{ found earlier.}$$

Hence, third approximation is not necessary and we can take :

$\lambda_1 = -2.89$ $\qquad \lambda_4 = +2.04$

$\lambda_2 = +3.91$ $\qquad \lambda_5 = -1.25$

$\lambda_3 = +1.94$ $\qquad \lambda_6 = +0.1373$

Substituting these values in equations for $e_1, e_2, \ldots\ e_{12}$, we get the following values of the corrections :

$$e_1 = \lambda_1 + 11.67\lambda_6 = -2.89 + 0.1373 \times 11.67 = -1''.29$$
$$e_2 = \lambda_1 - 12.18\lambda_6 = -2.89 - 0.1373 \times 12.18 = -4''.56$$
$$e_3 = \lambda_2 + 33.63\lambda_6 = +3.91 + 0.1373 \times 33.63 = -8''.53$$
$$e_4 = \lambda_2 - 37.15\lambda_6 = +3.91 - 0.1373 \times 37.15 = -1''.19$$
$$e_5 = \lambda_3 + 13.93\lambda_6 = +1.94 + 0.1373 \times 13.93 = +3''.85$$
$$e_6 = \lambda_3 - 10.73\lambda_6 = +1.94 - 0.1373 \times 10.73 = +0''.47$$
$$e_7 = \lambda_4 + 38.45\lambda_6 = +2.04 + 0.1373 \times 38.45 = +7''.32$$
$$e_8 = \lambda_4 + 37.57\lambda_6 = +2.04 - 0.1373 \times 37.57 = -3''.12$$
$$e_9 = \lambda_1 + \lambda_5 = -2.89 - 1.25 = -4''.14$$
$$e_{10} = \lambda_2 + \lambda_5 = +3.91 - 1.25 = +2''.66$$
$$e_{11} = \lambda_3 + \lambda_5 = +1.94 - 1.25 = +0''.69$$
$$e_{12} = \lambda_4 + \lambda_5 = +2.04 - 1.25 = +0''.79$$

Table 9.1 shows the original data along with calculated corrections.

9.20. METHOD OF EQUAL SHIFTS

In this method, the discrepancy in the angular measurements is equally divided between the three angles of a triangle. For any closed polygon with central stations, the equations of conditions to be satisfied are :

(i) *Figure equations*

The sum of angles of a triangle = 180°

(ii) *Station Equation or Local Equation* :

The sum of angles at a station = 360°

(iii) *Side equation* :

Σ (log sin left angle) = Σ (log sine right angles).

TABLE 9.1
METHOD OF LEAST SQUARES (EXAMPLE 9.26)

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Triangle	*Central Angle*	*Correction*	*L.H. angle*	*Log sine*	*Increase for 1″*	*Correction of angle*	*Correction of Log sine*
AEB	59° 03′ 10″	– 4″.14	61° 00′ 54″	9.9418823	11.67	– 1″.29	– 15
BEC	118° 23′ 50″	+ 2″.66	32° 03′ 54″	9.7249972	33.63	+ 8″.53	+ 287
CED	6° 32′ 05″	+ 0″.69	56° 28′ 01″	9.9209407	13.93	+ 3″.85	+ 54
DEA	122° 00′ 55″	+ 0″.79	28° 42′ 00″	9.6814434	38.45	+ 7″.32	+ 281
	360° 00′ 00″			39.2692636			+ 607 – 212 +
							+ 819

(1)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
Triangle	*R.H. angle*	*Log sine*	*Increase for 1″*	*Correction of angle*	*Correction of Log sine*	*Sum of angles*	*Total Correction*
AEB	59° 56′ 06″	9.9372458	12.18	– 4″.56	– 56	180° 00′ 10″	– 10″
BEC	29° 32′ 06″	9.6928074	37.15	– 1″.19	– 44	179° 59′ 50″	+ 10″
CED	62° 59′ 49″	9.9498691	10.73	+ 0″.47	+ 5	179° 59′ 55″	+ 5″
DEA	29° 17′ 00″	9.6894232	37.57	– 3″.12	– 117	179° 59′ 55″	+ 5″
		39.2693455 39.2692636			– 212		
		← 819					

The method of equal shifts indicates that any shift which is necessary to satisfy the local equation should be the same for each triangle of the polygon. Similarly, any shift necessary to satisfy the side equation should be the same for each triangle.

Solution of Example 9.26 by method of equal shifts

To illustrate the method of equal shifts, we will work out Example 9.26 by this method.

The solution is done in the following steps. See Table 9.2 and 9.3.

Step 1.

Fill up columns (1), (2), (4), (9), (14) and (15) of Table 9.2. Column (15) gives the total corrections to be applied to each triangle.

Step 2.

One-third of the corrections of column (15) will be the correction for the corresponding central angle. In column 16 (Table 9.3) for 1st trial correction for central angles, fill up these $\frac{1}{3}$ rd corrections with appropriate signs. The sum of column (16) comes out to be + 3″.34. To satisfy the station equation, the sum of all the four central angles should be 360°. Since the sum of the observed central angles was zero, the sum of the corrections of column (16) should be zero. To make it zero, apply a correction of $-\frac{3.34}{4} = -0''.84$ to each 1st trial corrections and enter the second trial corrections so obtained in column (17). The sum of corrections of column (17) is – 0″.02 which is distributed arbitrarily, and final corrections are entered in column (18). The sum of column (18) should be equal to zero. *Thus, the station equation is satisfied.* Complete column (19) by applying the correction to the corresponding central angles.

Step 3.

Fill up column (20) which is the difference of columns (15) and (18). This remaining correction is to be distributed equally to left and right hand angles of each triangle. Thus, columns (21) and (26) are completed.

Step 4.

From the seven-figure log tables, fill up columns (5), (6), (10) and (11). The sum of log sines of *unadjusted angles* shows a shift of 819 from right to the left. Let us first asertain the actual shift after applying the correction to the left and right angles as per columns (21) and (26). Knowing the difference for 1″, find the corresponding differences for the corrections of column (21) and (26), and thus fill up columns (22) and (27) respectively. For example, the log sine difference for 1″ of left angles of triangle *AEB* is 11.67. Hence the difference for – 2″.92 will be = 11.67 × (– 2″.92) = – 34. Enter these differences in columns (22) and (27) to the seventh figure only and not the decimal part.

Step 5.

Take the sum of column (22) and add it to the sum of column (5). Thus, we get $39.2692636 + \overline{211} = 39.2692847$. Similarly, take the sum of column (27) and add it to sum column (10). Thus, we get $39.2693455 + \overline{203} = 39.2693658$. Hence the total shift

TABLE 9.2
METHOD OF EQUAL SHIFTS : DATA SHEET (EXAMPLE 9.26)

	CENTRE	LEFT					
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Triangle	Central Angle ° ′ ″	Correction (″)	Observed Angles ° ′ ″	Log sine	Difference for 1″	Correction (″)	Correction Difference
AEB	59 03 10		61 00 54	9.9418823	11.67	+ 1.23	+ 14
BEC	118 23 50		32 03 54	9.7249972	33.63	+ 7.80	+263
CED	60 32 05		56 28 01	9.9209407	13.93	+ 6.23	+ 87
DEA	122 00 55		28 42 00	9.6814434	38.45	+ 6.23	+240
	360 00 00			39.2692636 604	97.68		+604
				39.2693240			

	RIGHT						
(1)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
Triangle	Observed Angles ° ′ ″	Log sine	Difference for 1″	Correction (″)	Correction Difference	Sum of observed angles ° ′ ″	Correction (″)
AEB	59 56 06	9.9372458	12.68	− 7.07	− 90	180 00 10	− 10
BEC	29 32 06	9.6928074	37.15	− 0.40	− 15	179 59 50	+ 10
CED	62 59 49	9.9498691	10.73	− 2.06	− 22	179 59 55	+ 5
DEA	29 17 00	9.6894232	37.57	− 2.06	− 78	179 59 55	+ 5
		39.2693455 205	+ 98.13		− 205		
		39.2693250 39.2693240					
		10					

TABLE 9.3
METHOD OF EQUAL SHIFTS : CORRECTION SHEET (EXAMPLE 9.26)

	CENTRE					LEFT				
(1)	(16)	(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)	(25)
Triangle	1 st Trial Correction (″)	2 nd Trial Correction (″)	Final Correction	Corrected Angles ° ′ ″	Remaining Correction for angle (″)	Trial Correction for angle (″)	Trial Correction Difference	Final Correction (″)	Final Correction Difference	Corrected Angles ° ′ ″
AEB	− 3.33	− 4.17	− 4.16	59 03 05.84	− 5.84	− 2.92	− 34	+ 1.23	+ 14	61 00 55.23
BEC	+ 3.33	+ 2.49	+ 2.50	118 23 52.50	+ 7.50	+ 3.75	+136	+ 7.80	+ 263	32 04 01.80
CED	+ 1.67	+ 0.83	+ 0.83	60 32 05.83	+ 4.17	+ 2.08	+ 29	+ 6.23	+ 87	56 28 07.23
DEA	+ 1.67	+ 0.83	+ 0.83	122 00 55.83	+ 4.17	+ 2.08	+ 80	+ 6.23	+ 240	28 42 06.23
	+ 3.34	− 0.02	0.00	360 00 00.00			+211		+ 604	

	RIGHT					
(1)	(26)	(27)	(28)	(29)	(30)	(31)
Triangle	Trial Correction for angle (″)	Trial Correction Difference	Final Correction for angle (″)	Final Correction Difference	Corrected Angles ° ′ ″	Sum of Corrected Angles ° ′ ″
AEB	− 2.92	− 37	− 7.07	− 90	59 55 58.93	180 00 00.00
BEC	+ 3.75	+140	− 0.30	− 15	29 32 05.70	180 00 00.00
CED	+ 2.09	+ 22	− 2.06	− 22	62 59 46.94	180 00 00.00
DEA	+ 2.09	+ 78	− 2.06	− 78	29 16 57.94	180 00 00.00
		+203		− 205		

required = 39.2693658 − 39.2692847 = 811 from right to left. Hence the left hand angles will have to be increased and the right hand angles will have to be decreased.

Step 6.

Take the sum of columns (6) and (11). The Σ (log sine diff. for 1″) for all the eight angles = 97.68 + 98.13 = 195.81. This shows that if we make a shift of 1″ in all the angles (increasing left angles and decreasing right angles), the total change in log sine angles will be 195.81. In other words, 195.81 corresponds to 1″ of shift. Hence the total of 811 {in the log (sine of angles) value} corresponds to $\frac{811}{195.81} = 4''.15$. Hence increase each left angle by 4″.15 and decrease each right angle by 4″.15 and complete columns (23) and (28). Columns(25) and (30) then be obtained by applying the corrections to the corresponding angles.

Step 7.

As a check, complete columns (24) and (29), by multiplying corrections by the corresponding differences of 1″ in the log sine. Add the sum of column (24) to the sum of column (5) and the sum of column (29) to the sum of column (10). Comparing the results so obtained, we observe that the required correction is 10 which is negligible since it causes an error of $\frac{10}{195.81} = 0.005''$ in each angle. Columns (7) and (8) correspond to column (23) and (24). Similarly, columns (12) and (13) correspond to columns (28) and (29).

Step 8.

Finally, complete column (31) by taking the sums of columns (19), (25) and (30).

PROBLEMS

1. An angle has been measured under different field conditions, with results as follows :

28° 24′ 20″	28° 24′ 00″
28° 24′ 40″	28° 24′ 40″
28° 24′ 40″	28° 24′ 20″
28° 25′ 00″	28° 24′ 40″
28° 24′ 20″	28° 25′ 20″

Find (*i*) the probable error of single observation, (*ii*) probable error of the mean.

2. The following values were recorded for a triangle *ABC*, the individual measurements being uniformly precise :

$A = 62° 28' 16''$; 6 obs.

$B = 56° 44' 36''$; 8 obs.

$C = 60° 45' 56''$; 6 obs.

Find the correct values of the angles. (*B.U.*)

3. At a station *O* in a triangulation survey, the following results were obtained :

Angle	Observed Values °	′	″	Weight
AOB	67	14	32.4	1.1
BOC	75	36	21.5	1.0
COD	59	56	02.0	0.7
DOE	83	24	17.1	0.8
EOA	73	48	45.0	1.4

The weights are proportional to the reciprocals of the squares of the probable errors. Adjust the angles. (*R.T.C.*)

4. The observations closing the horizon at a station are

$A = 24° 22' 18''.2$ Weight 1

$B = 30° 12' 24''.4$ Weight 2

$A + B = 54° 34' 48''.6$ Weight 3

$C = 305° 25' 13''.9$ Weight 2

$B + C = 335° 37' 38''.0$ Weight 3

Find the most probable values of the angles *A*, *B* and *C* (*P.U.*)

5. Adjust the angles α and β, observations of which give

$\alpha = 20° 10' 10''$ weight 6

$\beta = 30° 20' 30''$ weight 4

$\alpha + \beta = 50° 30' 50''$ weight 2 (U.B.)

6. The following values of angles were measured at a station :

$a = 20° 10' 14''$ weight 2

$b = 30° 15' 20''$ weight 2

$c = 42° 02' 16''$ weight 3

$a + b = 50° 25' 37''$ weight 3

$b + c = 72° 17' 34''$ weight 3

$a + b + c = 92° 27' 52''$ weight 1

Find the most probable values of the angles *a*, *b* and *c*.

7. *A*, *B*, *C*, *D* form a round of angles at a station so that $A + B + C + D = 360°$.

Their observed values were

$A = 76° 24' 40''$; $B = 82° 14' 25''$

$C = 103° 37' 50''$; $D = 97° 43' 15''$

The angle $B + C$ was also separately measured *twice* and found to average 185° 52′ 20″. Find the probable values of each of the four angles if all six measurements were of equal accuracy. (*U.L.*)

8. Adjust the following station observations :

$A = 34° 18' 20''.4$	weight	1
$B = 28° 32' 12''.8$	weight	2
$C = 22° 48' 32''.6$	weight	2
$A + B = 62° 50' 29''.6$	weight	2
$A + B + C = 85° 39' 08''.6$	weight	1.

(B.U.)

ANSWERS

1. (i) ± 19".83 ; 6".12
2. Correction : 3".69 ; + 2".77 ; + 5".54
3. *AOB* 32".74 ; *BOC*, 21".88 ; *COD*, 2".54 ; *DOE*, 17".57, *EOA*, 45".27 .
4. Correction for *A* : + 3".192
   Correction for *B* : + 0".685 .
5. α = 20° 10' 02" ; β = 30° 20' 48"
6. α = 20° 10' 15".9
   b = 30° 15' 20".2
   c = 42° 02' 14".9
7. *A* = 76° 24' 34".17 ; *B* = 82° 14' 25".83
   C = 103° 37' 50".83 ; *D* = 97° 43' 09".17
8. $C_A = -1''.07$; $C_B = -0''.53$; $C_C = +1''.47$.

10

Topographic Surveying

10.1. INTRODUCTION

Topographic surveying is the process of determining the positions, both on plan and elevation, of the natural and artificial features of a locality for the purpose of delineating them by means of conventional signs upon a *topographic map*. By *topography* is meant the shape or configuration of the earth's surface. The basic purpose of the topographic map is to indicate the three dimensional relationships for the terrain of any given area of land. Thus, on a topographic map, the relative positions of points are represented both horizontally as well as vertically. The representation of the difference in elevation is called the *relief*. On a plan, the relative altitudes of the points can be represented by *shading hachures, form lines or contour lines*. In addition to the relief, the topographic map depicts natural features such as streams, rivers, lakes, trees etc. as well as artifcial features such as highways, railroads, canals, towns, houses, fences and property lines. The topographic maps are very essential for the planning and designing of the most engineering projects such as location of railways, highways, design of irrigation and drainage systems, the development of water power, layout of industrial plants and city planning. Topographic maps are also very useful in directing military operations during a war.

10.2. METHODS OF REPRESENTING RELIEF

The system used for showing the relief on a topographic map must fulfil two purposes (*i*) the user of the map should be able to interpret the map as a model of the ground and (*ii*) it should furnish also definite information regarding the elevations of points shown on map. Relief may be represented on a map by hachures, form lines, tinting or contour lines. *Hachures* are a system of short lines drawn in the direction of slope. For a steep slope, the lines are *heavy* and closely spaced, while for a gentle slope, they are fine and widely spaced. While hachures show the surface form, they do not furnish exact information regarding the heights.

A *contour line* is an imaginary line on the ground joining the points of equal elevation. It is a line in which the surface of ground is intersected by a level surface. The relief on a topographic map is most commonly and accurately represented by contours. *Form lines* resemble contours, but are not drawn with the same degree of accuracy. Each form line represents an elevation but has not been determined by sufficient points to conform to the standard of accuracy usually required by contours. *Form lines* are sometimes used on the maps, intended for purpose of navigation, to show peaks and hill tops, along the coast. The relief or elevations may also be indicated by *tinting*. The area lying between two selected contours is coloured by one tint, that between the two others by another tint and so on.

10.3. CONTOURS AND CONTOUR INTERVAL

The system now in general use for representing the form of the surface is that employing contour lines. The elevations of the contours are known definitely, and hence the elevation of any point on ground may be derived from the map. At the same time, this system makes the form or relief apparant to the eye. Thus, this system fulfils both purposes discussed earlier.

The vertical distance between two consecutive contours is called the *contour interval.* The contour interval is kept constant for a contour plan, otherwise the general appearance of the map will be misleading. The horizontal distance between two points on two consecutive contours is known as the *horizontal equivalent* and depends upon the steepness of the ground. The choice of proper contour interval depends upon the following considerations:

(i) ***The nature of the ground.*** The contour interval depends upon whether the country is flat or highly undulated. A contour interval chosen for flat ground, will be highly unsuitable for undulated ground. For very flat ground, a small interval is necessary. If the ground is more broken, greater contour interval should be adopted, otherwise the contours will come too close to each other.

(ii) ***The scale of the map.*** The contour interval should be inversely proportional to the scale. If the scale is small, the contour interval should be large. If the scale is large, the contour interval should be small.

(iii) ***Purpose and extent of the survey.*** The contour interval largely depends upon the purpose and the extent of the survey. For example, if the survey is intended for detailed design work or for accurate earth work calculations, small contour interval is to be used. The extent of survey in such cases will generally be small. In the case of location surveys, for lines of communications, for reservoirs and drainage areas, where the extent of survey is large, a large contour interval is used.

(iv) ***Time and expense of field and office work.*** If the time available is less, greater contour interval should be used. If the contour interval is small, greater time will be taken in the field survey, in reduction and in plotting the map.

Considering all these aspects, the contour interval for a particular contour plan is selected. This contour interval is kept constant in that plan, otherwise it will mislead the general appearance of the ground. Table 10.1 suggests some suitable values of contour interval. Table 10.2 suggests the values of contour interval for various purposes.

For general topographical work, the general rule that may be followed is as follows:

$$\text{Contour interval (metres)} = \frac{25}{\text{No. of cm per km}}$$

or

$$\text{Contour interval (feet)} = \frac{50}{\text{No. of inches per mile}}$$

TABLE 10.1

Scale of map	*Type of ground*	*Contour interval (metres)*
Large (1 cm = 10 m or less)	Flat	0.2 to 0.5
	Rolling	0.5 to 1
	Hilly	1, 1.5 or 2
Intermediate (1 cm = 10 to 100 m)	Flat	0.5, 1 or 1.5
	Rolling	1, 1.5 or 2
	Hilly	2, 2.5 or 3
Small (1 cm = 100 m or more)	Flat	1, 2 or 3
	Rolling	2 to 5
	Hilly	5 to 10
	Mountainous	10, 25 or 50

TABLE 10.2

Purpose of survey	*Scale*	*Contour interval (metres)*
1. Building sites	1 cm = 10 m or less	0.2 to 0.5
2. Town planning schemes, reservoirs etc.	1 cm = 50 m to 100 m	0.5 to 2
3. Location surveys	1 cm = 50 m to 200 m	2 to 3

10.4. CHARACTERISTICS OF CONTOURS

The following characteristic features may be used while plotting or reading a contour plan or topographic map :

1. Two contour lines of different elevations cannot cross each other. If they did, the point of intersection would have two different elevations, which is absurd. However, contour lines of different elevations can intersect only in the case of an overhanging cliff or a cave.

2. Contour lines of different elevations can unite to form one line only in the case of a vertical cliff.

3. Contour lines close together indicate steep slope. They indicate a gentle slope if they are far apart. If they are equally spaced, uniform slope is indicated. A series of straight, parallel and equally spaced contour represent a plane surface.

4. A contour passing through any point is perpendicular to the line of steepest slope at that point. This agrees with (2) since the perpendicular distance between contour lines is the shortest distance.

5. A closed contour line with one or more higher ones inside it represents a hill. Similarly, a closed contour line with one or more lower ones inside it indicates a depression without an outlet.

6. Two contour lines having the same elevation cannot unite and continue as one line. Similarly, a single contour cannot split into two lines. This is evident because a

ngle line could, otherwise, indicate a knife-edge ridge or depression which does not occur n nature. However, two different contours of the same elevation may approach very near each other.

7. A contour line must close upon itself, though not necessarily within the limits f the map.

8. Contour lines cross a watershed or ridge line at right angles. They form curves f U-shapes round it with the concave side of the curve towards the higher ground.

9. Contour lines cross a valley line at right angles. They form sharp curves of -shape across it with convex side of the curve towards the higher ground. If there is stream, the contour on either sides, turning upstream, may disappear in coincidence with e edge of the stream and cross underneath the water surface.

10. The same contour appears on either side of a ridge or valley, for the highest rizontal plane that intersects the ridge must cut it on both sides. The same is true of e lower horizontal plane that cuts a valley.

0.5. PROCEDURE IN TOPOGRAPHIC SURVEYING

The field work in a topographic surveying consists of three parts : (*i*) establishing rizontal control as well as vertical control, (*ii*) locating the contours and (*iii*) locating etails such as rivers, streams, lakes, roads, railways, houses, trees etc. The establishment f horizontal and vertical control system is the most essential part and is the first step topographic survey, since the three co-ordinates of a point (*i.e.*, two co-ordinates in e horizontal plane to locate it horizontally, and one co-ordinate in the vertical plane to cate it vertically with respect to the datum) can be established or measured only with spect to well connected horizontal and vertical control systems.

(1) Horizontal Control. The horizontal control forms the skeleton of the survey from hich contours and other details are located. When the area to be surveyed is small, e horizontal control may consist of one single station, and the distance and direction each point can be measured with respect to this station. When the area is relatively rge, the horizontal control may consist of a traverse or a series of connected traverses. he traverses may be run with the help of tape-compass, plane table or tape-transit, depending on the extent of the area. Sometimes, length of the traverse sides are determined with e help of stadia measurements, specially when the land is uneven. A stadia traverse thus used in the survey of an uneven area of moderate size. On very extensive surveys, e horizontal control may be either a simple or a very elaborate triangulation system, lditional control being provided by traverses connecting the triangulation stations. Secondary averses are sometimes run with the plane table. In flat and densely wooded country, here triangulation is impracticable or very expensive, the primary horizontal control may established by precise traversing.

(2) Vertical Control. Vertical control establishes a frame-work with reference to hich the elevation differences are determined. This control is very important since the pographic map must indicate the relief or the third dimension. The object of the vertical ntrol is to determine the elevations of the primary control stations or to establish bench arks near them and at convenient interval. High order spirit level circuits are run to termine elevations accurately defining the position of all the control points. Trigonometric

levelling is often used to transfer elevations from precise levelling circuits to triangulation stations, these stations generally being located on high, commanding points, while the levels are run, in so far as possible, over level or gently sloping terrain. The secondary vertical control is then established by determining the elevations of traverse stations or bench marks near them. This can be established by tacheometric method or by spirit levelling. When level is used, the elevations of control points can be determined by running circuits of levels, or bench marks can be set in such positions that they can be seen from nearby horizontal-control points. For rough work, barometric levelling may be used.

Locating Details. After having located or established the horizontal and vertical control, the detail is located from the control points by the measurement of angles and distances to those points which are to appear on the finished map. The three co-ordinates of any point (or details) can be determined or computed by the measurement of (*i*) direction of that point from the control point, (*ii*) distance of the point from the control point and (*iii*) elevation of the point. Angles may be measured with the help of compass or transit, or graphically by plane table. Distance can be measured with a chain or a tape, or determined by tacheometric observations. The elevation of the point may be obtained by hand level or the engineer's level, or may be calculated from stadia or horizontal distances and vertical angles. The map is prepared by plotting first the control points and then the detail. The contour lines are drawn next and then the relief is depicted by means of conventional signs.

10.6. METHODS OF LOCATING CONTOURS

The location of the points in topographic survey involves both horizontal as well as vertical control. The methods of locating contours, therefore, depend upon the instruments used. In general, however, the field method may be divided into two classes :

(*a*) The direct method. (*b*) The indirect method.

In the *direct method*, the contour to be plotted is actually traced on the ground. *Only those points are surveyed which happen to be plotted.* After having surveyed those points, they are plotted and contours are drawn through them. The method is slow and tedious and is used for small areas and where great accuracy is required.

In the *indirect method*, some suitable guide points are selected and surveyed, the guide points need not necessarily be on the contours. These guide points, having been plotted, serve as a basis for the interpolation of contours. This is the method most commonly used in engineering surveys.

(*a*) Direct method.

As stated earlier, in the indirect method each contour is located by determining the positions of a series of points through which the contour passes. The operation is also sometimes called *tracing out contours*. The field work is two-fold :

(*i*) vertical control : location of points on the contour.

(*ii*) horizontal control : survey of those points.

(*i*) *Vertical control.* The points on the contours are traced either with the help of a level and staff or with the help of a hand level. In the former case, the level is set at a point to command as much area as is possible and is levelled. The staff is kept on the B.M. (Bench Mark) and the height of the instrument is determined. If the

B.M. is not nearby, fly-levelling may be performed to establish a Temporary Bench Mark (T.B.M.) in that area. Having known the height of the instrument, the staff reading is calculated so that the bottom of the staff is at an elevation equal to the value of the contour. For example, if the height of the instrument is 101.80 metres, the staff reading to set a point on the contour of 100.00 metres will be 1.80 m. Taking one contour at a time (say 100.0 m contour), the staff man is directed to keep the staff on the points on the contour so that readings of 1.80 m are obtained every time. In Fig. 10.1, the dots represent the points determined by this method.

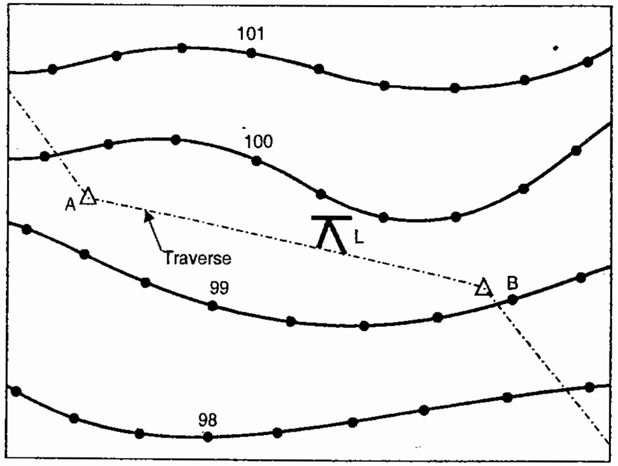


FIG. 10.1

If a hand level is used, slightly different procedure is adopted in locating the points on the contour. A ranging pole having marks at every decimetre interval may be used in conjunction with any type of hand level, preferably an Abney Clinometer. To start with, a point is located on one of the contours, by levelling from a B.M. The starting point must be located on the contour which is a mean of those to be commanded from that position. The surveyor then holds the hand level at that point and directs the rod man till the point on the rod corresponding to the height of the instrument above the ground is bisected. To do this conveniently, the level could be held against a pole at some convenient height, say, 1.50 m. If the instrument (*i.e.* the hand level) is at 100 m contour, the reading of the rod to be bisected at each point of 100.5 m, the rod reading to be bisected with the same instrument position will be (1.50 – 0.5) = 1.0 m. The work can thus be continued. The staff man should be instructed to insert a lath or twig at the points thus located. The twig must be split to receive a piece of paper on which the R.L. of the contour should be written.

(*ii*) ***Horizontal control.*** After having located the points on various contours, they are to be surveyed with a suitable control system ; the system to be adopted depends mainly on the type and extent of area. For small area, chain surveying may be used and the points may be located by offsets from the survey lines. In a work of larger nature, a traverse may be used. The traverse may use a theodolite, or a compass or a plane table as the principal instrument.

In the direct method, two survey parties generally work simultaneously—one locating the points on the contours and the other surveying those points. However, if the work is of a small nature, the points may be located first and then surveyed by the same party. Thus in Fig. 10.1, the points shown by dots are surveyed with respect to points *A* and *B* which may be tied by a traverse shown by chain-dotted lines.

(*b*) **Indirect method.**

In this method, some guide points are selected along a system of straight lines and their elevations are found. The points are then plotted and contours are drawn by interpolation. These guide points are not, except by coincidence, points on the contours to be located. While interpolating, it is assumed that slope between any two adjacent guide points is uniform. The following are some of the indirect methods of locating the ground points :

(1) ***By squares*** (Fig. 10.2). The method is used when the area to be surveyed is small and the ground is not very undulating. The area to be surveyed is divided into a number of squares. The size of the square may vary from 5 to 20 m depending upon the nature of the contours and contour interval. The elevations of the corners of the square are then determined by means of a level and a staff. The contour lines may then be drawn 'by interpolation.' It is not necessary that the squares may be of the same size. Sometimes rectangles are also used in place of squares. When there are appreciable breaks in the surface between corners, guide points in addition to those at corners may also be used. The squares should be as big as practicable, yet small enough to conform to the inequalities of the ground and to the accuracy required. The method is also known as *spot levelling*.

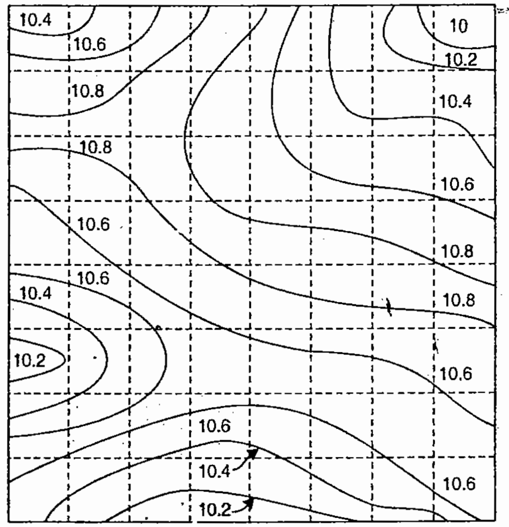


FIG. 10.2.

(2) ***By cross-sections.*** In this method, cross-sections are run transverse to the centre line of a road, railway or canal etc. The method is most suitable for route survey. The spacing of the cross-section depends upon the character of the terrain, the contour interval and the purpose of the survey. The cross-sections should be more closely spaced where the contours curve abruptly, as in ravines or on spurs. The cross-section and the points can then be plotted and the elevation of each point is marked. The contour lines are then interpolated on the assumption that there is uniform slope between two points on two adjacent contours. Thus, in Fig. 10.3, the points marked with dots are the points actually surveyed in the field while the points marked × on the first cross-section are the points interpolated on contours.

The same method may also be used in *direct method* of contouring with a slight modification. In the method described above, points are taken *almost* at regular intervals on a cross-section. However, the contour points can be located directly on the cross-section, as in the direct method. For example, if the height of the instrument is 101.80 m and if it is required to trace a contour of 100 m on the ground, the levelling staff readings placed on all such points are 1.80 m, and all these points will be on 100 m contour.

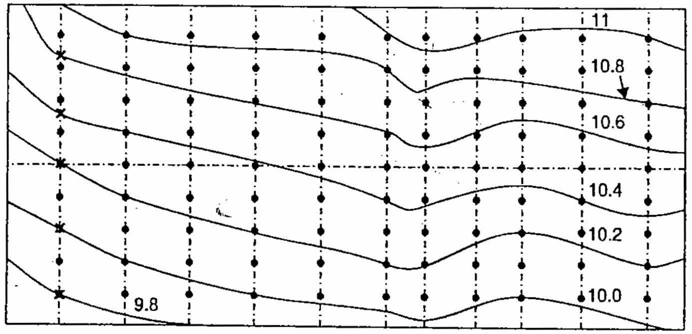


FIG. 10.3.

The guide points of different contours are determined first on one cross-line and then another instead of first on one contour and then on another, as in the direct method.

If there are irregularities in the surface between two cross-lines, additional guide points may be located on intermediate cross-lines. If required, some of the cross-lines may also be chosen at any inclination other than 90° to the main line.

(3) ***By tacheometric method.*** In the case of hilly terrain, the tacheometric method may be used with advantage. A tacheometer is a theodolite fitted with stadia diaphragm so that staff readings against all the three hairs may be taken. The staff intercept s is then obtained by taking the difference between the readings against the top and bottom wires. The line of sight can make any inclination with the horizontal, thus increasing the range of instrumental observations. The horizontal distances need not be measured, since the tacheometer provides both horizontal as well as vertical control.

A tacheometer may be set on a point from where greater control can be obtained. Radial lines can then be set making different angles with either the magnetic meridian or with the first radial line. On each radial line, readings may be taken on levelling staff kept at different points. The point must be so chosen that the approximate vertical difference in elevation between two consecutive points is less than the contour interval. Thus on the same radial line the horizontal equivalent will be smaller for those two points the vertical difference in elevation of which is greater and *vice versa*.

To survey an area connected by series of hillocks, a tacheometric traverse may be run, the tacheometric traverse stations being chosen at some commanding positions. At each traverse station, several radial lines may be run in various directions as required, the horizontal control being directly obtained by the tacheometer. The traverse, the radial lines and the points can then be plotted. The elevation of each point is calculated by tacheometric formulae and entered, and the contours can be interpolated as usual.

10.7. INTERPOLATION OF CONTOURS

Interpolation of the contours is the process of spacing the contours proportionately between the plotted ground points established by indirect method. The methods of interpolation are based on the assumption that the slope of ground between the two points is uniform. The chief methods of interpolation are :

(*i*) By estimation (*ii*) By arithmetic calculations
(*iii*) By graphical method.

(*i*) **By estimation.** This method is extremely rough and is used for small scale work only. The positions of contour points between the guide points are located by estimation.

(*ii*) **By arithmetic calculations.** The method, though accurate, is time consuming. The positions of contour points between the guide points are located by arithmetic calculations. For example, let *A*, *B*, *D* and *C* be the guide points plotted on the map, having elevations of 607.4, 617.3, 612.5 and 604.3 feet respectively (Fig. 10.4). Let $AB = BD = CD = CA = 1$ inch on the plan and let it be required to locate the position of 605, 610 and 615 feet contours on these lines. The vertical difference in elevation between *A* and *B* is $(617.3 - 607.4) = 9.9$ ft. Hence the distances of the contour points from *A* will be :

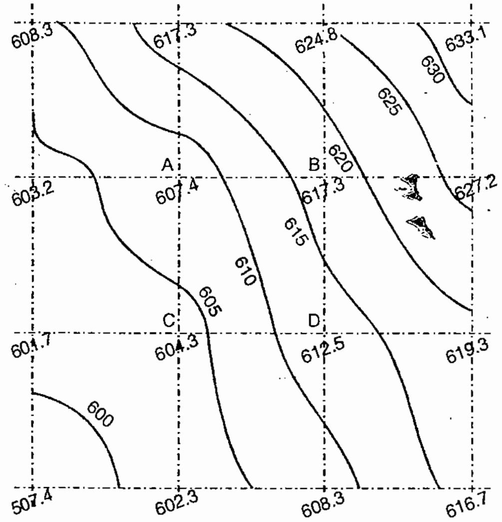


FIG. 10.4.

$$\text{distance of 610 ft. contour point} = \frac{1}{9.9} \times 2.6 = 0.26'' \text{ (approx.)}$$

$$\text{distance of 615 ft. contour point} = \frac{1}{9.9} \times 7.6 = 0.76'' \text{ (approx.)}$$

These two contour points may be located on *AB*. Similarly, the ~~position of the contour~~ points on the lines *AC*, *CD* and *BD*, and also on *AD* and *BC* may be located. Contour lines may then be drawn through appropriate contour points, as shown in Fig. 10.4.

(*iii*) **By graphical method.** In the graphical method, the interpolation is done with the help of a tracing paper on a tracing cloth. These are two methods :

First method.

The first method is illustrated in Fig. 10.5. On a piece of tracing cloth, several lines are drawn parallel to each other, say at an interval representing 0.2 metre. If required, each fifth line may be made heavier to represent each metre interval. Let the bottom line of the diagram so prepared on the tracing cloth represent an elevation of 99 m and let it be required to interpolate contours of 99.5, 100 and 100.5 m values between two points *A* and *B* having elevations of 99.2 and 100.7 m respectively.

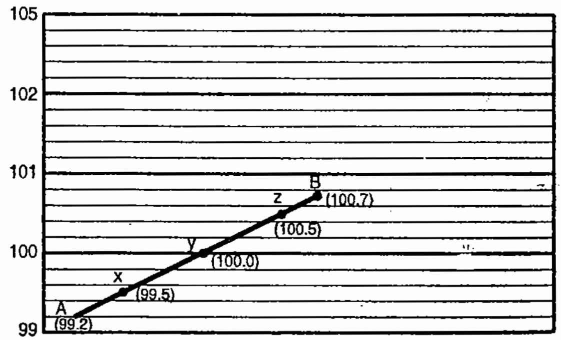


FIG. 10.5.

Keep the tracing cloth on the line in such a way that point A may lie on a *parallel* representing an elevation of 99.2 metres. Now rotate the tracing cloth on drawing in such a way that point B may lie on a parallel representing 100.5 metres. The points at which the parallels representing 99.5 (point x), 100.0 (point y) and 100.5 (point z) may now be pricked to get the respective positions of the contour points on the line AB.

Second method.

The second method is illustrated in Fig. 10.6. A line XY of any convenient length is taken on a tracing cloth and divided into several parts, each representing any particular interval, say 0.2 m. On a line perpendicular to XY at its midpoint, a pole O is chosen and radial lines are drawn joining the pole O and the division on the line XY. Let the bottom radial line represent an elevation of 97.0. If required, each fifth radial line representing one metre interval may be made dark. Let it be required to interpolate contours of 98, 99, 100 and 101 metres elevation between two points A and B having elevations of 97.6 and 101.8 metres. Arrange the tracing cloth on the line AB in such away that the points A and B lie simultaneously on radial lines representing 97.6 and 101.8 metres respectively. The points at which radial lines of 98, 99, 100 and 101 metres intersect AB may then be pricked through.

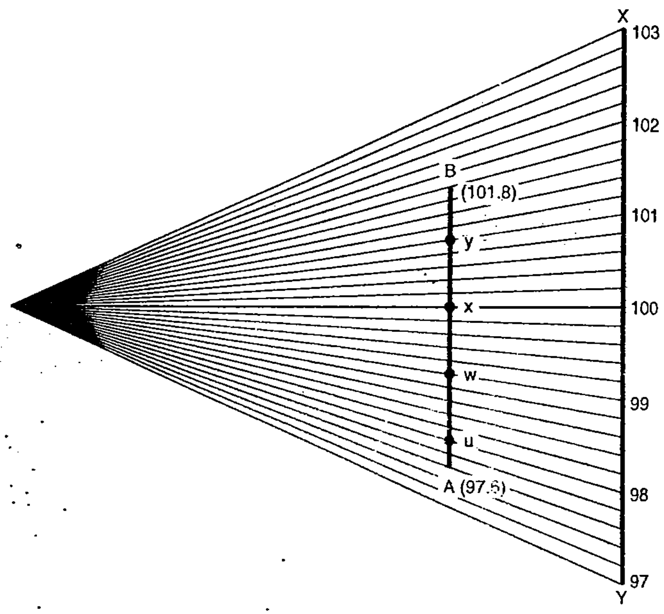


FIG. 10.6.

Contour drawing. After having interpolated the contour points between a network of guide points, smooth curve of the contour lines may be drawn through their corresponding contour points. While drawing the contour lines, the fundamental properties of contour lines must be borne in mind. The contour lines should be inked in either black or brown. If the contour plan also shows some of the features like roads etc., it is preferable to use brown ink for contour so as to distinguish it clearly from rest of the features. The value of the contours should be written in a systematic and uniform manner.

11

Route Surveying

11.1. INTRODUCTION

Surveys along a comparatively narrow strip of territory for the location, design and construction of any route of transportation, such as highways and railroads, aqueducts, canals and flumes, pipe-line for water, sewage, oil and gas, cableways and belt conveyors and power, telephone and telegraph transmission line is called *route surveying*. Route surveying includes all field work and requisite calculations, together with maps, profiles and other drawings.

Route surveys are done with two main aims : (1) determining the best *general route* between the termini and (2) fixing the alignment grades and other details of the *selected route*. Engineering principles require that the route be chosen in such a way that the project may be constructed and operated with the greatest economy and utility.

A comprehensive route survey consists of the following sequence of surveys :

(1) *Reconnaissance* of the terrain between the termini.
(2) *Preliminary surveys* over one or more locations along the general route recommended in the reconnaissance report.
(3) *Location survey*.
(4) *Construction survey*.

11.2. RECONNAISSANCE SURVEY

A reconnaissance survey is a rapid but thorough examination of an area or a strip of territory between the termini of the project to determine which of the several possible routes may be worthy of a detailed survey. Reconnaissance survey is the most important of the series of surveys mentioned above. A considerable amount of time and expense may be saved on unusable instrument surveys if the most desirable line is obtained during the reconnaissance. A very thorough and exhaustive examination of the whole area should be made to ensure that no possible route has been overlooked. Each route may be studied, and all but the most desirable eliminated. The work of reconnaissance must be entrusted to a very experienced engineer. The *locating engineer* should be endowed with a rare combination of technical thoroughness, business, judgement and visionary foresight, to enable him to select a route that will satisfy the future demand, in so far as this may be economically justified.

One of the first steps on reconnaissance is to assemble and study all the available maps of the territory to be covered, such as the Survey of India maps. If the available maps prove inadequate, aerial observation and photographs of the area or the several routes under consideration may be traversed on foot, horse-back, automobile or any other locally available conveyance.

Reconnaissance instruments. Distances are usually taken from the reference maps, or they may be checked roughly by pacing or pedometer, if the reconnaissance is made on foot. The directions of lines may be observed by means of a prismatic compass. In rough terrain or heavily wooded areas, some form of compass is indispensable to field reconnaissance. The relative elevations of points may be determined directly from the topographic map, if available. Whenever such maps are not available, aneroid barometer may be used to determine elevations. Recent improvements of the aneroid have furnished engineer with an instrument capable of measuring differences of elevations to an accuracy of from $\frac{1}{2}$ to $1\frac{1}{2}$ m. The relative slope of the ground or approximate difference in elevation may be obtained by the use of a level or a clinometer.

The notes and records may be marked directly on an existing map, or notes may be kept separately in a narrative form. The reconnaissance map, sketched on an existing map or made separately should show the several routes that are practicable, the controlling points, important topographic features, and all other details that may possibly be helpful in the selection of the route.

General information. The map is supplemented by reconnaissance notes which may contain the following :

(1) The general *topography* of the country or the character of the terrain between termini or major controlling points, as it is apt to impress to a characteristic pattern upon a route location, particularly in the case of a highway or a railroad. Terrain may be generally classified as *level, rolling* or *mountainous*.
(2) Possible ruling gradients.
(3) Stream crossings, which require a careful study of rate of flow, high-water elevation, flood conditions, character of banks, and the width of the stream. A suggested type of structure with most desirable points of crossing should be noted on the map.
(4) Information about railroad or other highway crossings.
(5) Obligatory points, such as intermediate towns, markets or production centres. Bypass locations should be indicated in the notes for all small towns and cities for the more important routes. Provisions should also be considered for connections of the route of these centres of population.
(6) Geological characteristics of the area affecting foundations for bridges etc. and stability of the line should be observed and noted. The presence of rock outcrop, swamps, varying soil types and dangerous possibilities of landslides is very important.
(7) Availability of building materials and labour, and sites of quarries etc. nearby the proposed route.
(8) Value of the land to be acquired.

Selection of route. From the reconnaissance survey and detailed notes collected, only one or two routes are selected, deserving further detail study. The recent development in the art of preparing topographic maps from aerial photographs has served as an invaluable aid in the selection of routes. The use of aerial maps for route selection does not obviate the need for ground surveys, but the expense of time-consuming reconnaissance and preliminary surveys may be reduced to minimum by the use of modern methods. A route may have three locations : (*i*) valley location, (*ii*) cross-country location, and (*iii*) ridge line location.

In the case of *valley location*, the route follows the valleys and the drainage lines of an area, and has few excessive grades. There is often danger of washouts and floods. A number of bridges may be required to cross the tributory streams. The reconnaissance of the route should include the entire valley since it often happens that a more advantageous location is achieved by crossing the valley at strategic points. In the case of cross-country location, the line is located in opposition to the drainage. Such a line crosses the ridges very often, and will have steep grade. The construction costs along such a line may also be excessive. Location along *ridges* are relatively free of drainage problems and major drainage structure. However, since ridges are seldom straight, considerable curvature may have to be employed in such a location. Also, steep grades are encountered when the location drops into valleys or when the ridge is regained.

11.3. PRELIMINARY SURVEY

A preliminary survey is detailed survey of a strip of territory through which the proposed line is expected to run. The preliminary survey is made of the best of the several lines investigated previously on the reconnaissance survey. The purpose is to prepare an accurate topographic map of the belt of country along the selected route, and thus arrive at a fairly close estimate of the cost of the line. Also construction plans are prepared from the preliminary survey.

Preliminary surveys differ greatly in method and precision. Invariably, however, there is at least one traverse (compass, stadia or transit-and-tape) which serves as a framework for the topographical details. Elevations along the traverse line and the measurements to existing physical features are essential. When there are two or more routes under consideration, a preliminary survey of each may be made and from a study of the maps and other data thus obtained, the final selection of a route is determined. The width of strip to be surveyed depends upon the type of the project. For highways, this width varies from 100 to 200 metres, while for railways it may be as big as 500 metres. The width of the strip also depends upon the character of the country. The strip is narrow in places where the final location is obviously restricted to a narrow area (such as in hilly country) and wide in places where the position of final location is not so evident (such as in flat country).

The following instruments are generally employed for preliminary survey : (*i*) the transit (*ii*) the compass (*iii*) the engineer's level (*iv*) hand level or Abney level or any other clinometer (*v*) levelling staves (*vi*) chains and tapes (*vii*) plane table (*viii*) substense bar (*ix*) miscellaneous equipment like ranging rods, pegs etc.

On small projects, the entire preliminary survey is done by one party. On big projects, the survey work is done by three parties under the general supervision of the *location* *engineer* : (1) the transit party, (2) the level party, and (3) the topography or cross-section party. The locating engineer is directly responsible for the location of the line, the prosecution of the work, the conduct of the survey, and all the other relating matters.

The *transit party* usually consists of four to seven men—chief of party, transitman, and two to four helpers. The survey work consists in *open traversing* with a transit along the selected route. In the case of highways, the traverse is usually run by the deflection angles, while in the case of railways, it is run by the method of back angles. The azimuths of the first and the last lines of the traverse are determined by astronomical observations. In the case of long traverse, azimuths are taken at about 20 km intervals. The *transitman* reads and records all angles, bearings and distances. Unless a special topography party is organised for the purpose, the transitman also records the topography notes that may have influence in determining the final location, such as the position and bearing of streams, drainage structures, property lines, intersecting roads or railroads and pipe lines. The plane table can sometimes be used to advantage particularly in rough country which is comparatively free from bushes, trees, or other obstructions to sightings. The transit-stadia method may also be used for making a preliminary survey. The transit-stadia method is rapid and economical, though less precise. If such a method is adopted, alignment, elevation and topographic details are carried in one operation by single party.

The *level party* includes three men : level man, rod man and note keeper. Sometimes, the level man himself records the data. The level party does important jobs : it establishes bench marks along the proposed route at regular and convenient places, (*ii*) it runs a longitudinal section of the traverse lines. The bench marks so obtained form the vertical control for the survey. Bench marks should be set on permanent objects and be carefully described in the notes. The elevations of the ground at all stakes on the transit line, points of change in slope and at intersections with roads, streams, railways etc. are determined. The levelling work is checked by taking observations on existing permanent bench marks and G.T.S bench marks. Each day's work is plotted from the profile level notes.

The *topography or cross-section* consists of three or four men —level man, rod man, chain man and recorder. The instruments used are ordinarily a hand level, rod and tape. If the ground is not very abrupt, an engineer's level is sometimes substituted for the hand level. However, a hand level is often used instead of the engineer's level with greater speed and with an accuracy within 0.05 m.

The topographer should prepare his note-book in advance with the alignment and elevation of the centre line points that he expects to cover the next day. This information can be supplied by the transit and level parties from their previous work. Cross-sections are set out at every 30 m stations, at right angles to the traverse lines and on either side of it by an optical square. In hilly and mountainous country, the distance between the cross-sectional lines may be reduced to 10 metres, while in flat or level country, it may be increased to 100 m. The topographer also records and sketches the natural and artificial features of the topography. The extent of the topography included in the notes will depend on the character of the ground being traversed, the speed and accuracy of the topo-grapher and his ability to visualise the features affecting the choice of the final line. At places where a change in horizontal direction is necessary, a wider strip of topography may enable the *locating engineer* to choose a more desirable final alignment when studying

the finished map. The most important features to be noted in taking of topography are the contours, streams and general character of the land traversed.

Paper location. The transit line of the preliminary survey is a series of straight lines, and no horizontal curves are introduced on this survey. After the survey is complete, a complete map is prepared on a scale of 1 cm=40 m or 1 cm=50 m. On this map, the final line can be located to include as much of the tangent lines as thought to be advisable, with horizontal curves introduced at points where changing in direction appear feasible. This new line drawn on the map is known as the *paper location*, and it can be located anywhere within the strip that has been given to all the features affecting the location. The factors affecting the choice of the *paper location* are : (1) minimum gradients and curvature, (2) balancing (equalization) of earth work, (3) heavy earthwork, (filling or cutting), (4) suitable crossing for rivers etc. The paper location of the alignment and grade line form a basis for the final location of the line. Several trials are necessary before the final location is marked on the preliminary survey map, fulfilling all the major requirements of the best location. In order to avoid excessive erasing of the pencil lines during the trial and error process of paper location, it is advisable to use a fine silk thread and needle. The first step is usually to stick pins at the controlling points and to stretch the thread from point to point. Portion of the thread lines between controlling points are then shifted back and forth until the line appears to fulfill the conditions of a good alignment. For each trial position of the thread-line location, a corresponding profile may be made from the topographic map and grade lines are then laid out with a thread on the profile. An adjustment of grade lines on the profile may require a corresponding shift in location on the map. When the thread on the map has been adjusted to show the final location of the centre line, the portions of the thread between pins represent tangents which are to be connected by curves. The line of location including curves, may now be *drawn* on the map to replace the thread lines and the so called *paper location* is completed.

11.4. LOCATION SURVEY

The location survey is the ground location of proposed line marked on the map, *i.e.*, it consists in laying out the *paper location* on the ground. The main purpose of location survey is to make minor improvements on the line as may appear desirable on the ground, and to fix up the final grades. The line as finally located on the ground is called the *field location*. The tangents, curves and drainage structure are established by means of a continuous transit survey, taking into account whatever adjustments in the line and grade appear to be advisable. Profile levels are run over the centre line, bench marks are established, and profile made which shows the ground line and the grade line. All other lines needed in construction are established with reference to the centre line. Cross-section notes are taken in order that the quantity of earth work may be computed. The notes are taken at every full station and at intermediate points along the line where the ground slope changes abruptly. These notes are taken by means of a rod and level on a line perpendicular to the centre line. On curves, the notes are taken on a radial line at the point of observations. Great care should be exercised to take the observations at 90° to the centre line to prevent serious error in the earthworth calculations, where the cut or fill is likely to be large. All important features in the close proximity of the located lines are also surveyed. The boundaries of private properties, with names of owners, are surveyed very accurately for purposes of acquisition of land securing rights of way. All necessary field data are obtained to permit the detailed design of miscellaneous structures.

11.5. CONSTRUCTION SURVEY

The purpose of the construction survey is to re-establish points, lines and grades on the ground during construction. It also consists in staking out various details such as culverts and bridges, and in carrying on such other surveying as may be needed for purpose of construction. Following are the surveying operation for construction purpose :

(1) retracing the centre line shown on the plan and referencing certain points on the curves ;

(2) checking bench marks and running centre-line levels over the retraced line ;

(3) taking elevations at all stations, at all breaks in the ground, and at other points where it is necessary to take cross-sections for volume quantities ;

(4) setting slope stakes and grade stakes ;

(5) setting stakes for the complete layout of culverts and bridges ;

(6) setting out curves ;

(7) reporting and making advantageous changes, if any, in line or grade or minor adjustments of the drainage structures ;

(8) progress reports ;

(9) final estimate etc.

As the work progresses, the stakes that have been destroyed must be reset.

12

Special Instruments

12.1. INTRODUCTION

We have, so far, studied the construction and working of usual instruments (such as compass, levels, theodolites, plane table etc.) commonly employed for surveying and setting out operations. However, we shall now study the special instruments employed for specific purposes. Some of such instruments were introduced in volume 1. Here, we will consider the following special instruments :

1. Jig telescope and Jig transit
2. Collimator
3. Telemeter
4. Altimeter
5. Electronic theodolite

12.2. JIG TELESCOPE AND JIG TRANSIT

Jig telescopes and jig transits are used for *optical tooling*– an essential part of industrial surveying. Originally, conventional transit and level were used for industrial surveying for many years. However, during World War II, rapid development of the aircraft and ship-building industries and other industries involved in the construction and installation of large and heavy machinery, including aligning journals in generators and turbines, needed high precision measurements. Modern industrial layouts and shop practices permit dimensional tolerances of only a few thousandth of a centimetre. This led the development of jig alignment telescope and jig tranist.

Jig Alignment Telescope

Fig 12.1 show a *jig alignment telescope* or *micrometer alignment scope* which is equipped with *optical micrometers* which measure the distances to the thousandth of an inch, that the line of sight is moved up or down, parallel to itself, when the micrometer knobs are turned.

The telescope is mounted in a socket at one end of the Jig. The target is similarly supported at the other end. An optical reference line is established when the cross-hairs of the telescope are centered on the target. The telescope is so designed that it can be focussed from infinity to a point in contact with the front end of the telescope. The optical micrometers when used in conjunction with precision optical tooling scales can be used with ranges 10 mm, 0.5 in. and 0.02 ft. The smallest interval on the drum is 0.2 mm, 0.01 in. and 0.001 ft. respectively.

12.3. COLLIMATORS

Collimators are used as reference marks in instrument workshops and for optical tooling. A telescope can be converted into a collimator by focussing to infinity and fitting an eye-piece lamp in place of the standard eye-piece.

A collimator manufactured by Otto Fennel can be levelled by means of four levelling screws. Similarly, Wild manufactures a *workshop* collimator for permanent mounting. However, there are T 2, T 3 or even T4 telescopes *without eye-piece* but with built-in reticule illumination. Special reticule patterns are available. Pointing to a collimator is like pointing to a perfect target at infinity. Neither the position of the instrument nor the distance from the collimator have any influence on the measurement or its acuracy. For interchangeability between eye-pieces and eye-piece accessories, the eye-piece mount has a bayonet fastening. Wild eye-piece No. 53 of 40X is useful for optical tooling and laboratory measurements, as well illuminated targets can be pointed with high accuracy. Because only the centre of field of view is used, the fall-off in image quality around the edge with very high magnification has no influence on the measurements.

Auto-collimation

Auto-collimation is the process of making the telescope line of sight perpendicular to a plane mirror. It is used in *optical tooling* and laboratory work for alignment and measuring small deflections. Wild theodolites converted into autocollimators are often used for measurements in laboratories and workshops. Autocollimation offers many advantages and is particularly suitable for the exact definition of reference directions and planes, the determination of minute angular changes and deviations, the setting out and checking of perpendiculars, the calibration of angle measuring devices etc. Fig. 12.6 shows an autocollimation eye piece, fitted to NA2 level, for setting machine parts and instrument components precisely vertical. With the autocollimation eyepiece fitted, the telescope magnification is 24 ×. Fig. 12.7 shows autocollimation eyepiece fitted to T1000 theodolite, which converts it into an autocollimator for measuring tasks in laboratories and industry. Fig. 12.8 shows different versions of autocollimation eyepieces fitted to T1 telescope. The autocollimation eye-piece (1) simply interchanges with the telescope eyepiece. *An autocollimation prism and autocollimation mirror are available.* For steep and vertical sights, there is the *diagonal autocollimation eyepiece* (3). A collimator provides a perfect reference target at infinity and can be obtained by fitting the eyepiece lamp (2) to the telescope. This is particularly useful in instrument workshops and laboratories as well as for *optical tooling*.

For observations, Wild GAS 1 autocollimation mirror is attached to the object. The telescope is focussed to infinity and then pointed to the mirror. A reflected image of the cross-pairs is seen in the field of view. By turning either the telescope or object with mirror, the reticule cross and its reflected image are made to coincide—auto-collimation. The line of sight is then at right angles to the mirror. The mirror is front silvered optically plane, 50 mm in diameter housed in stable titanium housing. Three tapped holes with M 4 thread allow mirror to be attached and adjusted to various mounts.

Angular deviations of the mirror from a preset line of sight can easily be measured. For this, read both circles, turn the telescope to achieve autocollimation and read the circles again. The difference in the readings gives the angular deviations in Hz and V. Autocollimation is independent of distance ; the mirror can even be directly in front of the objective.

If a reference for horizontal angles only is required, or if the theodolite has to be moved and its height changed, the *Wild GAP 1 Autocollimation prism* is used instead of a mirror. It is particularly suitable for machine assembly and alignment, and for checking the parallelism of rollers in steel and paper mills.

12.4. OPTICAL PLUMMETS : ZENITH AND NADIR PLUMMETS

Optical plummets are optical devices used for precise centring. We will discuss here two such devices :

(*i*) Telescope roof plummet (*ii*) ZNL Zenith and Nadir plummet

1. Telescope roof plummet

Fig. 12.9 shows the photograph of the telescope roof plummet. This special optical plummet is used for rapid centring under roof markers in mines and tunnels. It fits the telescope of the T-2 theodolite with the instrument in face right position. Centring accuracy is 1 to 2 mm in 10 m.

2. ZNL Zenith and Nadir plummet (Fig. 12.10)

The Wild ZNL zenith and nadir plummet is used for upward and downward plumbing with 1:30000 accuracy in construction, mining and industry. A separate instrument with detachable tribach, the ZNL interchanges with forced-centring against Wild T-1 and T2 accessories. Fig. 12.10 (*a*) shows the position of the ZNL for zenith (or upward) plumbing while Fig. 12.10 (*b*) shows the position for Nadir (or downward) plumbing.

Wild has two ***automatic plummets*** : the ZL Automatic Zenith plummet and the NL automatic Nadir plummet. These are the plummets of highest precision used for industry, deformation measurements, mining and precise construction. They define the plumb line with 1:200000 accuracy. They interchange with Wild T-1 and T-2 theodolites.

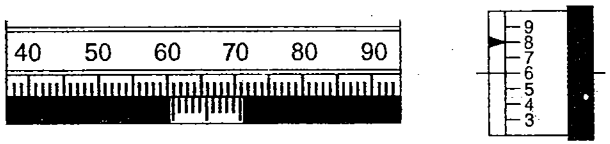
12.5. OBJECTIVE PENTAPRISM

Fig. 12.11 shows the photograph of objective pentaprism by Wild. The pentaprism turns the line of sight through 90°. It is used for plumbing up and down, transferring directions to different levels, and for setting out. The plumbing accuracy is 1:70000.

12.6. TELEMETER

'Telemeter' is a special device which is attached to the objective end of the telescope, to measure directly the horizontal distance. A corresponding counter weight is attached to the telescope near its eye-piece end. The method of measurement is similar to tacheometry, except that the observations are taken on a horizontal rod. Thus, the method use a horizontal base which is variable.

Fig. 12.12 shows special horizontal stadia rod with vernier, used with the telemeter. The image of the vernier is obtained which is displaced with respect to the main graduations of the rod by the amount of deflection. For field measurement over the rod kept at the point under observation, the image of the vernier is first brought into the middle of the field of view. Then the micrometer drum is rotated until a graduation on the vernier



Reading on staff .... 61 m.
Vernier .... 0.5 m.
Reading on drum .... 0.08 m.
Total reading .... 61.58 m.

Estimating the tenth of a drum division would give the millimetres of the distance (in our example 0.002 m).

Wild Heerbrugg Instruments, Inc.

FIG. 12.12. HORIZONTAL TELEMETER ROD WITH VERNIER.

is coincident with a main graduation on the rod. Fig. 12.12 illustrates the method of taking the reading.

It should be noted that the reading obtained with telemeter provide slope distance along the line of sight. This distance must be reduced to the horizontal by multiplying by the cosine of the observed vertical angle. Thus, if observed inclined distance is L and angle of inclination of line of sight is θ, the horizontal distance is given by

$$D = L \cos \theta.$$

12.7. ALTIMETER

An altimeter is an improved version of Aneroid barometer. As discussed in chapter 9, Vol 1, Barometric levelling is used to determine difference in elevation between points, for reconnaissance and preliminary surveys. The results obtained by older aneroid barometer may be erroneous ~~by as much as 8 to 15 m.~~ Requirements of the air plane as well as air survey operations led to the revival of interest in altimetry accompanied by the development of higher quality aneroids called *altimeters*.

Altimetry is the practice of determination of altitude by observations of atmospheric pressure. It depends upon the basic principle that the pressure caused by the weight of column of air above the observer decreases as the observer rises in altiude. A precise *surveying altimeter* is basically an improved version of old aneroid barometer. The instrument is remarkably sensitive to changes in atmospheric pressure. If required, an automatic photographic recorder, called *alti-recorder* can be used at base stations.

Since the altimeter surveying is dependent on the measurement of air density, all the factors, other than the elevation, must be considered that affect air density. One of the major factor is the temperature variation. Altimeters are generally calibrated at a temperature of 50° F. If the observational temperature is higher than 50° F the observed difference in elevation is too small. Conversely, at temperatures below 50° F, the observed difference in elevation is too great. As a thumb rule, the observed difference in elevation should be increased by 0.2 ft for each 100 ft of observed difference of elevation for each degree of temperature above 50° F.

There are two methods employed for altimeter surveys :

(*a*) Single base method and (*b*) Two base method.

Single Base Method

In single base method, two altimeters are employed. One altimeter is kept at a point of known elevation where altimeter readings and theromometer readings are taken at regular intervals. Other altimeter, known as moving altimeter, is taken to the stations where elevations are to be found. Altimeter readings as well as thermometer readings are taken at regular interval. Difference in elevation is then found after taking into account the temperature effects.

Two base method

This method was developed to eliminate the need for temperature and relative humidity corrections. In this method, general accuracy of altimeter surveying is increased. In this method, three altimeters are used. One altimeter is kept at a base station situated at a low point in the area to be surveyed while the other altimeter is kept at the second base situated at a high point. The elevations of both these stations are known from other methods, such as spirit levelling etc. Altimeters kept at these two base stations are read at regular time intervals. A third altimeter, called the *roving altimeter* is then transported to all these points where elevations are to be determined, and altimeter readings along with the time of observations are recorded.

The computations are based on the premise that the changes in the properties of the atmosphere takes place in a linear manner between the two bases. This means that the ratio between the known difference of elevation between the base stations and the difference of their altimeter readings is equal, at a given time, to that between the unknown difference of elevation between a base and field station and the difference of their altimeter readings. If the two base method is used, altimeter survey can be accomplished with an average error of about 3 ft and a maximum error of 10 ft. with the two base stations separated by 10 miles horizontally and 1000 ft vertically.

12.8. ELECTRONIC THEODOLITES

12.8.1. INTRODUCTION

Theodolites, used for angular measurements, can be classifed under three categories:

(*i*) Vernier theodolites
(*ii*) Microptic theodolites (optical theodolites)
and (*iii*) Electronic theodolites

Vernier theodolites (such as Vicker's theodolite) use verniers which have a least count of 10″ to 20″. However, microptic theodolites use optical micrometers, which may have least count of as small as 0.1″. Wild T-1 T-16, T-2, T-3 and T-4 and other forms of Tavistock theodolites fall under this category. Thus the optical theodolites are the most accurate instruments where in the readings are taken with the help of optical micrometers. However in electronic theodolites, absolute angle measurement is provided by a dynamic system with *opto-electronic scanning*. The electronic theodolites are provided with control panels with key boards and liquid crystal displays. The LCDs with points and symbols present the measured data clearly and unambiguously. The key board contains multi-function keys. The main operations require only a single key-stroke. The electronic theodolites work with electronic speed and efficiency. They measure electronically and open the way to elctronic data aquisition and data processing.

We shall consider here the following two models of electronic theodolites manufactured by M/s Wild Heerbrugg Ltd.

(*i*) Wild T-1000 electronic theodolite

(*ii*) Wild T-2000 and T-2000 S electronic theodolite

12.8.2. WILD T-1000 'THEOMAT'

Wild electronic theodolites are known as 'Theomat'. Fig 12.13 shows the photograph of Wild T-1000 electronic theodolite. Although it resembles a conventional theodolite (*i.e.*, optical theodolite) in size and weight, the T-1000 works with electronic speed and efficiency. It measures electronically and opens the way to electronic data aquisition and data processing. It has 30 × telescope which gives a bright, high-contrast, erect image. The coarse and fine focusing ensures that the target is seen sharp and clear. Pointing is fast and precise, even in poor observing conditions. The displays and reticle plate can be illuminated for works in mines and tunnels and at night.

The theodolite has two control panels, each with key-board and two liquid-crystal displays. It can be used easily and quickly in both positions. Fig. 12.14 shows the control panel of T-1000. The LCDs with points and symbols present the measured data clearly and unambiguously. The key-board has just six multifunction keys. The main operations require only a single keystroke. Accepted keystrokes are acknowledged by a beep. Colour-coding and easy-to-follow key sequences and commands make the instrument remarkably easy to use.

The theodolite has an absolute electronic-reading system with position-coded circles. There is no initialization procedure. Simply switch on and read the results. Circle reading is instantaneous. The readings up-date continuously as the instrument is turned. Readings are displayed to 1″. The standard deviation of a direction measured in face left and face right is 3″.

The theodolite has practice-tested *automatic index*. A well-damped pendulum compensator with 1″ setting accuracy provides the reference for T-1000 vertical circle readings. The compensator is built on the same principles as the compensator used in Wild automatic levels and optical theodolites. Thus with T-1000, one need not rely on a plate level alone. Integrated circuits and microprocessors ensure a high level to performance and operating comfort. Automatic self-checks and diagonostic routines makes the instrument easy to use.

T-1000 theodolite has electronic clamp for circle setting and repetition measurements. Using simple commands, one can set the horizontal circle reading to zero or to any value. The theodolite can be operated like a conventional theodolite using any observing procedure, including the repetition method. In addition to the conventional clockwise measurements, horizontal circle readings can be taken counter-clock-wise. Horizontal-collimation and vertical index errors can be determined and stored permanently. The displayed

DIST			Distance measurement
REC			Recording
ALL			Measurement and recording
DSP	Hz		Display Hz-circle and Hz-distance
SET	TRK		Tracking
SET	SET	Hz	Set horizontal-circle reading to zero

FIG. 12.15. TYPICAL COMMANDS IN T-1000 ELECTRONIC-THEODOLITE (WILD HEERBRUGG)

circle readings are corrected automatically. Displayed heights are corrected for earth curvature and mean refraction.

As stated earlier, the whole instrument is controlled from the key-board. Fig. 12.15 gives details of typical commands obtained by pressing corresponding keys. Fig. 12.16 gives typical display values obtained by pressing different keys. The power for T-1000 theodolite is obtained from a small, rechargeable 0.45 Ah Ni Cd battery which plugs into the theodolite standards.

Hz 1373452	V 913756	*Horizontal circle and vertical circle*
Hz 1373454	⊿ 118542	*Horizontal circle and horizontal distance*
V 913755	⊿ 3375	*Vertical circle and height differences*
V 913754	⊿ 118597	*Vertical circle and slope distance*

FIG. 12.16. TYPICAL-DISPLAYS ON THE PANELS OF T-1000

Wild T-1000 theodolite is fully compatible. It is perfectly modular, having the following uses :

(*i*) It can be used alone for angle measurement only.

(*ii*) It combines with Wild *Distomat* for angle and distance measurement.

(*iii*) It connects to GRE 3 data terminal for automatic data aquisition.

(*iv*) It is compatible with Wild theodolite accessories.

(*v*) It connects to computers with RS 232 interface.

Fig. 12.17 depicts diagrammatically, all these functions.

'Distomat' is a registered trade name used by Wild for their *electro-magnetic distance measurement* (EDM) instruments (see chapter 15). Various models of distomats, such as DI-1000, DI-5, DI-5S, DI-4/4L etc. are available, which can be fitted on the top of the telescope of T-1000 theodolite. The telescope can transit for angle measurements in both the positions. No special interface is required. With a Distomat fitted to it, the theodolite takes both angle and distance measurements. Wild DI-1000 distomat is a miniaturized EDM, specially designed for T-1000. It integrates perfectly with the theodolite to form the ideal combination for all day-to-day work. Its range is 500 m on to 1 prism and 800 m to 3 prisms, with a standard deviation of 5 mm + 5 ppm. For larger distances, DI-5S distomat can be fitted, which has a range of 2.5 km to 1 prism and 5 km to 11 prisms. For very long distances, latest long-range DI-3000 distomat, having a range of 6 km to 1 prism and maximum range of 14 km in favourable conditions can be fitted. Thus, with a distomat, T-1000 becomes *electronic total station.*

The T-1000 theodolite attains its full potential with the GRE 3 data terminal. This versatile unit connects directly to the T-1000. Circle readings and slope distances are transferred from the theodolite. Point numbering, codes and information are controlled from the GRE 3.

12.8.3. WILD T-2000 THEOMAT

Wild T-2000 Theomat (Fig. 12.18 *a*) is a high precision electronic angle measuring instrument. It has micro-processor controlled angle measurement system of highest accuracy. Absolute angle measurement is provided by a dynamic system with opto-electronic scanning

(Fig. 12.19). As the graduations around the full circle are scanned for every reading, circle graduation error cannot occur. Scanning at diametrically opposite positions eliminates the effect of eccentricity. Circle readings are corrected automatically for index error and horizontal collimation error. Thus angle measurements can be taken in one position to a far higher accuracy than with conventional theodolites. For many applications, operator will set the displays for circle reading to 1″, but for the highest precision the display can be set to read to 0.01″. For less precise work, circle readings can be displayed to 10″. Distances are displayed to 1 mm and 0.01 ft. With good targets, the standard deviation of the mean of a face-left and a face-right observations is better than 0.5″ for both the horizontal and vertical circles.

The theodolite has self-indexing maintenance free liquid compensator. The compensator provides the reference for vertical angle measurement. It combines excellent damping with high precision and allows accurate measurements unaffected by strong winds, vibrations etc.

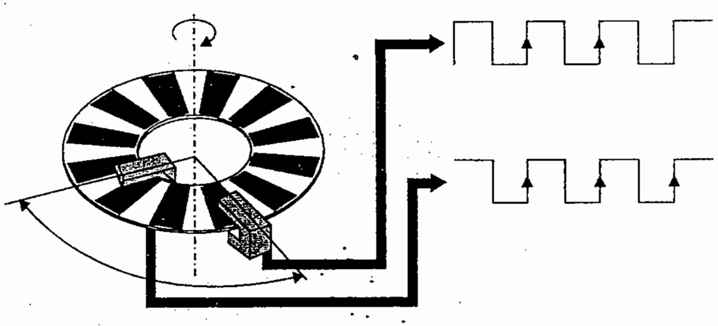


FIG. 12.19. MICROPROCESSOR CONTROLLED ANGLE MEASUREMENT SYSTEM

The instrument has two angle measuring modes : *single* and *tracking*. *Single mode* is used for angle measurements of highest accuracy. Hz and/or circle readings are displayed at the touch of a key. *Tracking* provides continuous single measurement with displays updated as the theodolite is turned. Tracking is used for rapid measurements, turning the theodolite to set a bearing or following a moving target. The horizontal circle reading can be set to zero or any value by means of the key-board.

The whole instrument is operated from a central panel comprising a water-proof key-board and three liquid-crystal displays, shown in Fig. 12.20. The key need only the slightest touch. One display guides the operator, the other two contain data. The displays and telescope reticle can be illuminated for work in the dark. Fig. 12.21 illustrates typical commands along with corresponding key to be used.

To measure angles, touch ________ HZV

To measure and record angles, touch ________ REC

To measure angles, distances heights and coordinates, touch ________ DIST

To measure and record angles, distances, heights and coordinates, touch ________ ALL

That's all there is to its a single keystroke for the main operations.

FIG 12.21. TYPICAL COMMANDS

Various parameters such as a circle orientation station co-ordinates and height scale correction and additive constant can be entered and stored. All are retained until over-written by new values. They cannot be lost even when the instrument is switched off. As circle readings are corrected for index error and horizontal collimation error, one control panel is in position. It is perfectly sufficient for many operations. However, for maximum convenience, particularly when measurements in both positions are required, the instrument is available with a control panel on each side.

The instrument uses rechargeable plug-in internal battery (NiCd, 2 Ah, 12 V DC) which is sufficient for about 1500 angle measurements or about 550 angle and distance measurements. The instrument switches off automatically after commmands and key sequences. The user can select a switch off time of 20 seconds or three minutes. This important power saving feature is made possible by the non-volatile memory. There is no loss of stored information when the instrument switches off.

Clamps and drives are coaxial. The drive screws have two speeds : fast for quick aiming, slow for fine pointing. Telescope focusing is also two-speed. An optical plummet is built into the alidade. The carrying handle folds back to allow the telescope to transit with Distomat fitted. Horizontal and vertical setting circles facilitate turning into a target and simplify setting-out work.

Modular Approach

The T-2000 offers all the benefits of the modular approach. It can be used as a theodolite combined with any distomat and connects to GRE 3 data terminal and computers. Fig. 12.22 illustrates diagrammetrically this modular approach which provides for easy upgrading at any time at minimum cost.

Wild theodolite accessories fit the T-2000: optional eye-pieces, filters, eye-piece prism, diagonal eye-piece, auto-collimation, eye-piece, parallel-plate micrometers, pentaprism, solar prism, auxiliary lenses etc. Wild tribachs, targets, distomat reflectors, target lamps, subtence bar, optical plummets and equipment for deformation measurements are fully compatible with the T-2000.

Two way data communication

Often, in industry and construction, one or more instruments have to be connected on line to a computer. Computation is in real time. Results are available immediately. To facilitate connection, interface parameters of the T-2000 instruments can be set to match those of the computer. Communication is two-way. The instrument can be controlled from

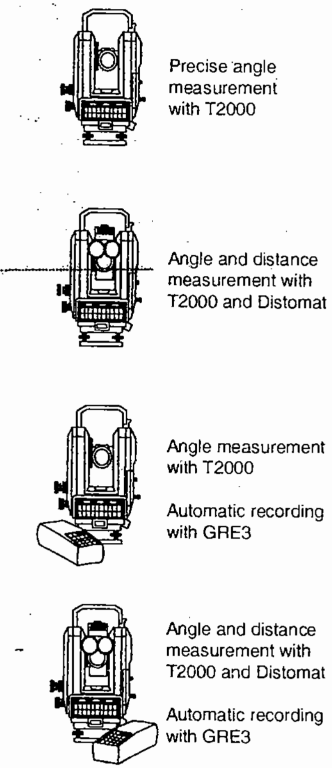


FIG. 12.22. T-2000 : MODULAR APPROACH.

the computer. Prompt messages and information can be transferred to the T-2000 displays. Of particular interest is the possibility of measuring objects by intersection from two theodolites (Fig. 12.23).

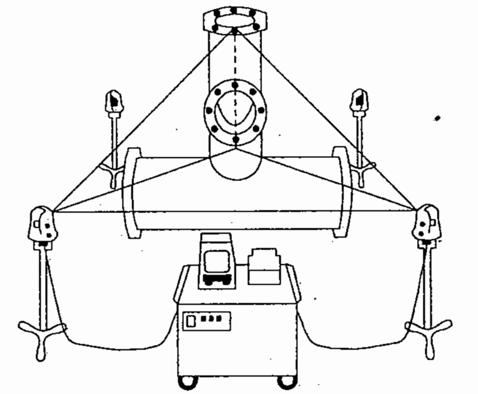


FIG. 12.23. RMS (REMOTE MEASURING SYSTEM) INTERSECTION METHOD.

Two T 2000 type instruments can be connected to the Wild GRE 3 Data Terminal. Using the Mini-RMS program, co-ordinates of intersected points are computed and recorded. The distance bet- ween any pair of object points can be calculated and displayed. For complex applications and special computations, two or more T 2000 or T 2000 S can be used with the *Wild-Leitz RMS 2000 Remote Measuring System.*

12.8.4. WILD T 2000 S 'THEOMAT'

Wild T 2000 S [Fig. 12.18 (*b*)] combines the pointing accuracy of a *special telescope* with the precision of T 2000 dynamic circle measuring system. This results in angle measurement of the highest accuracy. The telescope is *panfocal* with a 52 mm obejctive for an exceptionally bright, high contrast image. It focuses to object 0.5 m from the telscope. The focusing drive has coarse and fine movements.

Magnification and field of view vary with focusing distance. For observations to distant targets, the field is reduced and magnification increased. At close range, the field of view widens and magnification is reduced. This unique system provides ideal conditions for observation at every distance. With the standard eye-piece, magnification is 43 × with telescope focused to infinity. Optional eye-pieces for higher and lower magnification can also be fitted.

Stability of the line of sight with change in focusing is a special feature of the T 2000 S telescope. It is a true alignment telescope for metrology, industry and optical tooling industry. T 2000 S can also be fitted with a special target designed for pointing to small targets.

A special target can also be built into the telescope at the intersection of the horizontal and vertical axes. The target is invaluable for bringing the lines of sight of two T 2000 S exactly into coincidence. This is the usual preliminary procedure prior to measuring objects by the RMS intersection method.

For fatigue-free, maximum-precision auto-collimation measurements, the telescope is available with an auto-collimation eye-piece with negative reticle (green cross).

Like T 2000, the T 2000 S takes all Wild Distomats. It can also be connected the GRE 3 Data Terminal.

13

Field Astronomy

13.1. DEFINITIONS OF ASTRONOMICAL TERMS

1. The Celestial Sphere. The millions of stars that we see in the sky on a clear cloudless night are all at varying distances from us. Since we are concerned with their relative distances rather than their actual distance from the observer, it is exceedingly convenient to picture the stars as distributed over the surface of an imaginary spherical sky having its centre at the position of the observer. This imaginary sphere on which the stars appear to lie or to be studded is known as the *Celestial Sphere.* The radius of the celestial sphere may be of any value – from a few thousand metres to a few thousand kilometres. Since the stars are very distant from us, the centre of the earth may be taken as the centre of the celestial sphere.

2. The Zenith and Nadir. The Zenith (Z) is the point on the upper portion of the celestial sphere marked by plumb line above the observer. It is thus the point on the celestial sphere immediately above the observer's station. The *Nadir* (Z') is the point on the lower portion of the celestial sphere marked by the plumb line below the observer. It is thus the point on the celestial sphere vertically below the observer's station.

3. The Celestial Horizon. (also called *True* or *Rational horizon* or *geocentric horizon*). It is the great circle traced upon the celestial sphere by that plane which is perpendicular to the Zenith–Nadir line, and which passes through the centre of the earth. (*Great circle* is a section of a sphere when the cutting plane passes through the centre of the sphere).

4. The Terrestrial Poles and Equator. *The terrestrial poles* are the two points in which the earth's axis of rotation meets the earth's sphere. The *terrestrial equator* is the great circle of the earth, the plane of which is at right angles to the axis of rotation. The two poles are equidistant from it.

5. The Celestial Poles and Equator. If the earth's axis of rotation is produced indefinitely, it will meet the celestial sphere in two points called the *north and south celestial poles* (*P* and *P'*). The *celestial equator* is the great circle of the celestial sphere in which it is intersected by the plane of terrestrial equator.

6. The Sensible Horizon. It is a circle in which a plane passing through the point of observation and tangential to the earth's surface (or perpendicular to the Zenith–Nadir line) intersects with celestial sphere. The line of sight of an accurately levelled telescope lies in this plane.

7. The Visible Horizon. It is the circle of contact, with the earth, of the cone of visual rays passing through the point of observation. The circle of contact is a small circle of the earth and its radius depends on the altitude of the point of observation.

8. Vertical Circle. A vertical circle of the celestial sphere is great circle passing through the Zenith and Nadir. They all cut the celestial horizon at right angles.

9. The Observer's Meridian. The *meridian* of any particular point is that circle which passes through the Zenith and Nadir of the point as well as through the poles. It is thus a vertical circle.

10. The Prime Vertical. It is that particular vertical circle which is at right angles to the meridian, and which, therefore passes through the east and west points of the horizon.

11. The Latitude (θ). It is the angular distance of any place on the earth's surface north or south of the equator, and is measured on the meridian of the place. It is marked + or – (or N or S) according as the place is north or south of the equator. The latitude may also be defined as the angle between the zenith and the celestial equator.

12. The Co-latitude (c). The Co-latitude of a place is the angular distance from the zenith to the pole. It is the complement of the latitude and equal to $(90° - \theta)$.

13. The Longitude (ϕ). The longitude of a place is the angle between a fixed reference meridian called the prime or first meridian and the meridian of the place. The prime meridian universally adopted is that of Greenwich. The longitude of any place varies between 0° to 180°, and is reckoned as $\phi°$ east or west of Greenwich.

14. The Altitude (α). The altitude of celestial or heavenly body (*i.e.*, the sun or a star) is its angular distance above the horizon, measured on the vertical circle passing through the body.

15. The Co-altitude or Zenith Distance (z). It is the angular distance of heavenly body from the zenith. It is the complement of the altitude, *i.e.*, $z = (90° - \alpha)$.

16. The Azimuth (A). The azimuth of a heavenly body is the angle between the observer's meridian and the vertical circle passing through the body.

17. The Declination (δ). The declination of a celestial body is angular distance from the plane of the equator, measured along the star's meridian generally called the declination circle, (*i.e.*, great circle passing through the heavenly body and the celestial pole). Declination varies from 0° to 90°, and is marked + or – according as the body is north or south of the equator.

18. Co-declination or Polar Distance (p). It is the angular distance of the heavenly body from the nearer pole. It is the complement of the declination, *i.e.*, $p = 90° - \delta$.

19. Hour Circle. Hour circles are great circles passing through the north and south celestial poles. The declination circle of a heavenly body is thus its hour circle.

20. The Hour Angle. The hour angle of a heavenly body is the angle between the observer's meridian and the declination circle passing through the body. The hour angle is always measured *westwards*.

21. The Right Ascension (R.A.). It is the equatorial angular distance measured *eastward* from the First Point of Aries to the hour circle through the heavenly body.

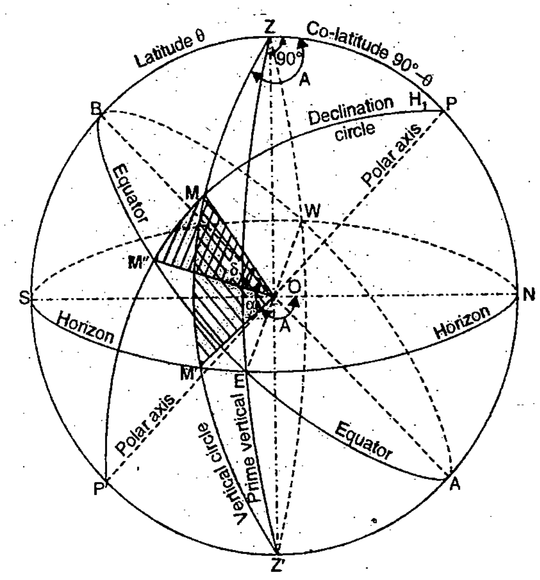


FIG. 13.1

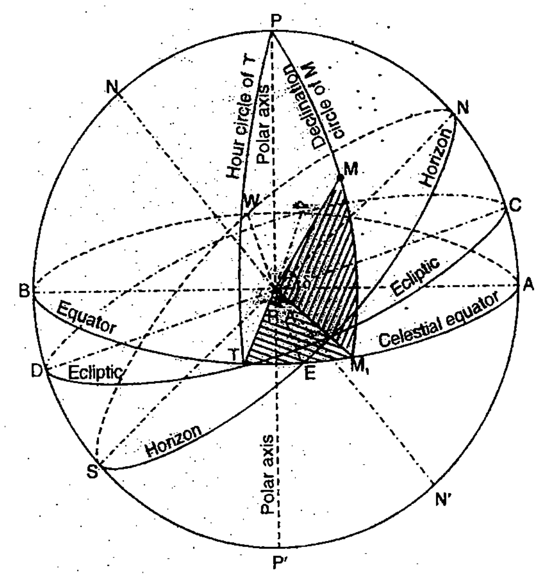


FIG. 13.2

22. Equinoctial Points. The points of the intersection of the ecliptic with the equator are called the equinoctial points. The declination of the sun is zero at the equinoctial points. The *Vernal Equinox* or the *First Point of Aries* (♈) is the point in which the sun's declination changes from south to north, and marks the commencement of spring. It is a fixed point on the celestial sphere. The *Autumnal Equinox* or the First Point of Libra (♎) is the point in which the sun's declination changes from north to south, and marks the commencement of autumn. Both the equinoctial points are six months apart in time.

23. The Ecliptic. Ecliptic is the great circle of the heavens which the sun appears to describe on the celestial sphere with the earth as a centre in the course of a year. The plane of the ecliptic is inclined to the plane of the equator at an angle (called the *obliquity*) of about 23° 27′, but is subjected to a diminution of about 5″ in a century.

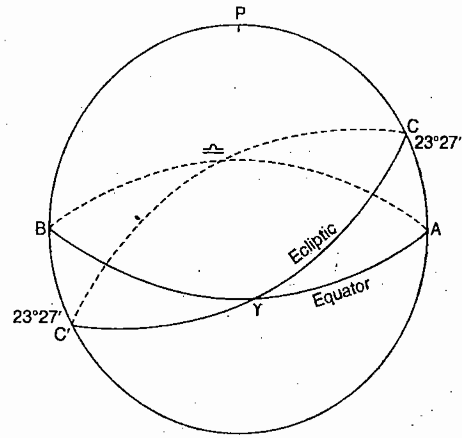


FIG. 13.3. THE ECLIPTIC.

24. Solastices. Solastices are the points at which the north and south declination of the sun is a maximum. The point *C* (Fig. 13.3) at which the north declination of the sun is maximum is called the *summer solastice,* while the point *C*′ at which south declination of the sun is maximum is known as the *winter solastice.* The case is just the reverse in the southern hemisphere.

25. North, South, East and West Directions. The north and south points correspond to the projection of the north and south poles on the horizon. The *meridian line* is the line in which the observer's meridian plane meets horizon plane, and the north and south points are the points on the extremities of it. The direction *ZP* (in plan on the plane of horizon) is the direction of north, while the direction *PZ* is the direction of south. The *east-west line* is the line in which the prime vertical meets the horizon, and east and west points are the extremities of it. *Since the meridian plane is perpendicular to both the equatorial plane as well as horizontal plane, the intersections of the equator and horizon determine the east and west points* (see Fig. 13.1).

13.2. CO-ORDINATE SYSTEMS

The position of a heavenly body can be specified by two spherical co-ordinates, *i.e.,* by two angular distances measured along arcs of two great circles which cut each other at right angles. One of the great circle is known as the primary circle of the reference and the other as the *secondary circle* of reference. Thus in Fig. 13.4, the position of the point *M* can be specified with reference to the plane *OAB* and the line *OA, O* being the origin of the co-ordinates. If a plane is passed through *OM* and perpendicular to the plane of *OAB,* it will cut the latter in the line *OB.* The two spherical co-ordinates of the point *M* are, therefore, angles *AOB* and *BOM* at the centre *O,* or the arcs *AB* and *BM.* In practical astronomy, the position of a celestial body can be specified by the following systems of co-ordinates :

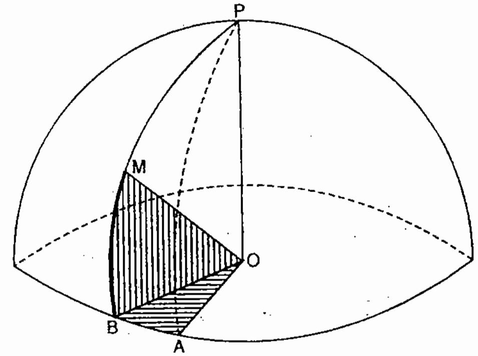


FIG. 13.4

1. The *horizon system*
2. The *independent equatorial system*
3. The *dependent equatorial system*
4. The *celestial latitude and longitude system.*

The *horizon system* is dependent on the position of the observer. The *independent equatorial system* is independent of the position of the observer and the positions apply to observers anywhere on the earth. In the *dependent equatorial system,* one of the great circle of reference is independent of the position of the observer while the other great circle perpendicular to the former is dependent on the position of the observer. There is yet another system of co-ordinates, known as the *celestial system,* in which the position of a body is specified by the *celestial latitude* and the *celestial longitude.*

1. THE HORIZON SYSTEM (ALTITUDE AND AZIMUTH SYSTEM)

In the horizon system, the horizon is the plane of reference and the co-ordinates of a heavenly body are (*i*) the *azimuth* and (*ii*) the *altitude.* This system is necessitated by the fact that we can measure only horizontal and vertical angles with the engineer's transit. The two great circles of reference are the horizon and the observer's meridian, the former being the primary circle and the latter the secondary circle.

In Fig. 13.5, *M* is the heavenly body in the Eastern part of the celestial sphere, *Z* is the observer's zenith and *P* is the celestial pole. Pass a vertical circle (*i.e.,* a great circle through *Z*) through *M* to intersect the horizon plane at *M*′. The first co-ordinate of *M* is, then, the *azimuth* (*A*) which is the angle between the observer's meridian and the vertical circle through the body. The azimuth can either be measured as the angular distance along the horizon, measured from the meridian to the foot of the vertical circle through the point. It is also equal to the angle at the zenith between the meridian and the vertical circle through *M.* The other co-ordinate of *M* is the *altitude* (α) which is the angular distance measured above (or below) the horizon, measured on the vertical circle through the body. Similarly, Fig. 13.6 shows the position (*M*) of the body in the Western part of the celestial sphere. It should be noted that, in the Northern hemisphere, the azimuth is always measured from the *north* either eastward, or westward, depending

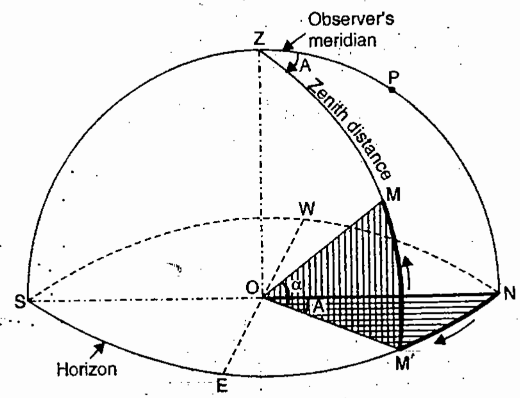


FIG. 13.5 BODY IN THE EASTERN PART OF THE CELESTIAL SPHERE.

upon whether the body is in the eastern part of celestial sphere or in the western part of the celestial sphere. In the southern hemisphere, the azimuth is measured from the south to the east or the west.

Alternatively, the position of a body is, sometimes specified in terms of *zenith distance and azimuth*. The zenith distance of any body is its angular distance from zenith, measured along the vertical circle. It is the complement of the altitude, *i.e.*,

zenith distance $(z) = 90° - \alpha$.

The horizon system of co-ordinates undergo constant and rapid changes due to the diurnal motions.

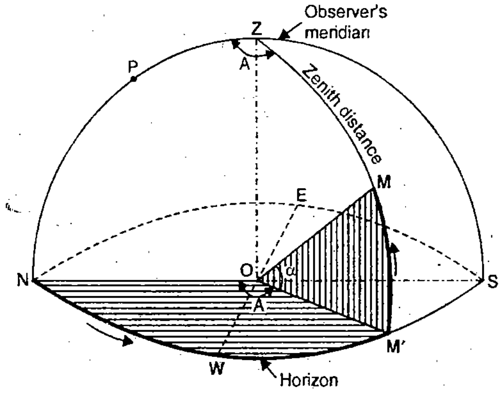


FIG. 13.6. BODY IN THE WESTERN PART OF THE CELESTIAL SPHERE.

2. THE INDEPENDENT EQUATORIAL SYSTEM (THE DECLINATION AND RIGHT ASCENSION SYSTEM)

This system is used in the publication of star catalogues, almanacs, or ephemerides in which the position of heavenly bodies are referred to spherical co-ordinates which are independent of the observer's position. The two great circles of reference are (*i*) the equatorial circle and (*ii*) the declination circle, the former being the primary circle and the latter the secondary circle of reference. For fixed stars, this system of co-ordinates is independent of the place of observation, and nearly independent of the time.

The first co-ordinate of the body (*M*) is the *right ascension,* which is the angular distance along the arc of the celestial equator measured from the first point of Aries (♈) as the point of reference towards East up to the declination circle passing through the body. It is also the angle, measured eastward at the celestial pole, between the hour circle through (♈) and the declination circle through *M*. The motion of the star is from East to West, and hence the Right Ascension is measured in a direction opposite to the motion of the heavenly body. It may be measured in degrees, minutes and seconds of arc or in hours, minutes and seconds of time. Thus in Fig. 13.7, ♈*P* is the hour circle through ♈, *M'MP* is the hour circle (or the declination circle) of *M*, and ♈*M'* is the R.A. measured along the arc of the equator.

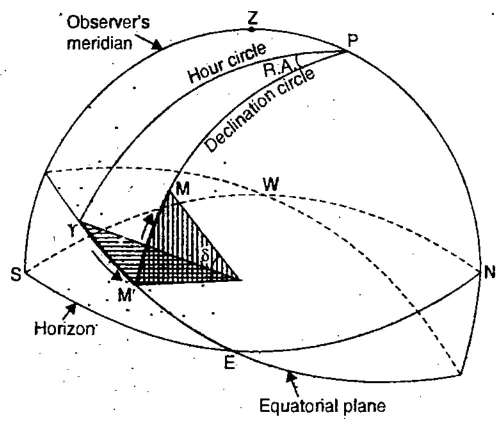


FIG. 13.7. THE DECLINATION–RIGHT ASCENSION SYSTEM.

The other co-ordinate in this system is the declination (δ). It is the distance of the body from the equator measured along the arc of the declination circle. The

declination circle accompanies the body in its diurnal course. The declination is considered positive when the body is north of the equator and negative when it is to south.

The *polar distance* (*p*) is the complement of the declination, *i.e.*, $p = (90° - \delta)$. In Fig. 13.7. *M'M* is the positive declination of the body (*M*).

The values of declination and right ascension of a fixed star in the heaven, although nearly constant, are not absolutely so. A register of these co-ordinates, together with their annual change (if any be found) will enable to identify a star once observed. Such a register is called a *catalogue of stars* and its correctness is of highest importance. The variation of the declination and right ascension of the sun is very much greater than for the stars.

3. THE DEPENDENT EQUATORIAL SYSTEM (THE DECLINATION AND HOUR ANGLE SYSTEM)

In this system, one co-ordinate is dependent of the observer's position and the other co-ordinate is independent of the observer's position. The two great circles of reference are (*i*) the horizon and (*ii*) the declination circle through the celestial body, the former being the primary circle and the latter the secondary circle of reference.

In this system, the first co-ordinate of *M* (Fig. 13.8) is the *hour angle.* Hour angle is the angluar distance along the arc of the horizon measured from the observer's meridian to the declination circle passing through the body. It is also measured as the angle, subtended at the pole, between the observer's me-

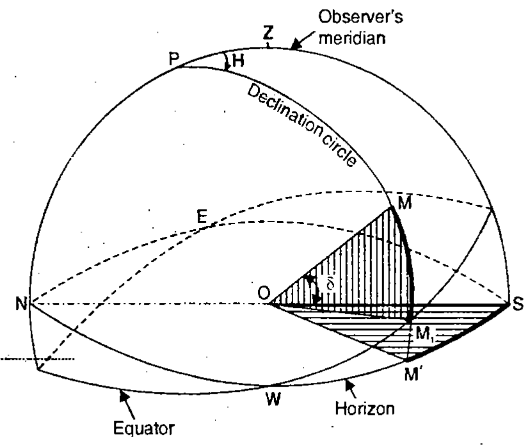


FIG. 13.8. THE DECLINATION–HOUR ANGLE SYSTEM.

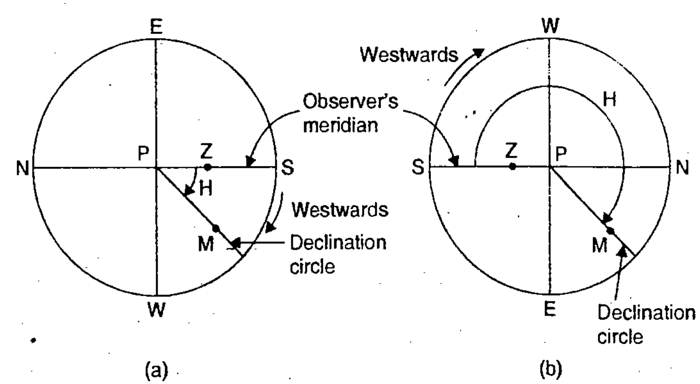


FIG. 13.9. PLAN ON THE PLANE OF THE EQUATOR.

ridian and the declination circle of the body. *In the northern hemisphere, the hour angle is always measured from the south and towards west upto the declination circle.* Its value varies from 0 ° to 360 °. If H varies from 0 ° to 180 °, the star is in the western hemisphere, otherwise in the eastern hemisphere. Fig. 13.9 shows the plan on the plane of the equator, illustrating how the hour angle is measured *westward* for two positions of the observer. The other co-ordinate is the declination, as in the second system. Thus, in Fig. 13.8, SM' is the hour angle, and M_1M is the declination of the celestial body (M), M' and M_1 being the projections of M on the horizon and equator respectively.

4. THE CELESTIAL LATITUDE AND LONGITUDE SYSTEM

In this system of the co-ordinates, the primary plane of reference is the ecliptic. The second plane of reference is a great circle passing through the First Point of Aries and perpendicular to the plane of the ecliptic. The two co-ordinates of a celestial body are (*i*) the celestial latitude and (*ii*) the celestial longitude.

The *celestial latitude* of a body is the arc of great circle perpendicular to the ecliptic, intercepted between the body and the ecliptic. It is positive or negative depending upon whether measured north or south of the ecliptic. The *celestial longitude* of a body is the arc of a ecliptic intercepted between the great circle passing through the First Point of Aries and the circle of the celestial latitude passing through the body. It is measured *eastwards* from 0° to 360°. Thus, in Fig. 13.10, M_1M is the celestial latitude (north) and $♈M_1$ is the celestial longitude for the heavenly body (M).

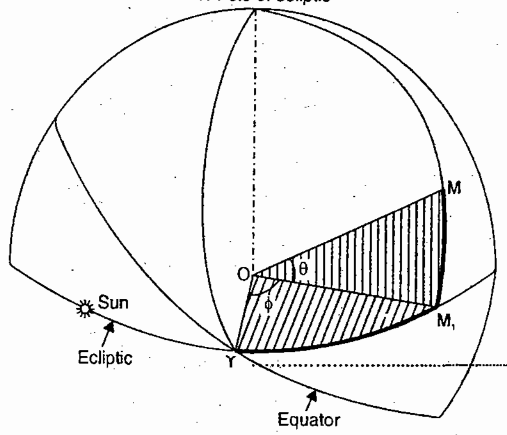


FIG. 13.10. THE CELESTIAL LATITUDE AND LONGITUDE.

Comparison of the Systems. As stated earlier, the azimuth and altitude of a star are not constant but are continuously changing due to diurnal motion. On the other hand, the right ascension and declination of a star are constant, because the reference point, the First Point of Aries, partakes of the diurnal motion of the stars. However, there is no instrument which can measure right ascension and declination of the star directly. The azimuth and the altitude of a star can be directly measured with the help of a theodolite. Knowing the hour angle and the azimuth of a star, its right ascension and declination can be computed from the solution of the astronomical triangle provided the instant of time at which the body was in a certain position (*i.e.*, the hour angle) is also determined. Thus, both the systems are necessary — the first one for the direct field observations and the second one for the computations required in respect of the preparation of the star catalogues.

13.3. THE TERRESTRIAL LATITUDE AND LONGITUDE

We have discussed the various systems of co-ordinates to establish the position of a heavenly body on the celestial sphere. In order to mark the position of a point on the earth's surface, it is necessary to use a system of co-ordinates. The terrestrial latitudes and longitudes are used for this purpose.

The terrestrial *meridian* is any great circle whose plane passes through the axis of the earth (*i.e.*, through the north and south poles). Terrestrial equator is the great circle whose plane is perpendicular to the earth's axis. The *latitude* θ of a place is the angle subtended at the centre of the earth north by the arc of meridian intercepted between the place and the equator. The latitude is north or positive when measured above the equator, and is south or negative when measured below the equator. The latitude of a point upon the equator is thus 0 °, while at the North and South Poles, it is 90 °N and 90 ° S latitude respectively. The *co-latitude* is the complement of the latitude, and is the distance between the point and pole measured along the meridian.

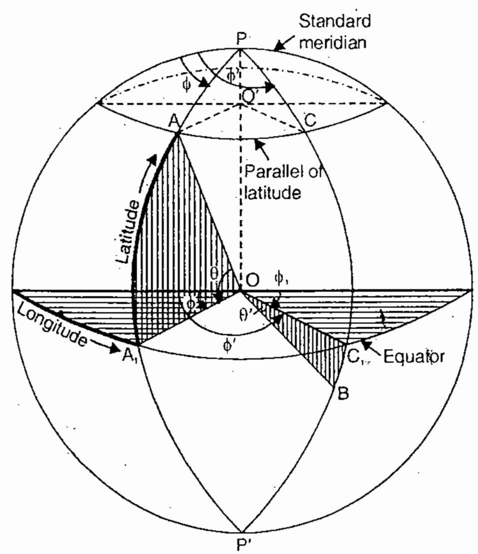


FIG. 13.11. THE TERRESTRIAL LATITUDE AND LONGITUDE.

The *longitude* (φ) of a place is the angle made by its meridian plane with some fixed meridian plane arbitrarily chosen, and is measured by the arc of equator intercepted between these two meridians. The prime meridian universally adopted is that of Greenwich. The longitude of any place varies between 0 ° to 180 °, and is reckoned as φ ° east or west of Greenwich. All the points on meridian have the same longitude.

The Parallel of Latitude

The *parallel of latitude* through a point is a small circle in which a plane through that point, and at right angles to the earth's axis, intersects the earth's surface. *All the points on the parallel of latitude have the same latitude.* The meridians are great circles of the same diameter while the parallel of a latitude are small circles, and are of different diameters depending upon the latitude of the place through which the parallel of the latitude is drawn. Due to this reason *a degree of longitude* has got different values at different latitudes — higher the latitude smaller the value. At the equator, a degree of longitude is equivalent to a distance of about 69 miles. However, a degree of latitude has the constant value of 69 miles everywhere.

To find the distance between two points A and C on a parallel of latitude, consider Fig. 13.11 in which θ = latitude of A = latitude of C, φ = longitude of A, and φ' = longitude of C. The angular radius PA of the parallel of latitude = 90 ° − θ.

Now arc $\frac{AC}{A_1C_1} = \frac{O'A}{OA_1}$ where O' is the centre of the parallel of latitude

$= \frac{O'A}{OA}$, since $OA_1 = OA$ = radius of the earth

$= \sin O'OA$, since $\angle AO'O = 90°$

$\therefore \quad AC = A_1C_1 \sin (90° - \theta) = \cos \theta \, . \, A_1C_1$

or $\quad$ AC = **cos latitude × difference of longitude.**

The shortest distance measured along the surface of the earth between two places is the length of the arc of the great circle joining them. The distance between two points in nautical miles measured along the parallel of latitude is called the *departure.*

Thus, ***departure = difference in longitude in minutes × cos latitude.***

The Zones of the Earth

The earth has been divided into certain zones depending upon the parallel of latitude of certain value above and below the equator. The parallel of latitude $23° 27\frac{1}{2}'$ north of equator is known as the *tropic of cancer.* The parallel of latitude $23° 27\frac{1}{2}'$ south of equator is known as the *tropic of capricorn.* The belt or zone of earth between these two tropics is known as the *torrid zone.* The parallel of latitude $66° 32\frac{1}{2}'$ north of equator is called the *arctic circle,* and of a similar value. South of equator is called the *anarctic circle.* The belt between the tropic of cancer and the arctic circle is known as the *north temperate zone* while the belt between the tropic of capricorn and the anarctic circle is known as the *south temperate zone.* The belt between the arctic circle and the north pole is called the *north frigid zone* and the belt between the anarctic circle and the south pole is called the *south frigid zone.*

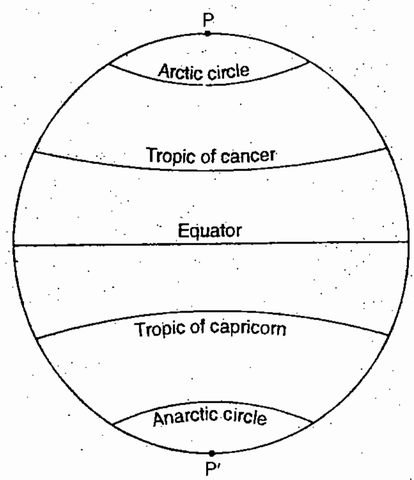


FIG. 13.12. THE ZONES OF THE EARTH.

The Nautical Mile. A nautical mile is equal to the distance on arc of the great circle corresponding to angle of 1 minute subtended by the arc at the centre of the earth.

Taking radius of earth = 6370 kilometres, we have

$$\text{One nautical mile} = \frac{\text{Circumference of the great circle.}}{360° \times 60} = \frac{2\pi \times 6370}{360 \times 60} = \mathbf{1.852\ km.}$$

13.4. SPHERICAL TRIGONOMETRY AND SPHERICAL TRIANGLE

Since in the astronomical survey many of the quantities involved are the parts of the celestial sphere, a simple knowledge of spherical trigonometry is essential.

Spherical Triangle

A spherical triangle is that triangle which is formed upon the surface of the the sphere by intersection of three arcs of great circles and the angles formed by the arcs at the vertices of the triangle are called the spherical angles of the triangle.

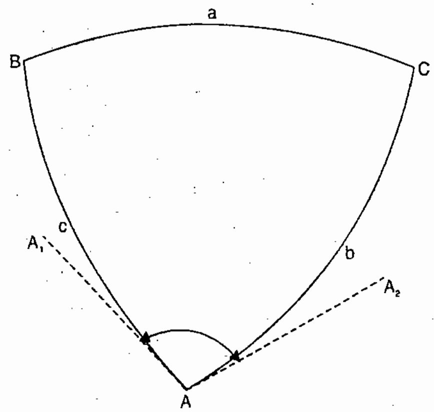


FIG. 13.13. SPHERICAL TRIANGLE.

Thus, in Fig. 13.13. AB, BC and CA are the three arcs of great circles and intersect each other at A, B and C. It is usual to denote the angles by A, B and C and the sides respectively opposite to them, as a, b and c. *The sides of spherical triangle are proportional to the angle subtended by them at the centre of the sphere and are, therefore, expressed in angular measure.* Thus, by $\sin b$ we mean the sine of the angle subtended at the centre by the arc AC. A spherical angle is an angle between two great circles, and is defined by the plane angle between the tangents to the circles at their point of intersection. Thus, the spherical angle at A is measured by the plane angle A_1AA_2 between the tangents AA_1 and AA_2 to the great circles AB and AC.

Properties of a spherical triangle

The following are the properties of a spherical triangle :

1. Any angle is less than two right angles or π.
2. The sum of the three angles is less than six right angles or 3π and greater than two right angles or π.
3. The sum of any two sides is greater than the third.
4. If the sum of any two sides is equal to two right angles or π, the sum of the angles opposite them is equal to two right angles or π.
5. The smaller angle is opposite the smaller side, and *vice versa.*

Formulae in Spherical Trigonometry

The six quantities involved in a spherical triangle are three angles A, B and C and the three sides a, b and c. Out of these, if three quantities are known, the other three can very easily be computed by the use of the following formulae in spherical trigonometry:

1. Sine formula : $\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$ ...(13.1)

2. Cosine formula : $\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$ ...(13.2)

or $\quad \cos a = \cos b \cos c + \sin b \sin c \cos A$ ...[13.2 (*a*)]

Also, $\quad \cos A = -\cos B \cos C + \sin B \sin C \cos a$ ...(13.3)

3. For computation purposes :

$$\sin\frac{A}{2}=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}} \qquad ...(13.4)$$

$$\cos\frac{A}{2}=\sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}} \qquad ...(13.5)$$

$$\tan\frac{A}{2}=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}} \qquad ...(13.6)$$

where $s=\frac{1}{2}(a+b+c)$

4. Similarly,

$$\sin\frac{a}{2}=\sqrt{\frac{-\cos S\cos(S-A)}{\sin B\sin C}} \qquad ...(13.7)$$

$$\cos\frac{a}{2}=\sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B\sin C}} \qquad ...(13.8)$$

$$\tan\frac{a}{2}=\sqrt{\frac{-\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)}} \qquad ...(13.9)$$

where $S=\frac{1}{2}(A+B+C)$

5. $$\tan\tfrac{1}{2}(a+b)=\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c \qquad ...(13.10)$$

$$\tan\tfrac{1}{2}(a-b)=\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c \qquad ...(13.11)$$

6. $$\tan\tfrac{1}{2}(A+B)=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C \qquad ...(13.12)$$

$$\tan\tfrac{1}{2}(A-B)=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C \qquad ...(13.13)$$

THE SOLUTION OF RIGHT-ANGLED SPHERICAL TRIANGLE

The relationships of right-angled spherical triangle are very conveniently obtained from '*Napier's rules of circular parts*'.

In [Fig. 13.14 (*a*)], *ABC* is a spherical triangle right-angled at *C*. Napier defines the circular parts as follows :

(*i*) the side *a* to one side of the right-angle,

(*ii*) the side *b* to the other side of the right-angle,

(*iii*) the complement (90° − *A*) of the angle *A*,

(*iv*) the complement (90° − *c*) of the side *c*,

and (*v*) the complement (90° − *B*) of the angle *B*.

These five parts are supposed to be arranged round a circle [Fig. 13.14 (*b*)] in order in which they stand in the triangle. Thus, starting with the side *a*, we have, *in*

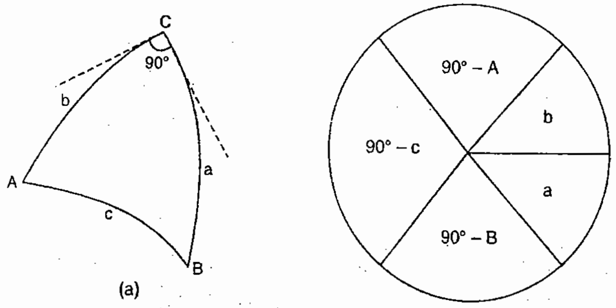


FIG. 13.14. NAPIER'S RULES OF CIRCULAR PARTS.

order, *b*, 90° − *A*, 90° − *c* and 90° − *B*. Then, if any part is considered as the '*middle part*' the two parts adjacent to it as '*adjacent parts*', and the remaining two as '*opposite parts*', we have the following rules by Napier :

sine of middle part = product of *tangents* of the *adjacent* parts ...(*i*)

and ***sine of middle part*** = product of *cosines* of opposite parts ...(*ii*)

Thus, $\sin b=\tan a\tan(90°-A)$

and $\sin b=\cos(90°-B)\cos(90°-c)$

By choosing different parts in turn as the middle parts, we can obtain all the possible relationships between the sides and angles.

THE SPHERICAL EXCESS

The spherical excess of a spherical triangle is the value by which the sum of three angles of the triangle exceeds 180°.

Thus, spherical excess $E=(A+B+C-180°)$ ...(13.14)

Also, $\tan^2\frac{1}{2}E=\tan\frac{1}{2}s\tan\frac{1}{2}(s-a)\tan\frac{1}{2}(s-b)\tan\frac{1}{2}(s-c)$ ...(13.15)

In geodetic work the spherical triangles on the earth's surface are comparatively small and the spherical excess seldom exceeds more than a few seconds of arc. The spherical excess, in such case, can be expressed by the approximate formula

$$E=\frac{\Delta}{R^2\sin 1''}\text{ seconds} \qquad ...[13.15\ (a)]$$

where *R* is the radius of the earth and Δ is the area of triangle expressed in the same linear units as *R*.

In order to prove the above expression for the spherical excess, let us consider the spherical triangle *ABC* [Fig. 13.14 (*c*)] which is formed by three great circles. These three great circles divide the whole sphere in eight divisions—the four in one hemisphere being similar to the other four in the other hemisphere because of symmetry.

Let Δ = area *ABC* ; Δ_1 = area *ACD*

Δ_2 = area *CDE* ; Δ_3 = area *BCE*

S = area of whole sphere = $4\pi R^2$;

R = radius of sphere

A, B, C = angles of the spherical triangle

Evidently, $(\Delta + \Delta_1) = \frac{B}{360°} \times S$

$\Delta + \Delta_3 = \frac{A}{360°} \times S$

and $\Delta + \Delta_2 = \frac{C}{360°} \times S$

Adding the three, we get

$$3\Delta + \Delta_1 + \Delta_2 + \Delta_3 = \frac{A+B+C}{360°} \times S \quad ...(1)$$

Also, $\Delta + \Delta_1 + \Delta_2 + \Delta_3$ = area of hemisphere

$$= \frac{S}{2} \quad ...(2)$$

Subtracting (2) from (1), we get

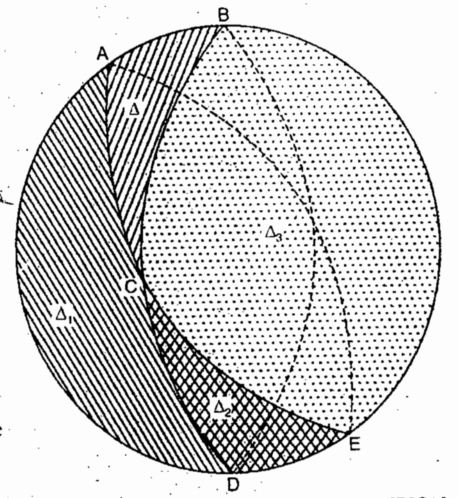


FIG. 13.14 (c) COMPUTATION OF SPHERICAL EXCESS.

$$2\Delta + \frac{S}{2} = \frac{A+B+C}{360°} \times S \quad \text{or} \quad 2\Delta = \frac{S}{360°}(A+B+C-180°)$$

or $2\Delta = \frac{S}{360°} \times E$, from Equation 13.14

which gives $E = (2 \times 360°)\frac{\Delta}{S} = \frac{720°\,\Delta}{4\pi R^2}$; or $E = 180° \frac{\Delta}{\pi R^2}$ degrees ...[13.15 (b)]

or $E = \frac{\Delta}{R^2 \sin 1''}$ seconds ...[13.15 (a)]

Area of sperical triangle :

The area of spherical triangle may be obtained from the formula

$$\text{Area } \Delta = \frac{\pi R^2 (A+B+C-180°)}{180°} = \frac{\pi R^2 E}{180°} \quad ...(13.16)$$

13.5. THE ASTRONOMICAL TRIANGLE (Fig. 13.15)

An astronomical triangle is obtained by joining the pole, zenith and any star M on the sphere by arcs of great circles. From this triangle, the relation existing amongst the spherical co-ordinates may be obtained.

Let α = altitude of the celestial body (M)

δ = declination of the celestial body (M)

θ = latitude of the observer.

Then ZP = co-latitude of the observer = $90° - \theta = c$

PM = co-declination or the polar distance of $M = 90° - \delta = p$

and ZM = zenith distance = co-altitude of the body = $(90° - \alpha) = z$

The angle at $Z = MZP$ = the azimuth (A) of the body

The angle at $P = ZPM$ = the hour angle (H) of the body

The angle at $M = ZMP$ = the parallactic angle.

If the three sides (*i.e.* MZ, ZP and PM) of the astronomical triangle are known, the angles A and H can be computed from the formulae of spherical trigonometry.

Thus, from Eq. 13.2, we have

$$\cos A = \frac{\sin \delta}{\cos \alpha \cdot \cos \theta} - \tan \alpha \cdot \tan \theta \quad ...[13.17\ (a)]$$

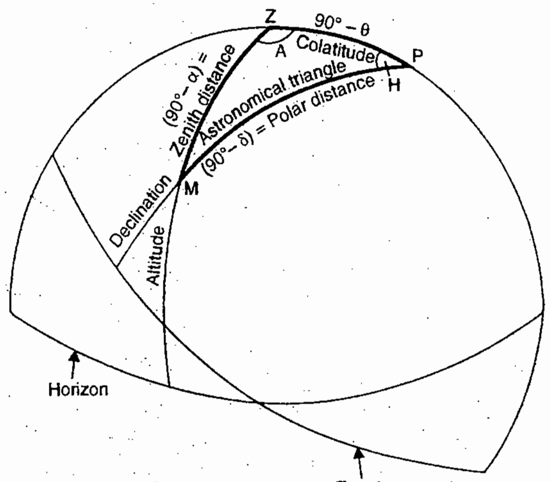


FIG. 13.15. THE ASTRONOMICAL TRIANGLE.

Also,

$$\tan \frac{A}{2} = \sqrt{\frac{\sin (s - ZM) \sin (s - ZP)}{\sin s \cdot \sin (s - PM)}} \quad ...(13.17)$$

$$= \sqrt{\frac{\sin (s - z) \sin (s - c)}{\sin s \cdot \sin (s - p)}} \quad ...[13.17\ (b)]$$

$$\sin \frac{A}{2} = \sqrt{\frac{\sin (s - z) \sin (s - c)}{\sin z \sin c}} \quad ...[13.17\ (c)]$$

$$\cos \frac{A}{2} = \sqrt{\frac{\sin s \cdot \sin (s - p)}{\sin z \sin c}} \quad ...[13.17\ (d)]$$

where $s = \frac{1}{2}(ZM + ZP + PM) = \frac{1}{2}(z + c + p)$

Similarly, $\cos H = \frac{\sin \alpha}{\cos \delta \cos \theta} - \tan \delta \tan \theta$ ...[13.18 (a)]

Also, $\tan \frac{H}{2} = \sqrt{\frac{\sin (s - ZP) \sin (s - PM)}{\sin s \cdot \sin (s - ZM)}} = \sqrt{\frac{\sin (s - c) \sin (s - p)}{\sin s \cdot \sin (s - z)}}$ ...(13.18)

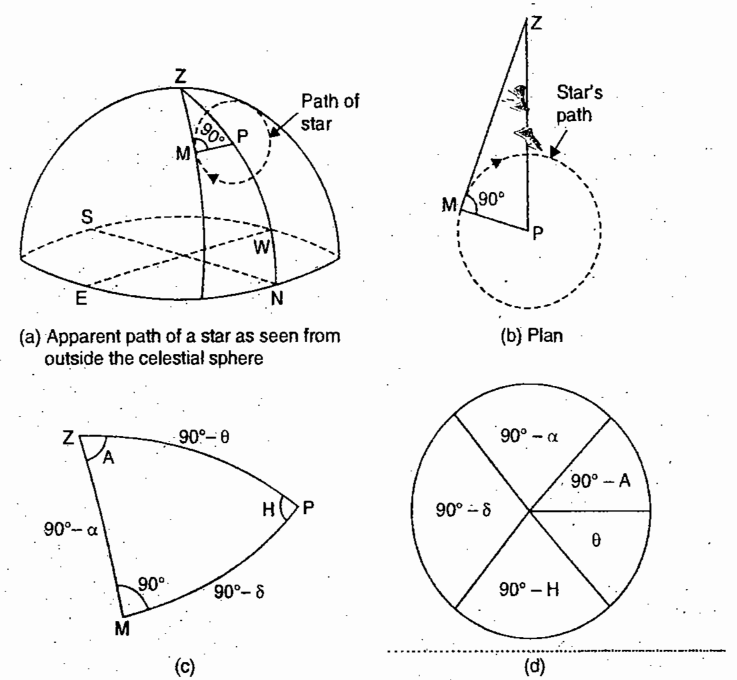
$$\sin \frac{H}{2} = \sqrt{\frac{\sin (s - c) \sin (s - p)}{\sin c \cdot \sin p}} \quad ...[13.18\ (b)]$$

$$\cos \frac{H}{2} = \sqrt{\frac{\sin s \cdot \sin (s - z)}{\sin c \cdot \sin p}} \quad ...[13.18\ (c)]$$

STAR AT ELONGATION

A star is said to be at elongation when it is at its greatest distance east or west of the meridian. In this position, the azimuth of the star is a maximum, and its diurnal circle is tangent to the vertical through the star. The triangle is thus right-angled at M.

The star is said to be at *eastern elongation*, when it is at its greatest distance to the east of the meridian, and at *western elongation*, when it is at its greatest distance to the west of the meridian. Fig. 13.16 (*a*) and (*b*) show the star *M* at its eastern elongation.



(a) Apparent path of a star as seen from outside the celestial sphere — (b) Plan — (c) — (d)

FIG. 13.16. STAR AT ELONGATION.

If the declination (δ) and the latitude of the place of observation is known, the azimuth (*A*), hour angle (*H*) and the altitude (α) of the body can be calculated from the Napier's rule [Fig. 13.16 (*c*) and (*d*)]. The five parts taken in order are, the two sides (90° – α), (90° – δ) and the complements of the rest of the three parts, *i.e.*, (90° – *H*), [90° – (90° – θ)] = θ and (90° – *A*).

Thus, sine of middle part = product of tangents of adjacent parts.

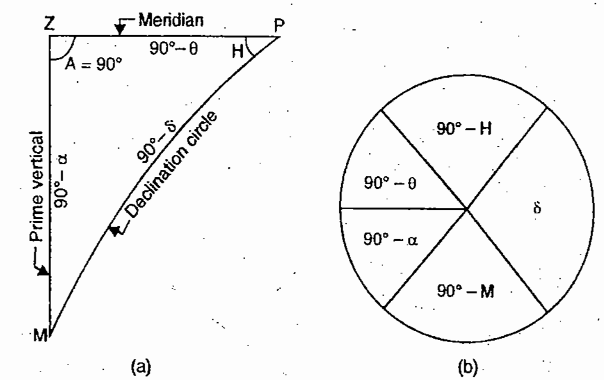
$$\sin(90° - H) = \tan(90° - \delta)\tan\theta \quad \text{or} \quad \cos H = \frac{\tan\theta}{\tan\delta} = \tan\theta \cdot \cot\delta \qquad ...(13.19)$$

Similarly,
$$\sin\theta = \cos(90° - \delta) \cdot \cos(90° - \alpha) \quad \text{or} \quad \sin\alpha = \frac{\sin\theta}{\sin\delta} = \sin\theta \cdot \operatorname{cosec}\delta \qquad ...(13.20)$$

and
$$\sin(90° - \delta) = \cos(90° - A)\cos\theta \quad \text{or} \quad \sin A = \frac{\cos\delta}{\cos\theta} = \cos\delta \cdot \sec\theta \qquad ...(13.21)$$

STAR AT PRIME VERTICAL

When the star is on the prime vertical of the observer, the astronomical triangle is evidently right-angled at *Z*.



(a) (b)

FIG. 13.17. STAR AT PRIME VERTICAL.

If the declination (δ) and the latitute (θ) of the place of observation are known, the altitude (α) and the hour angle (*H*) can be calculated by Napier's rule. The five parts taken in order are : the two sides (90° – θ) and (90° – α), and the complements of the rest of the three parts, *i.e.*, (90° – *M*), 90° – (90° – δ) = δ and (90° – *H*).

Now sine of middle part = product of cosine of opposite parts.

$$\therefore \quad \sin\delta = \cos(90° - \theta)\cos(90° - \alpha) = \sin\theta\sin\alpha \quad \therefore \quad \sin\alpha = \frac{\sin\delta}{\sin\theta} = \sin\delta\operatorname{cosec}\theta \qquad ...(13.22)$$

And
$$\sin(90° - H) = \tan(90° - \theta)\tan\delta \quad \text{or} \quad \cos H = \frac{\tan\delta}{\tan\theta} = \tan\delta\cot\theta \qquad ...(13.23)$$

STAR AT HORIZON

If a star (*M*) is at horizon, its altitude will be zero and the zenith distance will be equal to 90°.

If the latitude θ and the declination δ are known, the azimuth *A* and the hour angle *H* can be calculated by putting α = zero in equations 13.17 *a* and 13.18 *a*.

Thus,
$$\cos A = \frac{\sin\delta}{\cos\theta} = \sin\delta\sec\theta \qquad ...(13.24)$$

and
$$\cos H = -\tan\delta\tan\theta \qquad ...(13.25)$$

STAR AT CULMINATION

A star is said to culminate or transit when it crosses the observer's meridian. Each star crosses a meridian twice in its one revolution around the pole – the two culminations being designated as the upper culmination and the lower culmination. A star is to be at its *upper culmination* when its altitude is *maximum*, and at *lower culmination* when its altitude is *minimum*.

Thus, in Fig. 13.18, the star M culminates or transits the meridian at A and B, A being the point of upper culmination and B the point of lower culmination.

Similarly, the star M_1 culminates or transits the meridian at A_1 and B_1, A_1 being the point of upper culmination and B_1 the point of lower culmination.

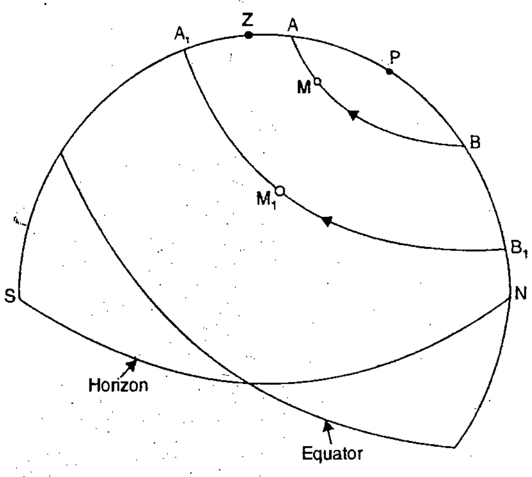


FIG. 13.18. STAR AT CULMINATION.

The upper culmination (A) of the star M occurs at the north side of the zenith, (*i.e.*, towards the pole) while the upper culmination (A_1) of the star M_1 occurs at the south side of zenith.

Now, at the upper culmination (A) of the star M, its zenith distance

$$= ZA = ZP - AP = (90° - \theta) - (90° - \delta) = (\delta - \theta) \quad ...(1)$$

Similarly, at the upper culmination (A_1) of the star M_1, the zenith distance

$$= ZA_1 = PA_1 - PZ = (90° - \delta) - (90° - \theta) = (\theta - \delta) \quad ...(2)$$

From (1) and (2), it follows that:

(*i*) The upper culmination of a star occurs to the *north side of the zenith* when the *declination of the star is greater than the latitude* of the place of observation.

(*ii*) The upper culmination of a star occur to the *south side of the zenith* when the *declination of the star is lesser than the latitude* of the place of observation.

CIRCUMPOLAR STARS

Circumpolar stars are those which are always above the horizon, and which do not, therefore, set. Such a star appears to the observer to describe a circle above the pole.

Thus, in Fig. 13.19, M_1 is a circumpolar star having its path A_1A_2 which is always above the horizon. In order that the circumpolar star does not set,

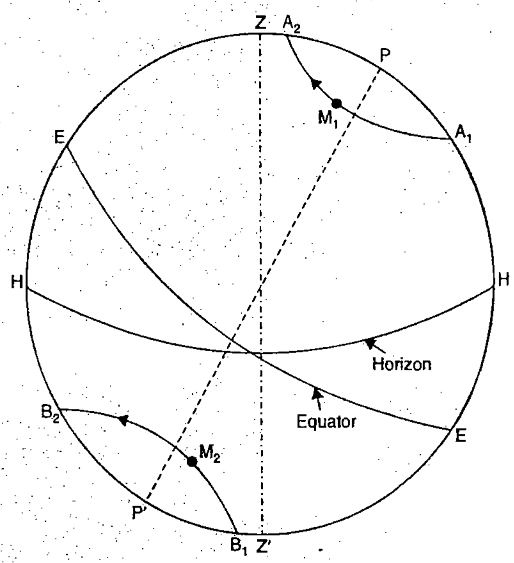


FIG. 13.19. CIRCUMPOLAR STARS.

distance above the pole (*i.e.*, PA_1) should be less than the distance of the pole from the horizon.

Hence $\quad PA_1 < PH \quad$ or $(90° - \delta) < \theta$ since $PH = \theta \quad$ or $\delta > (90° - \theta)$

Hence the declination of a circumpolar star is always greater than the co-latitude of the place of observation.

Similarly, M_2 is a circumpolar star having its path B_1B_2 which is always below the horizon and, therefore, never rises.

13.6. RELATIONSHIPS BETWEEN CO-ORDINATES

1. The Relation between Altitude of the Pole and Latitude of the Observer.

In Fig. 13.20. H-H is the horizon plane and E-E is the equatorial plane. O is the centre of the earth. ZO is perpendicular to HH while OP is perpendicular to EE.

Now latitude of place $= \theta = \angle EOZ$

And altitude of pole $= \alpha = \angle HOP$

$$\angle EOP = 90° = \angle EOZ + \angle ZOP$$
$$= \theta + \angle ZOP \quad ...(i)$$
$$\angle HOZ = 90° = \angle HOP + \angle POZ$$
$$= \alpha + \angle POZ \quad ...(ii)$$

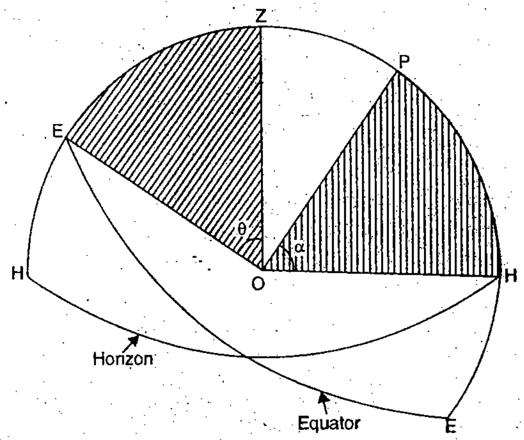


FIG. 13.20.

Equating the two, we get

$$\theta + \angle ZOP = \alpha + \angle POZ \quad \text{or} \quad \theta = \alpha$$

Hence *the altitude of the pole is always equal to the latitude of the observer.*

2. The Relation between Latitude of Observer and the Declination and Altitude of a Point on the Meridian.

For star M_1, $EM_1 = \delta$ = declination.

$SM_1 = \alpha$ = meridian altitude of star.

$M_1Z = z$ = meridian zenith distance of star.

$EZ = \theta$ = latitude of the observer.

Evidently, $EZ = EM_1 + M_1Z$

or $$\theta = \delta + z \quad ...(1)$$

The above equation covers all cases. If the star is below the equator, negative sign should be given to δ. If the star is to the north of zenith, negative sign should be given to z.

If the star is north of the zenith but above the pole, as at M_2, we have

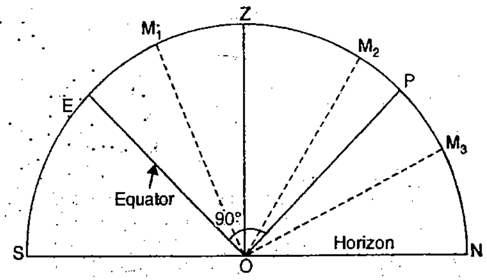


FIG. 13.21.

$$ZP = ZM_2 + M_2P$$

or $(90° - \theta) = (90° - \alpha) + p$, where p = polar distance = M_2P

or $\theta = \alpha - p$ ...(2)

Similarly, if the star is north of the zenith but below the pole, as at M_3, we have

$$ZM_3 = ZP + PM_3$$

or $(90° - \alpha) = (90° - \theta) + p$, where p = polar distance = M_3P

or $\theta = \alpha + p$ ...(3)

The above relations form the basis for the usual observation for latitude.

3. The Relation between Right Ascension and Hour Angle.

Fig. 13.22 shows the plan of the stellar sphere on the plane of the equator. M is the position of the star and $\angle SPM$ is its westerly hour angle. H_M. ♈ is the position of the First Point of Aries and angle SP♈ is its westerly hour angle. $\angle$♈PM is the right ascension of the star. Evidently, we have

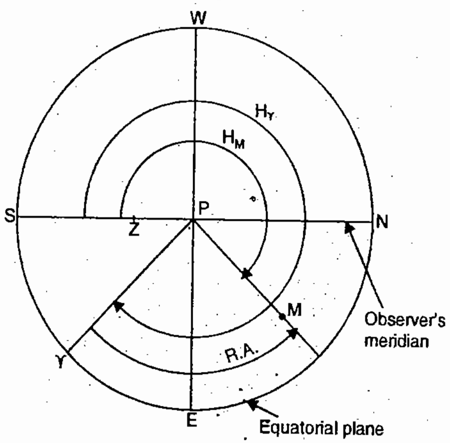


FIG. 13.22

∴ ***Hour angle of Equinox = Hour angle of star + R.A. of star.***

Example 13.1. *Find the difference of longitude between two places A and B from their following longitudes :*

(*1*) *Longitude of A = 40° W*
Longitude of B = 73° W

(*2*) *Long. of A = 20° E*
Long. of B = 150° E

(*3*) *Longitude of A = 20° W*
Longitude of B = 50° W

(*4*) *Long. of A = 40° E*
Long. of B = 150° W

Solution.

(1) The difference of longitude between A and B = 73° – 40° = **33°**

(2) The difference of longitude between A and B = 150° – 20° = **130°**

(3) The difference of longitude between A and B = 20° – (– 50°) = **70°**

(4) The difference of longitude between A and B = 40° – (– 150°) = **190°**

Since it is greater than 180°, it represents the obtuse angular difference. The acute angular difference of longitude between A and B, therefore, is equal to 360° – 190° = **170°**.

Example 13.2. *Calculate the distance in kilometers between two points A and B along the parallel of latitude, given that*

(*1*) *Lat. of A, 28° 42′N ; longitude of A, 31° 12′W*
Lat. of B, 28° 42′N ; longitude of B, 47° 24′W

(*2*) *Lat. of A, 12° 36′S ; longitude of A, 115° 6′W*
Lat. of B, 12° 36′S ; longitude of B, 150° 24′E.

Solution.

The distance in nautical miles between A and B along the parallel of latitude = difference of longitude in minutes × cos latitude.

(1) Difference of longitude between A and B = 47° 24′ – 31° 12′ = 16° 12′ = 972 minutes

∴ Distance = 972 cos 28° 42′ = 851.72 nautical miles
= 851.72 × 1.852 = **1577.34 km.**

(2) Difference of longitude between A and B
= 360° – {115° 6′ – (– 150° 24′)} = 94° 30′ = 5670 min.

∴ Distance = 5670 cos 12° 36′ = 5533.45 nautical miles
= 5533.45 × 1.852 = **10,247.2 km.**

Example 13.3. *Find the shortest distance between two places A and B, given that the longitudes of A and B are 15° 0′N and 12° 6′ N and their longitudes are 50° 12′E and 54° 0′E respectively. Find also the direction of B on the great circle route. Radius of earth = 6370 km.*

Solution.

In Fig. 13.23, the positions of A and B have been shown.

In the spherical triangle ABP,

$b = AP = 90°$ – latitude of $A = 90° - 15° \, 0' = 75°$

$a = BP = 90°$ – latitude of B
$= 90° - 12° \, 6' = 77° \, 54'$

$P = \angle APB$ = difference of longitude
$= 54° \, 0' - 50° \, 12' = 3° \, 48'$.

The shortest distance between two points is the distance along the great circle passing through the two points.

Knowing the two sides one angle, the third side AB (= p) can be easily computed by the cosine rule.

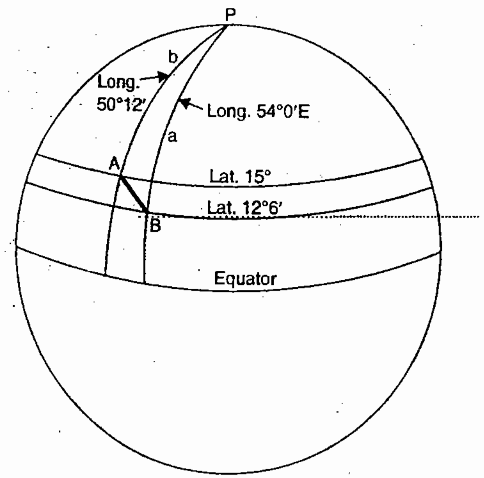


FIG. 13.23.

Thus $\cos P = \dfrac{\cos p - \cos a \cos b}{\sin a \sin b}$

or $\cos p = \cos P \sin a \sin b + \cos a \cos b$

$= \cos 3° \, 48' \sin 77° \, 54' \sin 75° + \cos 77° \, 54' \cos 75° = 0.94236 + 0.05425 = 0.99661$

∴ $p = AB = 4° \, 40' = 4°{\cdot}7$

Now, arc ≈ radius × central angle $= \dfrac{6370 \times 4°{\cdot}7 \times \pi}{180°} = 522.54$ km.

Hence distance AB = **522.54 km.**

Direction of A from B :

The direction of A from B is the angle B, and the direction of B from A is the angle A.

Angles A and B can be found by the tangent semi-sum and semi-difference formulae (Eqs. 13.12 and 13.13).

Thus $$\tan \tfrac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \tfrac{1}{2} P$$

and $$\tan \tfrac{1}{2}(A-B) = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \cot \tfrac{1}{2} P$$

Here $$\frac{a-b}{2} = \frac{77°\,54' - 75°}{2} = \frac{2°\,54'}{2} = 1°\,27'$$

$$\frac{(a+b)}{2} = \frac{77°\,54' + 75°}{2} = \frac{152°\,54'}{2} = 76°\,27' ; \quad \frac{P}{2} = \frac{3°\,48'}{2} = 1°\,54'$$

$\therefore$ $$\tan \tfrac{1}{2}(A+B) = \frac{\cos 1°\,27'}{\cos 76°\,27'} \cot 1°\,54'$$

From which, $$\frac{A+B}{2} = 89°\,35' \qquad \ldots(i)$$

and $$\tan \tfrac{1}{2}(A-B) = \frac{\sin 1°\,27'}{\sin 76°\,27'} \cot 1°\,54'$$

From which, $$\frac{A-B}{2} = 38°\,6' \qquad \ldots(ii)$$

$\therefore$ Direction of B from A = angle A = 89° 35′ + 38° 6′ = 127° 41′ = **S 52° 19′ E**

Direction of A from B = angle B = 89° 35′ − 38° 6′ = 51° 29′ = ***N* 51° 29′ *W*.**

Example 13.4. *At a point A in latitude 45° N, a straight line is ranged out which runs due east at A. This straight line is prolonged for 300 nautical miles to B. Find the latitude of B, and if it be desired to travel due north from B so as to meet the 45° parallel again at C, find the ABC at which we must set out, and the distance BC.*

Solution

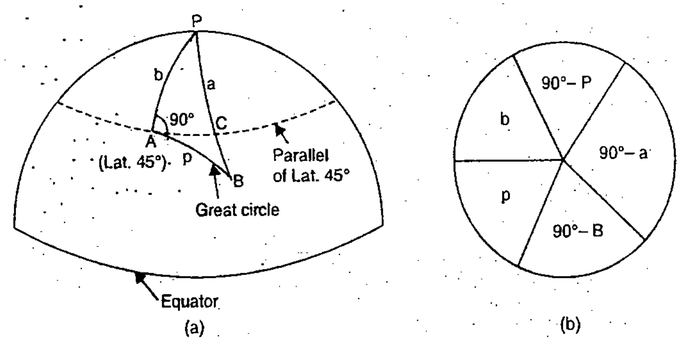


FIG. 13.24.

In Fig. 13.24, AB is straight line portion of a great circle ; since its length is 300 nautical miles, it subtends 300 minutes (= 5°) at the centre of the earth. AP is the meridian through A. Since AB is due east of the meridian, $\angle PAB = 90°$. Similarly, BP is the meridian through B, and meets the parallel to latitude through A (45° N) in C. PAB is, therefore, an astronomical triangle in which

side $PA = b$ = co-latitude of $A = 90° - 45° = 45°$; side $AB = p = 5°$; $\angle A = 90°$

The side $PB = a$ can be calculated by Napier's rule. Thus, sine of middle part = product of cosines of opposite parts.

$\therefore$ $\sin(90° - a) = \cos b \cos p$ or $\cos a = \cos 45° \cos 5°$

$\log \cos 45° = \bar{1}.8494850$

$\log \cos 5° = \bar{1}.9983442$

$\log \cos a = \bar{1}.8478292$

$\therefore$ $a = PB = 45°\,13'{\cdot}108$

$BC = PB - PC = 45°\,13'{\cdot}108 - 45° = 13'{\cdot}108$

Hence distance BC = **13.108** nautical miles = 13.108 × 1.852 = **24.275 km.**

The angle at B can be found by the application of the sine formula,

i.e. $$\frac{\sin B}{\sin b} = \frac{\sin A}{\sin a} \quad \text{or} \quad \frac{\sin B}{\sin 45°} = \frac{\sin 90°}{\sin 45°\,13'{\cdot}108}$$

or $$\sin B = \frac{\sin 45°}{\sin 45°\,13'{\cdot}108}$$

$\log \sin 45° = \bar{1}{\cdot}8494850$

$\log \sin 45°\,13'{\cdot}108 = \bar{1}{\cdot}8511345$ (subtract)

$\log \sin B = \bar{1}{\cdot}998505$; **B = 85° 0′ 34″.**

Example 13.5. *Two ports have the same latitude l and their longitudes differ by 2d. Prove that the length of the shortest route between them is* $2R \sin^{-1}(\sin d \cos l)$, *where R is the mean radius of the earth.*

Find the greatest distance, along a meridian, between the shortest route and the parallel of latitude through the ports. (U.L.)

Solution

In Fig. 13.25, A and B are the two ports. AFB is the arc of the great circle through A and B and F is the middle point. Due to symmetry, therefore, $\angle AFP = BFP = 90°$. ACB is the arc of parallel of latitude. AP and BP are the two meridians through A and B, FP is the meridian through the middle point of AB. Hence, triangles APF and BPF are astronomical triangles.

In triangle PFB,

$PB = f$ = co-latitude of $B = (90° - l)$; $PF = b$; $\angle FPB = d$.

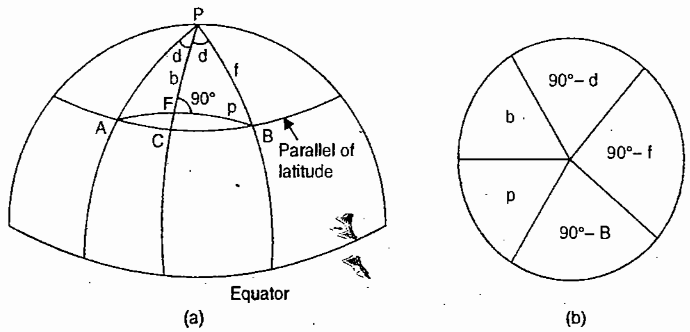


FIG. 13.25.

Distance $FB = p$ can be calculated by Napier's rule for the circular parts shown in Fig. 13.25 (*b*).

∴ sine middle part = Product of cosines of opposite parts

∴ $\sin p = \cos (90° - d) \cos (90° - f) = \sin d \sin f = \sin d \sin (90° - l) = \sin d \cos l$

or $FB = p = \sin^{-1} (\sin d \cos l)$

Hence $AB = 2\,FB = 2\,p = 2 \sin^{-1} (\sin d \cos l)$ radians.

∴ Distance AB along great circle = radius × angle at the centre of earth = $R \times 2\,p$

$= 2\,R \sin^{-1} (\sin d \cos l)$ (Proved).

The greatest distance between the great circle AFB and the parallel of latitude ACB will evidently be along CF (since $\angle F = 90°$).

The distance $PF = b$ can be found by Napier's rule.

sin middle part = product of tangents of adjacent parts

or $\sin (90° - d) = \tan b \tan (90° - f)$

or $\cos d = \tan b \cot f = \tan b \cot (90° - l) = \tan b \tan l$

∴ $\tan b = \cos d \cot l$ or $b = PF = \tan^{-1} (\cos d \cot l)$

Now $CF = CP - PF$

But $CP = (90° - l) = \frac{\pi}{2} - l$ radians

∴ $CF = \left(\frac{\pi}{2} - l\right) - \tan^{-1} (\cos d \cot l)$ radians

∴ Distance along CF = Radius × angle at the centre = $R \left\{ \left(\frac{\pi}{2} - l\right) - \tan^{-1} (\cos d \cot l) \right\}$ **Ans.**

Example 13.6. *Find the zenith distance and altitude at the upper culmination of the stars from the following data :*

(*a*) *Declination of star* = *42° 15′ N* *Latitude of observer* = *26° 40′ N*

(*b*) *Declination of star* = *23° 20′ N* *Latitude of observer* = *26° 40′ N*

(*c*) *Declination of star* = *65° 40′ N* *Latitude of observer* = *26° 40′ N*

Solution. (Fig. 13.18)

(*a*) Since the declination of the star is greater than the latitude of the observer ($\delta > \theta$), the upper culmination of the star occurs to the north side of zenith, *i.e.*, between Z and P.

Hence zenith distance at upper culmination = $ZA = ZP - AP$

$= (90° - \theta) - (90° - \delta) = (\delta - \theta) = 42° 15' - 26° 40' = \mathbf{15° 35'}$

∴ Altitude of the star at upper culmination = $90° - 15° 35' = \mathbf{74° 25'}$.

(*b*) Since the declination of the star is lesser than the latitude of the observer, the upper culmination of the star occurs at the south side of the zenith.

∴ Zenith distance of the star at upper culmination = $ZA_1 = A_1P - ZP$

$= (90° - \delta) - (90° - \theta) = \theta - \delta = 26° 40' - 23° 20' = \mathbf{3° 20'}$

∴ Altitude of the star at the upper culmination = $90° - 3° 20' = \mathbf{86° 40'}$.

(*c*) Fig. 13.19, $\delta = 65° 40'$ N ; $90° - \theta = 90° - 26° 40' = \mathbf{63° 20'}$

Since the declination of the star is greater than the co-latitude, the star is circumpolar, and will never set. The upper culmination will occur at the north side of zenith, *i.e.*, between Z and P.

∴ Zenith distance at the upper culmination = $ZA_2 = ZP - A_2P$

$= (90° - \theta) - (90° - \delta) = \delta - \theta = 85° 40' - 26° 40' = \mathbf{39°}$.

∴ Altitude of the star at the upper culmination = $90° - 39° = \mathbf{51°}$.

Example 13.7. *Find the zenith distance and altitude at the lower culmination for a star having declination = 85° 20′ if the latitude of the place of observation = 46° 50′.*

Solution.

$\delta = 85° 20'$; $90° - \theta = 90° - 46° 50' - 43° 10'$

Since the declination of the star is greater than the co-latitude of the place, it is circumpolar and will not set.

In Fig. 13.19, let A_1 be the lower culmination of a circumpolar star M_1. Its zenith distance at the lower culmination = $ZA_1 = ZP + PA_1$

$= (90° - \theta) + (90° - \delta) = 180° - \delta - \theta = 180° - 85° 20' - 46° 50' = \mathbf{47° 50'}$

The altitude of the star = $90° - 47° 50' = \mathbf{42° 10'}$.

Example 13.8. *A star having a declination of 56° 10′ N has its upper transit in the zenith of the place. Find the altitude of the star at its lower transit.*

Solution. (Fig. 13.18)

Let M be the star having A and B as its upper and lower transits. Since the upper culmination is at the zenith, Z and A coincide.

Hence zenith distance of star = zero

and Polar distance of the star = $AP = ZP$ = co-latiude of place

∴ $90° - \delta = 90° - \theta$ or $\theta = \delta = 56° 10'$

At the lowest transit of the star at B, its zenith distance = $ZB = ZP + PB$

$= (90° - \theta) + (90° - \delta) = 180° - \theta - \delta = 180° - 2\delta = 180° - 112° 20' = 67° 40'$

$\therefore$ Altitude of the star at lower transit = 90° – 67° 40′ = **22° 20′**.

Example 13.9. *The altitudes of a star at upper and lower transits of a star are 70° 20′ and 20° 40′, both the transits being on the north side of zenith of the place. Find the declination of the star and the latitude of the place of observation.*

Solution. (Fig. 13.18)

Let M be the star having A and B as its upper and lower culminations.

At the upper culmination, zenith distance = $ZA = ZP - AP$

$$= (90° - \theta) - (90° - \delta) = \delta - \theta$$

$\therefore$ Altitude of star = 90° – zenith distance = 90° – (δ – θ).

But this is equal to 70° 20′ (given)

$$70° \, 20' = 90° - (\delta - \theta)$$

or $$\delta - \theta = 90° - 70° \, 20' = 19° \, 40' \quad \ldots(1)$$

At the lower culmination of the star, the zenith distance of the star

$$= ZB = ZP + PB = (90° - \theta) + (90° - \delta) = 180° - (\theta + \delta)$$

$\therefore$ Altitude of the star = 90° – zenith distance = θ + δ – 90°

But this is equal to 20° 40′ (given)

$$\therefore \quad \theta + \delta - 90° = 20° \, 0' \quad \text{or} \quad \theta + \delta = 110° \, 40' \quad \ldots(2)$$

Solving equation (1) and (2), we get δ = **65° 10′** and θ = **45° 30′**.

Note. *Since the altitudes of the star at both the culminations are positive, the star is circumpolar.*

Example 13.10. *Determine the azimuth and altitude of a star from the following data:*

(i)	*Declination of star*	*= 20° 30′ N*
(ii)	*Hour angle of star*	*= 42° 6′*
(iii)	*Latitude of the observer*	*= 50° N.*

Solution. (Fig. 13.26)

The hour angle of the star = 42° 6′ and since it is measured towards west, the star is in the western part of the hemisphere as shown in Fig. 13.26.

In the astronomical Δ PZM, we have

$$PZ = \text{co-latitude} = 90° - 50° = 40°$$

$$PM = \text{co-declination of star} = 90° - 20° \, 30' = 69° \, 30'$$

$$\angle ZPM = H = 42° \, 6'.$$

It is required to find angle A and ZM.

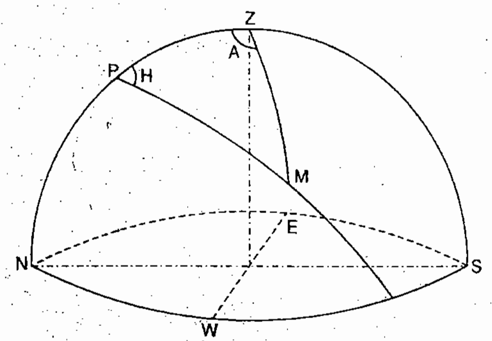


FIG. 13.26.

Using the cosine rule (Eq. 13.2 *a*)

$$\cos ZM = \cos PZ \cos PM + \sin PZ \sin PM \cos H$$
$$= \cos 40° \cos 69° \, 30' + \sin 40° \sin 69° \, 30' \cos 42° \, 6'$$
$$= 0.26828 + 0.44673 = 0.71501$$

$\therefore$ $ZM = 44° \, 21'$

$\therefore$ Altitude of the star = α = 90° – ZM = 90° – 44° 21′ = **45° 39′**

Again, using the cosine rule (Eq. 13.2), we have

$$\cos A = \frac{\cos PM - \cos PZ \, . \cos ZM}{\sin PZ \, . \sin ZM}$$

$$= \frac{\cos 69° \, 30' - \cos 40° \, . \cos 44° \, 21'}{\sin 40° \, . \sin 44° \, 21'} = \frac{0.35021 - 0.54780}{0.44934} = -0.43972.$$

Since cos A is negative the angle A lies between 90° and 180°.

$\therefore$ cos (180° – A) = – cos A = 0.43972

$\therefore$ 180° – A = 63° 55′ or A = 180° – 63° 55′ = **116° 5′ W**.

Example 13.11. *Determine the azimuth and altitude of a star from the following data :*

(i)	*Declination of star*	*= 8° 30′ S*
(ii)	*Hour angle of star*	*= 322°*
(iii)	*Latitude of the observer*	*= 50° N.*

Solution. (Fig. 13.27)

Since the hour angle of the star is more than 180°, it is in the eastern hemi-sphere and its azimuth will be eastern as shown in Fig. 13.27 where ZPM is the astronomical triangle. The star M is below the equator since its declination is negative.

Now, ZP = co-latitude = 90° – 50° = 40°

$$PM = 90° - (-8° \, 30') = 98° \, 30';$$

$$H_1 = 360° - H = 360° - 322° = 38°$$

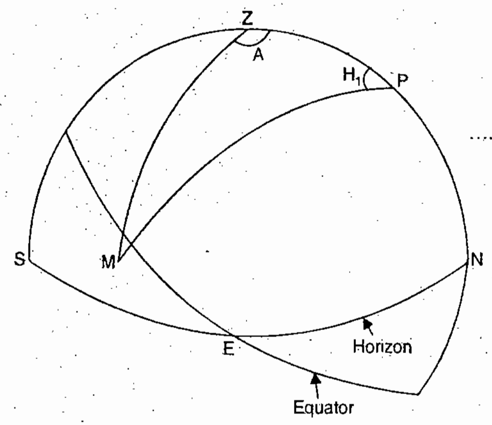


FIG. 13.27

Knowing the two sides and the included angle, the third side can be calculated by the cosine rule (Eq. 13.2 *a*).

Thus $\cos ZM = \cos PZ \, . \cos PM + \sin PZ \times \sin PM \cos H_1$

$$= \cos 40° \, . \cos 98° \, 30' + \sin 40° \sin 98° \, 30' . \cos 38°$$
$$= -0.11323 + 0.50094 = 0.38771$$

$\therefore$ $ZM = 67° \, 11'$

$\therefore$ Altitude of the star = 90° – 67° 11′ = **22° 49′**

(The star is thus above the horizon)

Again, from the cosine rule [Eq. 13.2]

$$\cos A = \frac{\cos PM - \cos PZ \cdot \cos PM}{\sin PZ \cdot \sin ZM} = \frac{\cos 98° 30' - \cos 40° \cos 67° 11'}{\sin 40° \sin 67° 11'}$$

$$= \frac{-0.14781 - 0.29687}{0.59250} = -0.75051.$$

Since $\cos A$ is negative, the value of A is between 90° and 180°

$\therefore$ $\cos (180° - A) = -\cos A = 0.75051$

$\therefore$ $(180° - A) = 41° 22'$ or $A = 138° 38'$

$\therefore$ Azimuth of star = **138° 38′ E.**

Example 13.12. *Determine the hour angle and declination of a star from the following data :*

(*i*) *Altitude of the star* = *22° 36′*

(*ii*) *Azimuth of the star* = *42° W*

(*iii*) *Latitude of the place of observation* = *40° N.*

Solution. (Fig. 13.26)

Since the azimuth of the star is 42° W, the star is in the western hemi-sphere. In the astronomical ΔPZM, we have

PZ = co-latitude = 90° − 40° = 50° ; ZM = co-altitude = 90° − 22° 36° = 67 24′; angle A = 42°

Knowing the two sides and the included angle, the third side can be calculated from the cosine formula (Eq. 13.2 *a*).

Thus, $\cos PM = \cos PZ \cdot \cos ZM + \sin PZ \cdot \sin ZM \cdot \cos A$

$= \cos 50° \cdot \cos 67° 24' + \sin 50° \cdot \sin 67° 24' \cdot \cos 42°$

$= 0.24702 + 0.52556 = 0.77258$

$\therefore$ $PM = 39° 25'$

$\therefore$ Declination of the star = $\delta = 90° - PM = 90° - 39° 25' =$ **50° 35′ *N*.**

Similarly, knowing all the three sides, the hour angle H can be calculated from Eq. 13.2.

$$\cos H = \frac{\cos ZM - \cos PZ \cdot \cos PM}{\sin PZ \cdot \sin PM} = \frac{\cos 67° 24' - \cos 50° \cdot \cos 39° 25'}{\sin 50° \cdot \sin 39° 25'}$$

$$= \frac{0.38430 - 0.49659}{0.48640} = -0.23086$$

$\therefore$ $\cos (180° - H) = 0.23086$ $\therefore$ $180° - H = 76° 39'$

$\therefore$ $H =$ **103° 21′.**

Example 13.13. *Determine the hour angle and declination of a star from the following data :*

(*1*) *Altitude of the star* = *21° 30′*

(*2*) *Azimuth of the star* = *140° E*

(*3*) *Latitude of the observer* = *48° N.*

Solution

Refer Fig. 13.27. Since the azimuth of the star is 140° E, it is in eastern hemi-sphere.

In the astronomical triangle ZPM, we have

$ZM = 90° - \alpha = 90° - 21° 30' = 68° 30'$; $ZP = 90° - \theta = 90° - 48° = 42°$; $A = 140°$

Knowing the two sides and the included angle, the third side can be calculated by the cosine rule (Eq. 13.2 *a*).

Thus $\cos PM = \cos ZM \cos ZP + \sin ZM \sin ZP \cos A$

$= \cos 68° 30' \cos 42° + \sin 68° 30' \sin 42° \cos 140°$

$= 0.27236 - 0.47691 = -0.20455$

$\therefore$ $\cos (180° - PM) = 0.20455$ or $180° - PM = 78° 12'$

$\therefore$ $PM = 101° 48'$

$\therefore$ Declination of the star = $90° - 101° 48' = -11° 48' = -$ **11° 48′** S

Again, knowing all the three sides, the angle H_1 can be calculated from the cosine formula, (Eq. 13.2). Thus

$$\cos H_1 = \frac{\cos MZ - \cos ZP \cdot \cos MP}{\sin ZP \sin MP} = \frac{\cos 68° 30' - \cos 42° \cos 101° 48'}{\sin 42° \sin 101° 48'}$$

$$= \frac{0.36650 + 0.15198}{0.65499} = 0.79161 \quad \therefore \quad H_1 = 37° 40'$$

But H_1 is the angle measured in the eastward direction.

$\therefore$ Hour angle of the star = $360° - H_1 = 360° - 37° 40' =$ **322° 20′.**

Example 13.14. *Calculate the sun's azimuth and hour angle at sunset at a place in latitude 42° 30′ N, when its declination is (a) 22°12′ N and (b) 22° 12′ S.*

Solution

Let us consider the astronomical triangle ZPM, where M is the position of the sun. Since the sun is on the horizon at its setting, it altitude is zero, and hence $ZM = 90°$.

Also, $ZP = 90° - 42° 30' = 47° 30'$

(*a*) $PM = 90° - 22° 12' = 67° 48'$

From the triangle ZPM, we get by cosine rule

$\cos PM = \cos ZP \cdot \cos ZM + \sin ZP \cdot \sin ZM \cdot \cos A$

But $\cos ZM = \cos 90° = 0$ and $\sin ZM = \sin 90° = 1$

$\therefore$ $\cos A = \frac{\cos PM}{\sin ZP} = \frac{\cos 67° 48'}{\sin 47° 30'}$ Hence $A = 59° 10'$

Hence azimuth of the sun at setting = **59° 10′ West.**

Again, from the cosine rule, we get

$\cos ZM = \cos ZP \cdot \cos PM + \sin ZP \cdot \sin PM \cdot \cos H$

But $\cos ZM = \cos 90° = 0$

Hence $\cos H = -\cot ZP \cdot \cot PM = -\cot 47° 30' \cot 67° 48'$

or $\cos (180° - H) = +\cot 47° 30' \cot 67° 48'$

$\therefore$ $180° - H = 68° 03'$ or $H = 180° - 68° 03' = 111° 57'$

Hence sun's hour angle at sunset = $111° 57'$ = **$7^h 27^m 48^s$.**

(*b*) As before, the azimuth is given by

$$\cos A = \frac{\cos PM}{\sin ZP}. \quad \text{Here, } PM = 90° - (-22° 12') = 122° 12'$$

and $ZP = 47° 37'$ and $ZM = 90°$ as before

$$\therefore \quad \cos A = \frac{\cos 112° 12'}{\sin 47° 30'} = -\frac{\cos 67° 48'}{\sin 47° 30'}$$

or $$\cos (180° - A) = +\frac{\cos 67° 48'}{\sin 47° 30'}$$

From which, $180' - A = 59° 10'$ or $A = 120° 50'$

$\therefore$ Azimuth of the sun at sunset = **120° 50'** West.

Similarly, $\cos H = -\cot ZP \cdot \cot PM = -\cot 47° 30' \cot 112° 12' = \cot 47° 30' \cot 67° 48'$

$\therefore \quad H = 68° 3'$

Hence sun's hour angle at sunset = $68° 3' = \mathbf{4^h\ 32^m\ 12^s}$.

Example 13.15. *Calculate the sun's hour angle and azimuth at sunrise for a place in latitude 42° 30′ S when the declination is 22° 12′ N.*

Solution

Consider the astronomical triangle $Z'P'M$, where M is the position of the sun at the horizon and P' is the south pole.

$$Z'P' = 90° - \theta = 90° - 42° 30' = 47° 30'$$

$$Z'M = 90°, \text{ since the sun is at horizon}$$

$$MP' = 90° + 22° 12' = 112° 12'.$$

By the cosine rule, $\cos Z'M = \cos Z'P' \cdot \cos P'M + \sin Z'P' \sin P'M \cdot \cos H$

But $\cos Z'M = \cos 90° = 0$

Hence $\cos H = -\cot Z'P' \cot P'M = -\cot 47° 30' \cot 112° 12'$

$$= \cot 47° 30' \cot 67° 48'$$

Hence $H = 68° 3'$

Since the sun is at its setting, its hour angle is eastern.

Hence westerly hour angle of sun = $180° - 68° 3' = 111° 57' = \mathbf{7^h\ 27^m\ 48^s}$

Again, as before, $$\cos A = \frac{\cos P'M}{\sin Z'P'} = \frac{\cos 112° 12'}{\sin 47° 30'} = -\frac{\cos 67° 48'}{\sin 47° 30'}$$

$$\therefore \quad \cos (180° - A) = -\frac{\cos 67° 48'}{\sin 47° 30'}$$

$\therefore \quad 180° - A = 59° 10'$ or $A = 180° - 59° 10'$

Hence the azimuth of the sun = **120° 50' East.**

13.7. THE EARTH AND THE SUN

1. The Earth. The Earth is considered approximately spherical in shape. But actually, it is very approximately an *oblate spheroid*. Oblate spheroid is the figure formed by revolving an ellipse about its minor axis. The earth is flattened at poles – its diameter along the polar axis being lesser than its diameter at the equator. The equatorial radius a of the earth, according to Hayford's spheroid is 6378.388 km and the polar radius b of the earth is 6356.912 km. The *ellipticity* is expressed by the ratio $\frac{a-b}{a}$, which reduces to $\frac{1}{297}$. For the Survey of India, Everest's first constants were used as follows : $a = 20{,}922.932$ ft and $b = 20{,}853{,}642$ ft, the ellipticity being $\frac{1}{311.04}$.

The earth revolves about its minor or shorter axis (*i.e.* polar axis), on an average, once in twenty-four hours, from West to East. If the earth is considered stationary, the whole celestial sphere along with its celestial bodies like the stars, sun, moon etc. appear to revolve round the earth from East to West. The axis of rotation of earth is known as the *polar axis*, and the points at which it intersects the surface of earth are termed the North and South *Geographical* or *Terrestrial Poles*. In addition to the motion of rotation about its own polar axis, the earth has a motion of rotation relative to the sun, in a plane inclined at an angle of 23° 27' to the plane of the equator. The time of a complete revolution round the sun is one year. The apparent path of the sun in the heavens is the result of both the diurnal and annual real motions of the earth.

The earth has been divided into certain *zones* depending upon the parallels of latitude of certain value above and below the equator. The zone between the parallels of latitude $23° 27\frac{1}{2}'$ N and $23° 27\frac{1}{2}'$ S is known as the *torrid zone* (see Fig. 13.12). This is the hottest portion of the earth's surface. The belt included between $23° 27\frac{1}{2}'$ N and $66° 32\frac{1}{2}'$ N of equator is called the *north temperate zone*. Similarly, the belt included between $23° 27\frac{1}{2}'$ S and $66° 32\frac{1}{2}'$ S is called *south temperate zone*. The belt between $66° 32\frac{1}{2}'$ N and the north pole is called the *north frigid zone* and the belt between $66° 32\frac{1}{2}'$ S and the south pole is called *south frigid zone*.

2. The sun. The sun is at a distance of 93,005,000 miles from the earth. The distance is only about $\frac{1}{250{,}000}$ of that of the nearest star. The diameter of the sun is about 109 times the diameter of the earth, and subtends and angle of 31′ 59″ at the centre of the earth. The mass of the sun is about 332,000 times that of the earth. The temperature at the centre of the sun is computed to be about 20 million degrees.

The sun has two *apparent* motions, one with respect to the earth from east to west, and the other with respect to the fixed stars in the celestial sphere. The former apparent path of the sun is in the plane which passes through the centre of the celestial sphere and intersects it in a great circle called the *ecliptic*. The apparent motion of the sun is along this great circle. The angle between the plane of equator and the ecliptic is called the *Obliquity of Ecliptic*, its value being 23° 27'. The obliquity of ecliptic changes with a mean annual diminution of 0′.47.

The points of the intersection of the ecliptic with the equator are called the equinoctial points, the declination of the sun being zero at these points. The *Vernal Equinox* or the First point of Aries (♈) is the point in which the sun's declination changes from south to north. The *Autumnal Equinox* or the *First point of Libra* (♎) is the point in which the sun's declination changes from north to south. The points at which sun's declinations are a maximum are called *solstices*. The point at which the north declination of sun is maximum is called the *summer solstice*, while the point at which the south declination of the sun is maximum is known as the *winter solstice*.

The Earth's Orbital Motion Round the Sun — The Seasons

The earth moves *eastward* around the sun once in a year in a path that is *very nearly* a huge circle with a radius of about 93 millions of miles. *More accurately*, the path is described as an *ellipse*, one focus of the ellipse being occupied by the sun. The

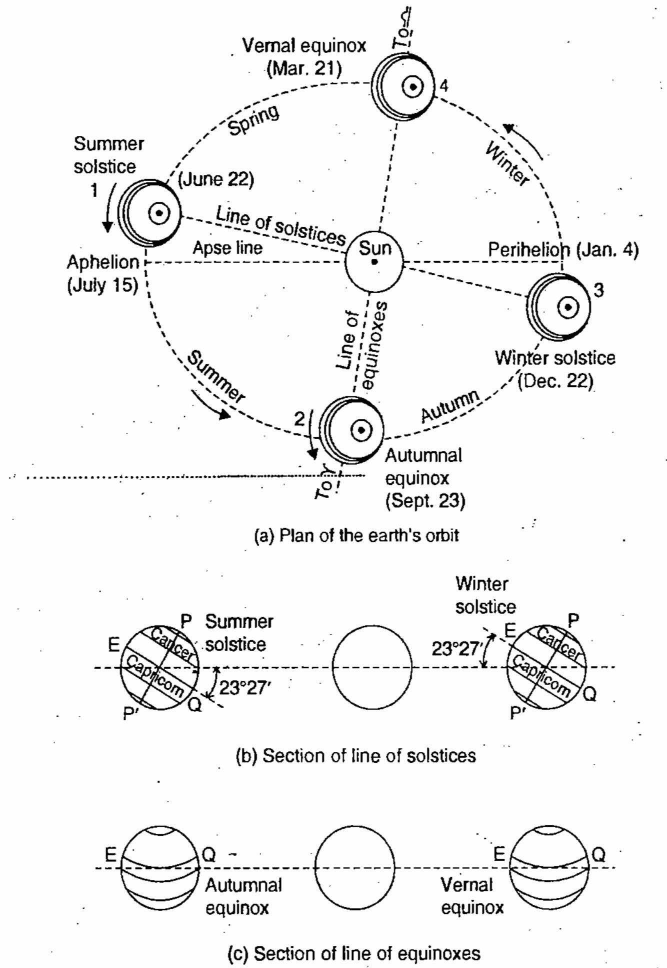


FIG. 13.28. EFFECT OF EARTH'S ANNUAL MOTION.

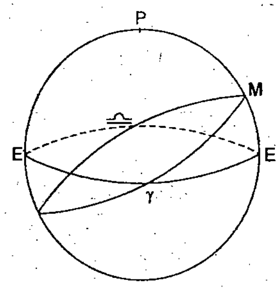
earth is thus at varying distances from the sun. The orbit lies (very nearly) in one *plane*. The apparent path of the sun is in the same plane. The plane passes through the centre of the celestial sphere and intersects it in a great circle called the *ecliptic*. The plane of the ecliptic is inclined at about 23° 27′ to that of the equator. Hence, the *axis of the earth is inclined to the plane of the ecliptic at an angle of 66° 33′, and remains practically parallel to itself throughout the year*. The inclination of the axis of the earth round its orbit causes variations of seasons. Fig. 13.28 shows the diagrammatic plan and sections of earth's orbit.

As previously mentioned, the earth's orbit is an ellipse with the sun at one of its foci. The earth is thus at varying distances from the sun. The earth is at a point nearest the sun (called the *perihelion* of the earth's orbit) on about January 4 and at a point farthest from the sun (called the *aphelion* of the earth's orbit) on about July 5. The earth's rate of angular movement around the sun is greatest at perihelion and least at aphelion.

In position I, the earth is in that part of the orbit where the northern end of the axis is pointed towards the sun. The sun appears to be farthest north on about June 22, and at this time the days are longest and nights are shortest. The summer begins in the northern hemisphere. This position of the earth is known as the summer *solstice*. In position 2 (Sept. 23), the sun is in the plane of the equator. The nights are equal everywhere. The instant at which this occurs is called the *Autumnal Equinox*. The axis of the earth is perpendicular to the line joining the earth and the sun. In position 3, the earth is in that part of the orbit where the northern end of axis is pointed away from the sun. The sun appears to be farthest south (Dec. 22) and at this time winter begins in the northern hemisphere. The days are shortest and nights are longest. The position of the earth is known as the *winter solstice*. In position 4 (March 21), the sun is again in the plane of the equator. The day and night are equal everywhere. The instant at which this occurs is called the *Vernal Equinox*. The line of the equinoxes is the intersection of the planes of the ecliptic and the equator, and is at right angles to the line of solstices.

Fig. 13.29 (*b*) shows the sun's apparent positions at different seasons. Let us study this in conjunction with Fig. 13.29 (*a*). Thus, on Fig. 13.29 (*a*), we shall trace the annual motion of the sun, while on Fig. 13.29 (*b*), we shall trace the apparent diurnal paths of the sun at different seasons. As is clear from Fig. 13.29 (*a*), the sun's declination changes daily as it progresses along the ecliptic. Due to the change in the declination, its apparent path of each day is different from that of the day before. *The apparent path thus ceases to be circular and all the daily paths taken together will give rise to one continous spiral curve*. However, for explanation purposes, we shall assume that throughout each day, the sun's declination is constant – retaining the same value it has at sunrise. On this assumption the sun's daily paths will consist of a series of parallels instead of a spiral as illustrated in Fig. 13.29 (*b*).

On 21st March, the sun is at ♈ [Fig. 13.29 (*a*)] and its declination is zero. The sun's daily path on this day will be along the equator rising at E and setting at W of the horizon. Its hour angle at E will be $EPZ = 90°$ when it rises. At W, it will again have an hour angle of 90° when it sets. *Thus, day and night will be of equal duration*. On that day, the meridian altitude SB of the sun is equal to the co-latitude. As the sun advances along the ecliptic, its declination increases. At the solstitial point M, it attains its maximum declination (23° 27′) on about June 22. The parallel A_1AA_2 represents sun's path on that day. The sun rises at A_1 when its hour angle is equal to A_1PZ which is greater than 90°. The sun sets at A_2 when its hour angle is greater than 90°. *The day is thus longest* on 22nd June. The meridian altitude SA also attains its maximum value. On Sept. 23, the declination of the sun is again zero, the sun's daily path is along the equator and the day and night are of equal length. As the motion of the sun continues along the ecliptic, its declination increases to the south of the equator. On December 22, its southern declination is maximum, C_1CC_2 represents sun's path on that day. The sun rises at C_1 when it has the hour angle C_1PZ which is evidently less than 90° and sets at C_2 when its hour angle is less than 90°. The day is thus shorter than the night.



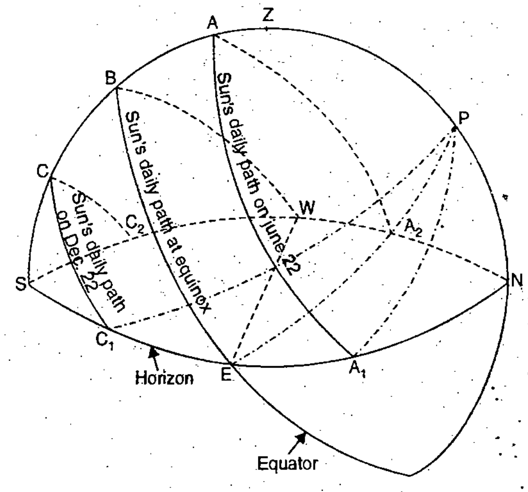


FIG. 13.29. SUN'S APPARENT POSITIONS AT DIFFERENT SEASONS.

It is colder in winter due to two main reasons :

(1) the days are shorter in winter.

(2) the rays of sunlight strike the surface of the ground more obliquely, thus weakening the heating power of the sun's rays.

Though the earth is nearer to the sun in winter it has very small effect in making the winter hotter. The amount of heat received from the sun depends upon the time it remains above the horizon, and also on the altitude it attains during the day.

MEASUREMENT OF TIME

Due to the intimate relationship with hour angle, right ascension and longitude, the knowledge of measurement of time is most essential. The measurement of time is based upon the apparent motion of heavenly bodies caused by earth's rotation on its axis. *Time* is the interval which lapses, between any two instants. In the subsequent pages, we shall use the following abbreviations.

G.M.T.	... Greenwhich Mean Time	G.M.M.	... Greenwich Mean Midnight
G.A.T.	... Greenwich Apparent Time	L.A.N.	... Local Apparent Noon
G.S.T.	... Greenwich Sidereal Time	L.M.M.	... Local Mean Midnight
L.M.T.	... Local Mean Time	L.Std.T.	... Local Standard Time
L.A.T.	... Local Apparent Time	N.A.	... Nautical Almanac
L.S.T.	... Local Sidereal Time	S.A.	... Star Almanac
G.M.N.	... Greenwich Mean Noon		

13.8. UNITS OF TIME

There are the following systems used for measuring time :

1. Sidereal Time
2. Solar Apparent Time
3. Mean Solar Time
4. Standard Time

1. Sidereal Time

Since the earth rotates on its axis from west to east, all heavenly bodies (*i.e.* the sun and the fixed stars) appear to revolve from east to west (*i.e.* in clock-wise direction) around the earth. Such motion of the heavenly bodies is known as *apparent* motion. We may consider the earth to turn on it axis with absolute regular speed. Due to this, the stars appear to complete one revolution round the celestial pole as centre in constant interval of time, and they cross the observer's meridian twice each day. For astronomical purposes the sidereal day is one of the principal units of time. *The sidereal day is the interval of time between two successive upper transits of the first point of Aries* (♈). It begins at the instant when the first point of Aries records $0^h 0^m 0^s$. At any other instant, the *sidereal time* will be the hour angle of ♈ reckoned westward from 0^h to 24^h. The sidereal day is divided into 24 hours, each hour subdivided into 60 minutes and each minute into 60 seconds. However, the position of the Vernal Equinox is not fixed. It has slow (and variable) westward motion caused by the *precessional* movement of the axis, the actual interval between two transits of the equinox differs about 0.01 second of time from the true time of one rotation.

Local Sidereal Time (L.S.T.) The local sidereal time is the time interval which has elapsed since the transit of the first point of Aries over the meridian of the place.

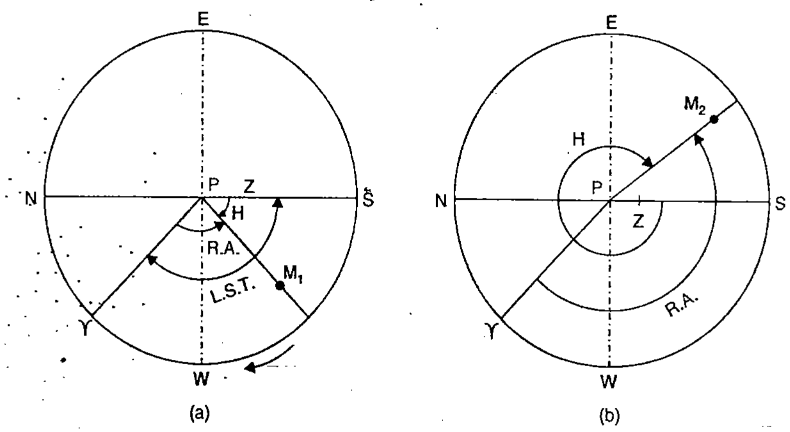


FIG. 13.30

It is, therefore, a measure of the angle through which the earth has rotated since the equinox was on the meridian. *The local sidereal time is, thus, equal to the right ascension of the observer's meridian.*

Since the sidereal time is the hour angle of the firsr point of Aries, *the hour angle of a star is the sidereal time that has elapsed since its transit.* In Fig 13.30, M_1 is the position of a star having SPM_1 (= H) as its hour angle measured *westward* and ΥPM_1 is its right ascension (R.A.) measured *eastward*. $SP\Upsilon$ is the hour angle of Υ and hence the local sidereal time.

Hence, we have $$SPM_1 + M_1P\Upsilon = SP\Upsilon$$

or **star's hour angle + star's right ascension = local sidereal time** ...(1)

If this sum is greater than 24 hours, deduct 24 hours, while if it is negative *add*, 24 hours,

In Fig. 13.30 (*b*), the star M_2 is in the other position. ΥPM_2 is its Right Ascension (eastward) and ZPM_2 is its hour angle (westward). Evidently,

$$ZPM_2 \text{ (exterior)} + \Upsilon PM_2 - 24^h = SP\Upsilon = \text{L . S . T .}$$

or **star's hour angle + star's right ascension** $- 24^h$ = L . S . T .

This supports the preposition proved with reference to Fig. 13.30 (*a*). The relationship is true for all positions of the star.

When the star is on the meridian, its hour angle is zero. Hence equation 1 reduces to

Star's right ascesion = local sidereal time at its transit.

A sidereal clock, therefore, records the right ascension of stars as they make their upper transits.

The hour angle and the right ascension are generally measured in *time* in preference to angular units. Since one complete rotation of celestial sphere through 360° occupies 24 hours. we have

$$24 \text{ hours} = 360° \quad ; \quad 1 \text{ hour} = 15°$$

The difference between the local sidereal times of two places is evidently equal to the difference in their longitudes.

2. Solar Apparent Time

Since a man regulates his time with the recurrence of light and darkness due to rising and setting of the sun, the sidereal division of time is not suited to the needs of every day life, for the purposes of which the sun is the most convenient time measurer. A *solar day* is the interval of time that elapes between two successive *lower* transits of the sun's centres over the meridian of the place. The lower transit is chosen in order that the date may change at mid-night. The solar time at any instant is the hour angle of the sun's centre reckoned westward from 0^h to 24^h. This is called the *apparent solar time*, and is the time indicated by a sun-dial. Unfortunately, the apparent solar day is not of constant length throughout the year but changes. Hence our modern clocks and chronometers cannot be used to give us the apparent solar time. The non-uniform length of the day is due to two reasons :

(1) The orbit of the earth round the sun is not circular but elliptical with sun at one of its foci. The distance of the earth from the sun is thus variable. In accordance with the law of gravitation, the apparent angular motion of the sun is not uniform – it moves faster when is nearer to the earth and slower when away. Due to this, the sun reaches the meridian sometimes earlier and sometimes later with the result that the days are of different lengths at different seasons.

(2) The apparent diurnal path of the sun lies in the ecliptic. Due to this, even though the eastward progress of the sun in the ecliptic were uniform, the time elapsing between the departure of a meridian from the sun and its return thereto would vary because of the obliquity of the ecliptic.

The sun changes its right ascension from 0^h to 24^h in one year, advancing eastward among the stars at the rate of about 1° a day. Due to this, the earth will have to turn nearly 361° about its axis to complete one solar day, which will consequently be about 4 minutes longer than a sidereal day. Both the obliquity of the ecliptic and the sun's unequal motion cause a *variable rate of increase* of the sun's right ascension. If the rate of change of the sun's right ascension were uniform, the solar day would be of constant length throughout the year.

3. Mean Solar Time

Since our modern clocks and chronometers cannot record the variable apparent solar time, a *fictitious sun* called the *mean sun* is imagined to move at a uniform rate along the equator. The motion of the mean sun is the average of that of the true sun in its right ascension. It is supposed to start from the vernal equinox at the same time as the true sun and to return the vernal equinox with the true sun. The *mean solar day* may be defined as the interval between successive transit of the mean sun. The mean solar day is the average of all the apparent solar days of the year. The mean sun has the constant rate of increase of right ascension which is the average rate of increase of the true sun's right ascension.

The *local mean noon* (L.M.N.) is the instant when the mean sun is on the meridian. The mean time at any other instant is given by the hour angle of the mean sun reckoned westward from 0 to 24 hours. For civil purposes, however, it is found more convenient to begin the day at midnight and complete it at the next midnight, dividing it into two periods of 12 hours each. Thus, the zero hour of the mean day is at the *local mean midnight* (L.M.M.). The *local mean time* (L.M.T.) is that reckoned from the local mean midnight. The difference between the local mean time between two places is evidently equal to the difference in the longitudes.

From Fig. 13.30 (*a*) if M_1 is the position of the sun, we have

Local sidereal time = R.A. of the sun + hour angle of the sun ...(1)

Similarly, Local sidereal time = R.A. of the mean sun + hour angle of the mean sun ...(2)

The hour angle of the sun is zero at its upper transit. Hence

Local sidereal time of apparent noon = R.A. of the sun ...(3)

Local sidereal time of mean noon = R.A. of the mean sun ...(4)

Again, since the hour angle of the sun (true or mean) is zero at its upper transit while the solar time (apparent or mean) is zero as its lower transit, we have

The apparent solar time = the hour angle of the sun + 12^h ...(5)

The mean solar time = the hour angle of mean sun + 12^h ...(6)

Thus, if the hour angle of the mean sun is 15° (1 hour) the mean time is 12 + 1 = 13 hours, which is the same thing as 1 o'clock mean time in the afternoon; if the hour angle of the mean sun is 195° (13 hours), the mean time is 12 + 13 = 25 hours, which is the same as 1 o'clock mean time after the midnight (*i.e.*, next. day).

The Equation of Time

The difference between the mean and the apparent solar time at any instant is known as the *equation of time*. Since the mean sun is entirely a fictitious body, there is no means to directly observe its progress. Formerly, the apparent time was determined by solar observations and was reduced to mean time by equation of time. Now-a-days, however, mean time is obtained more easily by first determining the sidereal time by steller observations and then converting it to mean time through the medium of wireless signals. *Due to this reason it is more convenient to regard the equation of time as the correction that must be applied to mean time to obtain apparent time.* The nautical almanac tabulates the value of the equation of time for every day in the year, in this sense (*i.e.* apparent – mean). Thus, we have

Equation of time = Apparent solar time – Mean solar time.

The equation of time is *positive* when the apparent solar time is *more* than the mean solar time ; to get the apparent solar time, the equation of time should then be added to mean solar time. For example, at 0^h G.M.T. on 15 October 1949, the equation of the time is + $13^m\ 12^s$. This means that the apparent time at 0^h mean time is $0^h\ 13^m\ 12^s$. In other words, the true sun is $13^m\ 12^s$ ahead of the mean sun. Similarly, the equation of time is negative when the apparent time is less than the mean time. For example, at 0^h G.M.T. on 18 Jan., 1949, the equation of time is – $10^m\ 47^s$. This means that the apparent time at 0^h mean time will be $23^h\ 49^m\ 13^s$ on January 17. In other words, the true sun is behind the mean sun at that time.

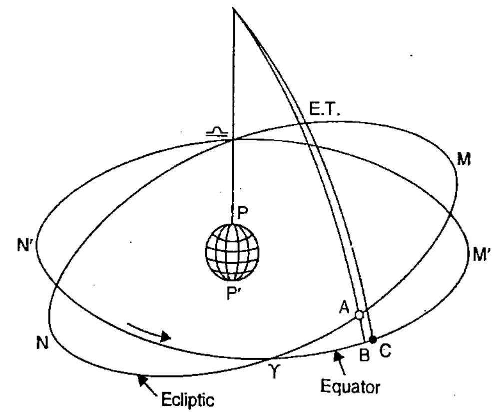
The value of the equation of time varies in magnitude throughout the year and its value is given in the Nautical Almanac at the instant of apparent midnight for the places on the meridian of Greenwich for each day of the year. For any other time it must be found by adding or subtracting the amount by which the equation has increased or diminished since midnight.

It is obvious that the equation of time is the value expressed in time, of the difference at any instant between the respective hour angles or right ascensions of the true and mean suns.

The amount of equation of the time and its variations are due to two reasons : *(1) obliquity of the ecliptic, and (2) ellipticity of the orbit*. We shall discuss both the effects separately and then combine them to get the equation of time.

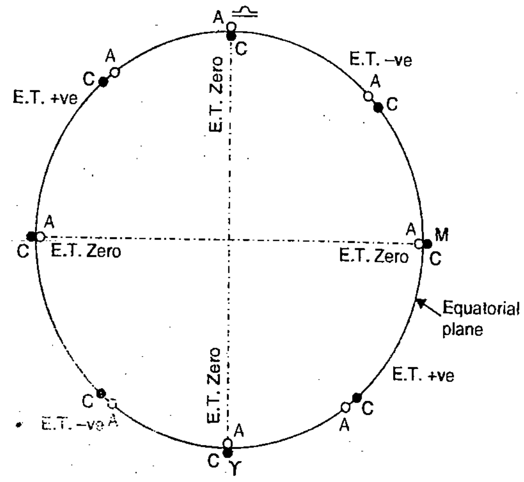
(1) Obliquity of the ecliptic

Neglecting the elliptic motion, let the true sun describe the ecliptic orbit ♈M ♎N with *uniform angular velocity*. Let the mean sun describe the equatorial orbit ♈M′♎N′ with the same uniform angular velocity. Let both the suns start from ♈ at the same instant in the direction of the arrow. The earth axis PP' also turns in the same direction once in a day. When the true sun is at A, the mean sun will be at C such that ♈A = ♈C. If a declination circle is drawn through A, it will meet the equator in B. The difference between the declination circles of A and C will then be the equation of time. The points



(a)

A = True Sun ; C = Mean Sun



(b)

FIG. 13.31. EFFECT OF OBLIQUITY OF THE ECLIPTIC.

B and *C* will coincide only at equinoxes and solstices. Between the equinox to solstice, *C* will be in advance of *B*, and any given meridian will (as the earth rotates in the direction of the arrow) overtake first the true sun *A* and then the mean sun. *That is, apparent noon will precede mean noon and hence the equation of time will be additive.* Similarly, between the solstice to equinox, *C* will be behind *A* and the equation of time is subtractive. In Fig. 13.33, the curve *A-A* denotes the equation of time due to the obliquity of ecliptic. *It may be noted that the equation of time due to this reason vanishes four times in a year — at equinoxes and solstices.* Fig. 13.31 (*b*) shows the plan, on equatorial plane, of the positions of the true and mean sun at different parts of the year.

Thus, to conclude, *the equation of time due to obliquity of the ecliptic is due to the fact that the uniform motion along the ecliptic does not represent uniform motion in the right ascension.*

2. Ellipticity or the Eccentricity of the Orbit

Let us now neglect the obliquity of ecliptic so that the orbit of the sun is in the equator, and its apparent path is elliptical as shown in Fig. 13.32. At the Perihelion (December 31), the true sun (*A*) and the mean sun (*C*) start at the same instant. The mean sun (*C*) rotates with uniform rate while the true sun (*A*) moves with the greater angular velocity since it is nearer the earth at Perihelion. Due to this, the true sun precedes the mean sun. Now, since the earth rotates from west to east (*i.e.*, in the same direction as that of the motion of the sun along its orbit indicated by the arrow), any meridian at a place on it will overtake the mean sun before the true sun. The mean noon will thus occur before the apparent noon, the mean time will exceed the apparent time and hence the *equation of time will be negative.* After 90° from the Perihelion, the true sun, though ahead of the mean sun, will have decrease in its angular velocity so that the distance between the sun and the mean sun goes on decreasing. At the Aphelion (July 1), both the suns meet and the equation of time becomes zero. Between December 31 to July 1, equation of time thus remains negative. After July 1, the true sun has lesser angular velocity than the uniform velocity of the mean sun, and the mean sun precedes the true sun. The apparent noon will thus occur earlier than the mean noon at a particular meridian, the apparent time exceeds the mean time, and the *equation of time becomes positive.* After about 90° (October 1) from the Aphelion, the gap between the mean sun and true sun gradually reduces due to gradual increase in the angular velocity of the true sun, till both the suns reach perihelion at the same

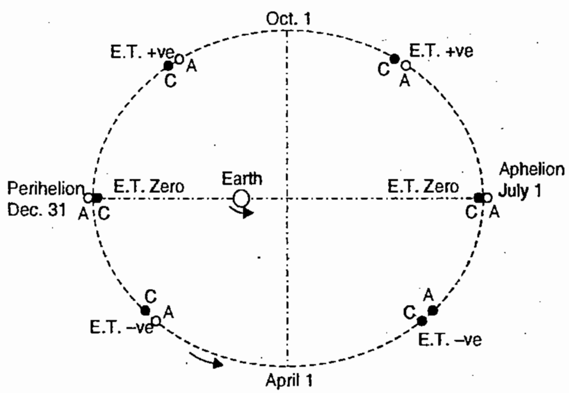


FIG. 13.32. EFFECT OF ELLIPTICITY OF THE ORBIT.

instant. The equation of time is thus positive from July 1 to December 31. In Fig. 13.33, the curve *B-B* denotes the equation of time due to ellipticity of the orbit.

The Final Curve for Equation of Time

In Fig. 13.33, the curve *C-C* shows the final equation of time obtained by combining the curves *A-A* and *B-B*. It will be seen that the equation of time vanishes four times a year, on or about April 16, June 14, September 2, and December 25. From December 25 till April 16, it is negative having a maximum value of about $14^m 20^s$ on February 12. From April 16 to June 14 it is positive, having its maximum value of about $3^m 44^s$ on May 15. From June 14 to September 2, it is again negative with a maximum value of $6^m 24^s$ on July 27. Between September 2 and December 25, it is again positive, attaining its greatest positive value for the year 1951, about $16^m 23^s$ on November 4.

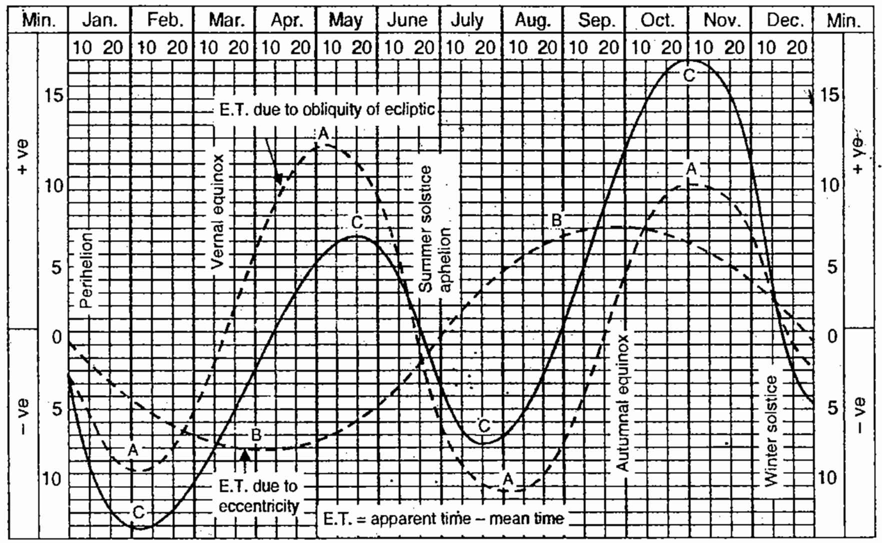


FIG. 13.33. THE EQUATION OF TIME : THE CORRECTION TO BE ADDED TO THE MEAN TIME TO OBTAIN APPARENT TIME.

4. Standard Time

We have seen that the local mean time at a particular place is reckoned from the lower transit of the mean sun. Thus, at different meridians there will be different local mean times. In order to avoid confusion arising from the use of different local mean time it is necessary to adopt the mean times on a particular meridian as the standard time for the whole of the country. Such a *standard meridian* lies an exact number of hours from Greenwich. The mean time associated with the standard meridian is known as the *standard time*. The difference between standard time and local mean time at any place is that due to the difference of longitude between the given place and the standard meridian used. For places east of the standard meridian, local mean time is later (or greater) than andard time, and for places to the west, the local time is earlier (or lesser). *The following re the standard meridians of the some of the countries* :

	Country	*Longitude of standard meridian*	
		Degrees	*Times* Hrs. Mts.
.	Great Britain, Belgium, Spain	0°	0 – 00
2.	Germany, Switzerland	15° *E*	1 – 00
3.	India	$82\frac{1}{2}°$ *E*	5 – 30
4.	Western Australia	120° *E*	8 – 00
5.	New Zealand	180° *E*	12 – 00
6.	Central Zones of U.S.A.	90° *W*	6 – 00
7.	British Columbia	120° *W*	8 – 00

The civil time for the meridian of Greenwich reckoned from midnight, is known s the *Universal Time* (U.T.)

'he Astronomical and Civil Time

The astronomers count the mean solar day as beginning at midnight and divide it ontinuously from 0^h to 24^h. However, for ordinary purposes, it is preferable to divide he day into halves and to count from two zero points : (1) From midnight to noon s called A.M. (*ante meridiem*), and (2) from noon to midnight is called **P.M.** (*post neridiem*).

Example 13.16. *Find the equation of time at 12^h G.M.T. on July 1, 1951 from he following data obtained from N.A.*

(a) E.T. at Greenwich mean midnight on July 1, 1951 = -3^m 28.41^s.

(b) Change between the value for 0^h July 1, and that for 0^h July 2 = -11.82^s.

Solution

The change in the equation of time for 24 hours = -11.82^s

$\therefore$ Change in equation of time for $12^h = \frac{11.82}{24} \times 12 = -5.91^s$

$\therefore$ E.T. at 12^h G.M.T. = $-3^m\ 28.41^s - 5.91^s = -\mathbf{3^m\ 34.32^s}$.

Example 13.17. *Find the G.A.T. on February 16, 1951, when the G.M.T. is $0^h\ 30^m$ A.M. Given E.T. at G.M.N. on Feb. 16, 1951 = $-14^m\ 10^s$ increasing at the rate f 1 second per hour.*

Solution. E.T. at G.M.N. = $-14^m\ 10^s$. Since the E.T. is increasing after G.M.N., ts value will be less than $14^m\ 10^s$ before noon.

Now, $10^h\ 30^m$ A.M. occurs $1^h\ 30^m$ before the noon.

Change in E.T. in $1^h\ 30^m$ = 1 sec × 1.5 = 1.5 seconds.

$\therefore$ Equation of time at $10^h\ 30^m$ A.M. = $-[14^m\ 10^s - 1.5^s] = -14^m\ 8.5^s$

Now G.A.T. = G.M.T. + E.T. = $10^h\ 30^m - 14^m\ 8.5^s = \mathbf{10^h\ 15^m\ 51.5^s}$.

13.9. INTERCONVERSION OF TIME

13.9.1. RELATION BETWEEN DEGREES AND HOURS OF TIME

The degrees may be converted into hours and *vice versa* by the following relation:

360° = 24 hours.

15° = 1 h	1 h = 15°
1° = 4 m	1 m = 15′
15′ = 1 m	1 s = 15″
1′ = 4 s	
15″ = 1 s	

Example 13.18. *Express the following angles in hours, minutes and seconds : (a) 50° 12′ 48″, (b) 8° 18′ 6″, (c) 258° 36′ 30″.*

Solution.

(a) $50° = \frac{50}{15}h = 3^h\ 20^m\ 0^s$
$12' = \frac{12}{15}m = 0^h\ 0^m\ 48^s$
$48'' = \frac{48}{15}s = 0^h\ 0^m\ 3.2^s$
Total = $3^h\ 20^m\ 51.2^s$

(b) $8° = \frac{8}{15}h = 0^h\ 32^m\ 0^s$
$18' = \frac{18}{15}m = 0^h\ 1^m\ 12^s$
$6'' = \frac{6}{15}s = 0^h\ 0^m\ 0.4^s$
Total = $0^h\ 33^m\ 12.4^s$

(c) $258° = \frac{258}{15}h = 17^h\ 12^m\ 0^s$
$36' = \frac{36}{15}m = 0^h\ 2^m\ 24^s$
$30'' = \frac{30}{15}s = 0^h\ 0^m\ 2^s$
Total = $17^h\ 14^m\ 26^s$

Example 13.19. *Express the following hours etc. into degrees, minutes and seconds: (a) $4^h\ 34^m\ 13^s$, (b) $18^h\ 11^m\ 38^s$.*

Solution.

(a) $4^h = 4 \times 15° = 60°\ 0'\ 0''$
$34^m = 34 \times 15' = 8°\ 30'\ 15°$
$13^s = 13 \times 15'' = 0°\ 3'\ 15''$
Total = 68° 33′ 15″

(b) $18^h = 18 \times 15° = 270°\ 0'\ 0''$
$11^m = 11 \times 15' = 2°\ 45'\ 0''$
$38^s = 38 \times 15'' = 0°\ 9'\ 30''$
Total = 272° 54′ 30″

13.9.2. CONVERSION OF LOCAL TIME TO STANDARD TIME AND VICE VERSA

The difference between the standard time and the local mean time at a place is equal to the difference of longitudes between the place and the standard meridian.

If the meridian of the place is situated *east* of the standard meridian, the sun, while moving apparently from east to west, will transit the meridian of the place earlier than the standard meridian. Hence the local time will be greater than the standard time. Similarly, if the meridian of the place is to the *west* of the standard meridian, the sun will transit the standard meridian earlier than the meridian of the place and hence the local time will be lesser than the standard time. Thus, we have

L.M.T. = Standard M.T. ± Difference in the longitudes $\left(\frac{E}{W}\right)$ ...(1)

L.A.T. = Standard A.T. ± Difference in the longitudes $\left(\frac{E}{W}\right)$ ...(2)

L.S.T. = Standard S.T. ± Difference in the longitudes $\left(\frac{E}{W}\right)$ ...(3)

Use (+) sign if the meridian of place is to the east of the standard meridian, and (−) sign if it to the west of the standard meridian.

If the local time is to be found from the given Greenwich time, we have

$$L.M.T. = G.M.T. \pm \text{Longitude of the place} \left(\frac{E}{W}\right).$$

Example 13.20. *The standard time meridian in India is 82° 30′ E. If the standard time at any instant is 20 hours 24 minutes 6 seconds, find the local mean time for two places having longitudes (a) 20° E, (b) 20° W.*

Solution

(*a*) The longitude of the place = 20° *E*

Longitude of the standard meridian = 82° 30′ *E*

∴ Difference in the longitudes = 82° 30′ − 20° = 62° 30′, *the place being to the west of the standard meridian.*

Now 62° of longitude $= \frac{62°}{15}$ h $= 4^h\ 8^m\ 0^s$

30′ of longitude $= \frac{30}{15}$ m $= 0^h\ 2^m\ 0^s$

∴ Total $= 4^h\ 10^m\ 0^s$

Now L.M.T. = Standard time − Difference in longitude (*W*)

$= 20^h\ 24^m\ 6^s - 4^h\ 10^m\ 0^s = 16^h\ 14^m\ 6^s$ past midnight $= 4^h\ 14^m\ 6^s$ P.M.

(*b*) Longitude of the place = 20° *W*

Longitude of the standard meridian = 82° 30′ *E*.

∴ Difference in the longitude = 20° + 82° 30′ = 102° 30′, the meridian of the place being to the west to the standard meridian.

Now 102° of longitude $= \frac{102}{15}$ h $= 6^h\ 48^m\ 0^s$

30′ of longitude $= \frac{30}{15}$ m $= 0^h\ 2^m\ 0^s$

Total $= 6^h\ 50^m\ 0^s$

Standard time $= 20^h\ 24^m\ 6^s$

Subtract the difference in longitude $= 6^h\ 50^m\ 0^s$

∴ Local mean time $= 13^h\ 34^m\ 6^s$ past mid-night $= 1^h\ 34^m\ 6^s$ P. M.

Example 13.21. *Find the G.M.T. corresponding to the following L.M.T.*

(*a*) *$9^h\ 40^m\ 12^s$ A.M. at a place in longitude 42° 36′ W.*

(*b*) *$4^h\ 32^m\ 10^s$ A.M. at a place in longitude 56° 32′ E.*

Solution.

(*a*) Longitude of the place is 42° 36′ *W*

Now 42° $= \frac{42}{15}$ h $= 2^h\ 48^m\ 0^s$

36′ $= \frac{36}{15}$ m $= 0^h\ 2^m\ 24^s$

Total $= 2^h\ 50^m\ 24^s$

Now since the place is to the west of Greenwich, the Greenwich time will be more.

∴ G.M.T. = L.M.T. + Longitude (*W*)

L.M.T. $= 9^h\ 40^m\ 12^s$ (A.M.)

Add the longitude $= 2^h\ 50^m\ 24^s$

∴ G.M.T. $= 12^h\ 30^m\ 36^s$

or G.M.T. $= 0^h\ 30^m\ 36^s$ (P.M.)

(*b*) Longitude of the place = 56° 32′ *E*

Now 56° $= \frac{56}{15}$ h $= 3^h\ 44^m\ 0^s$

32′ $= \frac{32}{15}$ m $= 0^h\ 2^m\ 8^s$

Total $= 3^h\ 46^m\ 8^s$

Since the place is to the east of Greenwich, the Greenwich time will be lesser than the local time.

∴ G.M.T. = L.M.T. − Longitude (*E*)

L.M.T. $= 4^h\ 32^m\ 10^s$ (A.M.)

Subtract longitude $= 3^h\ 46^m\ 8^s$

G.M.T. $= 0^h\ 46^m\ 2^s$ (A.M.)

Example 13.22. *Given the Greenwich civil time (G.C.T.) as $6^h\ 40^m\ 12^s$ P.M. on July 2, 1965, find the L.M.T. at the places having the longitudes (a) 72° 30′ E, (b) 72° 30′ W, and (c) 110° 32′ 30″ E.*

Solution

(*a*) Longitude of the place = 72° 30′ *E*

Now $72° = \frac{72}{14}$ h = $4^h\ 48^m\ 0^s$

$30' = \frac{30}{15}$ m = $0^h\ 2^m\ 0^s$

Total = $4^h\ 50^m\ 0^s$

Since the place is to the east of Greenwich, the local mean time will be more than the standard time.

Now G.M.T. = $18^h\ 40^m\ 12^s$ Past mid-night

Add longitude = $4^h\ 50^m\ 0^s$

∴ L.M.T. = $23^h\ 30^m\ 12^s$

= $11^h\ 30^m\ 12^s$. P.M. on July 2.

(*b*) Longitude of the place = 72° 30′ *W* = $4^h\ 50^m$ of time

Since the place is to the west of the Greenwich, the local mean time will be lesser than the standard time.

Now G.M.T. = $6^h\ 40^m\ 12^s$ P.M. = $18^h\ 40^m\ 12^s$ Past mid-night

Subtract longitude = $4^h\ 50^m\ 0^s$

∴ L.M.T. = $13^h\ 40^m\ 12^s$ = $1^h\ 40^m\ 12^s$ P.M. on July 2.

(*c*) Longitude of the place = 110° 32′ 30″ *E*

Now $110° = \frac{110}{15}$ h = $7^h\ 20^m\ 0^s$

$32' = \frac{32}{15}$ m = $0^h\ 2^m\ 8^s$

$30'' = \frac{30}{15}$ s = $0^h\ 0^m\ 2^s$

Total = $7^h\ 22^m\ 10^s$

Since the longitude is to the east to Greenwich, the local mean time will be more than the G.M.T.

G.M.T. = $18^h\ 40^m\ 12^s$ Past mid-night

Add longitude = $7^h\ 22^m\ 10^s$

∴ L.M.T. = $26^h\ 02^m\ 22^s$

= $2^h\ 02^m\ 22^s$ on July 3

∴ L.M.T. = $2^h\ 02^m\ 22^s$ A.M. on July 3.

Example 13.23. *Find the local apparent time of an observation at a place in longitude 60° 18′ E, corresponding to local mean time $10^h\ 20^m\ 30^s$, the equation of time at G.M.N. being $5^m\ 4.35^s$ additive to the mean time, and decreasing at the rate of 0.32^s per hour.*

Solution.

The equation of time is given at G.M.N. In order to calculate the E.T. at the given L.M.T., we will have to first calculate the corresponding G.M.T. and convert it to G.A.T. Knowing G.A.T., L.A.T. can be calculated.

Longitude of place = 60° 18′ *E* = $4^h\ 1^m\ 12^s$ *E*

L.M.T. of observation = $10^h\ 20^m\ 30^s$

Subtract longitude in time = $4^h\ 1^m\ 12^s$

∴ G.M.T. of observation = $6^h\ 19^m\ 18^s$

∴ Mean time interval before G.M.N. = $12^h - (6^h\ 19^m\ 18^s) = 5^h\ 40^m\ 42^s$ = 5.68 hours

Since the E.T. decreases at the rate of 0.32^s per hour after G.M.N., it will have increased value for any time instant before G.M.N.

∴ Increase for 5.68 hours @ 0.32^s per hour = $(5.68 \times 0.32)^s = 1.82^s$

∴ E.T. at G.M.N. = $5^m\ 4.35^s$

Add increase = $0^m\ 1.82^s$

∴ E.T. at observation = $5^m\ 6.17^s$

Now G.A.T. = G.M.T. + E.T.

G.M.T. of observation = $6^h\ 19^m\ 18^s$

Add E.T. = $0^h\ 5^m\ 6.17^s$

∴ G.M.T. of observation = $6^h\ 24^m\ 24.17^s$

Add longitude in time = $4^h\ 1^m\ 12^s$

∴ L.A.T. of observation = $10^h\ 25^m\ 36.17^s$

Example 13.24. *Find the L.M.T. of observation at a place from the following data:*

L.A.T. of observation = $15^h\ 12^m\ 40^s$

E.T. at G.M.N. = $5^m\ 10.65^s$ additive to apparent time and increasing at 0.22^s per hour.

Longitude of the place = 20° 30′ W.

Solution.

Longitude of the place = 20° 30′ *W* = $1^h\ 22^m\ 0^s$ *W*

L.A.T. of observation = $15^h\ 12^m\ 40^s$

Add longitude in time = $1^h\ 22^m\ 0^s$

∴ G.A.T. of observation = $16^h\ 34^m\ 40^s$

E.T. at G.M.N. = $5^m\ 10.65^s$

Time interval after G.M.N. = $4^h\ 34^m\ 40^s = 4.578^h$

(The above time interval is approximate, since it has been calculated by subtracting G.M.N. from the G.A.T. while actually the G.M.N. should be subtracted from G.M.T. which is not known at present).

$\therefore$ Increase for 4.578^h @ 0.22^s per hour = $(4.578 \times 0.22)^s = 1.01^s$

$\therefore$ E.T. at observation = $5^m\ 10.65^s + 1.01^s = 5^m\ 11.66^s$

Now G.A.T. of observation = $16^h\ 34^m\ 40^s$

Add E.T. = $0^h\ 5^m\ 11.66^s$

G.M.T. of observation = $16^h\ 39^m\ 51.66^s$

Deduct longitude in time = $1^h\ 22^m\ 0^s$

L.M.T. of observation = $15^h\ 17^m\ 51.66^s$

13.9.3. CONVERSION OF MEAN TIME INTERVAL TO SIDEREAL TIME INTERVAL AND VICE VERSA

The tropical year: A year is the period of earth's revolution about the sun, from some determinate position back again to the same position. The reference point chosen for the use of man is the first point of Aries (♈). The year so chosen is the *tropical year* or *the solar year.* A *Sidereal year* is the time taken by the earth in making one complete revolution round the sun with reference to a fixed star.

The first point of Aries has a retrograde motion westwards through an arc of 50.22" per year. The retrograde motion of the first point of Aries is due to the attraction of the moon and the sun which causes the direction of the axis of the earth alter its position very gradually in such a way that earth arrives at the position of the vernal equinox a little earlier each year. This phenomenon is known as the *Precession of Equinoxes.* Due to the precession of Equinoxes, therefore, the earth does not revolve by 360° round the sun from the positions of vernal equinox to vernal equinox, but revolves through (360° – 50".22)

The sun advances among the stars in the same direction — west to east — as the earth revolves about the axis. Any given meridian, therefore, crosses the first point of Aries exactly once oftener than it does the sun, in the course of a tropical year. According to Bassel, there are 365.2422 mean solar days in a tropical year, and in the same period there are 366.2422 sidereal days.

Thus, we have the relation

365.2422 mean solar day = 366.2422 sidereal days

or 1 mean solar day $= 1 + \frac{1}{365.2422}$ sidereal days $= 24^h\ 3^m\ 56.56^s$ sidereal time ...(I)

Thus, the mean solar day is $3^m\ 56.56^s$ longer than the sidereal day.

Hence 1 hour mean solar time = $1^h + 9.8565^s$ sidereal time

1 minute mean solar time = $1^m + 0.1642^s$ sidereal time

1 second mean solar time = $1^s + 0.0027^s$ sidereal time

Thus, to convert the mean solar time to the sidereal time, we will have to add a correction of 9.8565^s per hour of mean time. This correction is called the *acceleration.*

To get the concept how a mean solar day is of a longer time interval than the sidereal time, let us study Fig. 13.34.

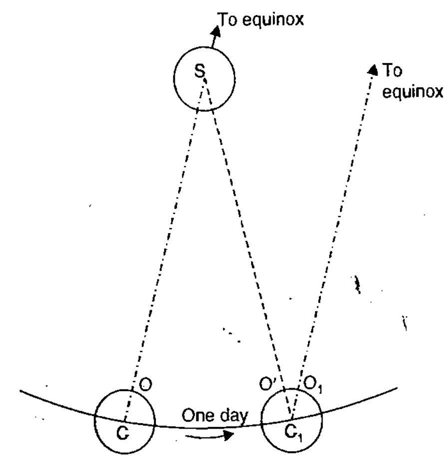


FIG. 13.34

Let C be the centre of the earth and O be the position of the observer at noon of its meridian at the date of the equinox. Let C_1 be the position of the earth's centre the next day. After the earth makes one complete rotation (with reference to ♈), the observer will be at O_1 and *the sidereal time will be the same as it was the day before when he was at O.* However, the solar day is the time interval between two successive transits of the centre of the sun over the meridian. *In order that the sun transits the observer's meridian, the earth will have to revolve additionally by the arc O_1O'.* The time taken for this additional rotation is 3 minutes 56.66 seconds.

Thus, we have

366.2422 sidereal days = 365.2422 solar days.

To convert sidereal time into mean time, we have

1 sidereal day $= \frac{365.2422}{366.2422}$ mean solar day $= 1 - \frac{1}{366.2422}$ mean solar day

or 1 sidereal day = $23^h\ 56^m\ 4.09^s$ mean solar time

$\therefore$ 1^h sidereal time = $1^h - 9.8296^s$ mean solar time

1^m sidereal time = $1^m - 0.1638^s$ mean solar time

1^s sidereal time = $1^s - 0.0027^s$ mean solar time

Thus, *to convert 1 hour sidereal time to the mean solar time, a correction of 9.8296 seconds per hour will have to be subtracted from the sidereal time.* This correction is called the *retardation.*

Example 13.25. *Convert 4 hours 20 minutes 30 seconds of mean solar time into equivalent interval of sidereal time.*

Solution.

To convert the mean solar time to the sidereal time, we will have to first calculate the acceleration at the rate of 9.8565^s per hour of mean time.

Thus $4 \text{ hours} \times 9.8565 = 39.426$ seconds
$20 \text{ min.} \times 0.1642 = 3.284$ seconds
$30 \text{ sec.} \times 0.0027 = 0.081$ seconds

Total = 42.791 seconds

Mean time interval = $4^h\ 20^m\ 30^s$

Add acceleration = 42.791^s

∴ Sidereal time interval = $4^s\ 21^m\ 12.791^s$,

Example 13.26. *Convert 8 hours 40 minutes 50 seconds sidereal time interval into corresponding mean time interval.*

Solution.

To convert the sidereal time to mean solar time, we will have to first calculate the retardation at the rate of 9.8296^s per sidereal hour.

Thus, $8 \text{ hours} \times 9.8296 = 78.637$ seconds
$40 \text{ min.} \times 0.1638 = 6.552$ seconds
$50 \text{ sec.} \times 0.0027 = 0.135$ seconds

Total = 85.324 seconds = $1^m\ 25.324^s$

Sidereal time interval = $8^h\ 40^m\ 50^s$

Subtract retardation = $1^m\ 25.324^s$

∴ Mean time interval = $8^h\ 39^m\ 24.676^s$.

13.9.4. GIVEN GREENWICH SIDEREAL TIME AT GREENWICH MEAN MIDNIGHT, TO FIND THE LOCAL SIDEREAL TIME AT LOCAL MEAN MIDNIGHT AT ANY OTHER PLACE ON THE SAME DATE

(*i.e. Given G.S.T. at G.M.M., to find L.S.T. at L.M.M.*)

From the discussions of the previous article, it is clear that if we have two clocks, one set to keep sidereal time and other to keep mean time, *the sidereal clock will complete its day in a shorter period than the other.* Since 24 hours of solar time are equal to $24^h\ 3^m\ 56.56^s$ of sidereal time, the sidereal clock will be continually gaining over the mean clock at the rate of 9.8565 seconds for every mean solar hour. *The G.S.T. at G.M.M. is then the difference between the sidereal clock and the mean clock at that instant.* The L.S.T. at L.M.M. will then be the difference between these two clocks at the meridian under consideration at the instant.

If the place is to the west of Greenwich, it will have its L.M.M. certain hours after the G.M.M. depending upon the longitude of the meridian. Naturally, by the time there is L.M.M., the sidereal clock will have gained over the mean clock at the rate 9.8565^s for every hour of longitude. *Hence the L.S.T. at L.M.M. will be greater than the G.S.T. at G.M.M. by an amount calculated at 9.8565^s per hour* of western longitude.

Similarly, if the place is to the east of Greenwich meridian, the L.M.M. will occur few hours earlier than the G.M.M., depending upon the longitude of the place. The L.S.T. at L.M.M. will then be lesser than G.S.T. at G.M.M. at the rate of 9.8565 seconds per hour of longitude. Thus, we have the relation:

L.S.T. at L.M.M.= G.S.T. at G.M.M. ± 9.8565^s per hour of longitude $\left(\frac{W}{E}\right)$

Use (+) sign if the longitude is to the west and (−) sign if it is to the east.

Similarly,

L.S.T. at L.M.N. = G.S.T. at G.M.N. ± 9.8565^s per hour of longitude $\left(\frac{W}{E}\right)$

Example 13.27. *If the G.S.T. of G.M.N. on a certain day is $16^h\ 30^m\ 12^s$, what will be the L.S.T. of L.M.M. at a place in longitude :*

(*a*) *160° 30′ 30″ W of Greenwich* (*b*) *160° 30′ 30″ E of Greenwich.*

Solution

(*a*) As the longitude is to the west, the event of which the time is required occurs *later* than G.M.M. by an amount corresponding to the longitude.

		h	m	s
Now	$160° = \frac{160}{15}$ h =	10	40	0
	$30' = \frac{30}{15}$ m =	0	2	0
	$30'' = \frac{30}{15}$ s =	0	0	2

∴ Difference of longitude in terms of time. = 10 42 2

Thus, L.M.M. occurs $10^h\ 42^m\ 2^s$ mean time later than G.M.M. *In the interval between L.M.M. and G.M.M., the ♈ will gain on the mean sun at 9.8565 seconds per hour.*

∴ Gain in sidereal time :

$10^h \times 9.8565 = 98.565$ seconds
$42^m \times 0.1642 = 6.896$ seconds
$2^s \times 0.0027 = 0.005$ second

Total gain = $105.466^s = 1^m\ 45.466^s$

∴ L.S.T. at L.M.N. = G.S.T. of G.M.N. + Gain

$= 16^h\ 30^m\ 12^s + 1^m\ 45.466^s = 16^h\ 31^m\ 57.466^s$.

(*b*) Since the longitude is to the east, the L.M.N. occurs $10^h\ 42^m\ 2^s$ mean time earlier than the G.M.M.

Hence L.S.T. at L.M.M. = G.S.T. of G.M.M. − 9.8565^s per hour of eastern longitude

$= 16^h\ 30^m\ 12^s - 1^m\ 45.466^s = 16^h\ 28^m\ 26.534^s$.

13.9.5. GIVEN THE LOCAL MEAN TIME AT ANY INSTANT, TO DETERMINE THE LOCAL SIDEREAL TIME

At a given meridian, let us have two clocks, one showing the mean time and the other the sidereal time. At the local mean mid-night, the mean time in the mean clock will be zero. At that time (*i.e.* L.M.M.) the L.S.T. can easily be computed if the G.S.T. at G.M.M. is known. If the place is to the west of the Greenwich, the sidereal clock will have a gain over the mean time at L.M.M. at the rate of 9.8565 seconds per hour, as discussed in § 13.9.4 above. At any other instant at the given meridian, the mean clock will show the time that has elapsed since the lower transit of the sun over the meridian. This mean time interval can be easily converted into sidereal time interval as discussed in § 13.9.3 above. Thus, the L.S.T. at L.M.T. will be equal to L.S.T. at L.M.M. plus the sidereal time interval. Hence *the rules for finding the L.S.T. at L.M.T. are:*

(*a*) From the given G.S.T. at G.M.M., calculate L.S.T. at L.M.M.

(*b*) Convert the given L.M.T. (or mean time interval) into sidereal time interval since L.M.M.

(*c*) L.S.T. at L.M.T.= L.S.T. at L.M.M. + S.I. from L.M.M.

Example 13.28. *Find the L.S.T. at place in longitude 85° 20′ E at $6^h\,30^m$ P.M., G.S.T. at G.M.N. being $6^h\,32^m\,12^s$.*

Solution.

Longitude = 85° 20′ *E*

	h	m	s
$85° = \frac{85}{15}$ h =	5	40	0
$20' = \frac{20}{15}$ m =	0	1	20
∴ Longitude in hours =	5	41	20 *E*

Since the place is to the east of Greenwich, let us calculate the loss of sidereal time for $5^h\,41^m\,20^s$ of longitude.

$5^h \times 9.8565^s = 49.283$ seconds

$41^m \times 0.1642^s = 6.732$ seconds

$20^s \times 0.0027^s = 0.054$ second

Total = 56.069 seconds

L.S.T. at L.M.N. = G.S.T. at G.M.N. – retardation

$= 6^h\,32^m\,12^s - 56.069^s = 6^h\,31^m\,15.931^s$ ...(1)

Now, L.M.T. = $6^h\,30^m$ P.M.

∴ M.T. interval from L.M.N. = $6^h\,30^m$.

Let us convert it into sidereal time interval by adding the acceleration to the mean time interval.

Thus, $6^h \times 9.8565^s = 59.139$ seconds

$30^m \times 0.1642^s = 4.926$ seconds

Total acceleration $= 64.065^s = 1^m\,4.065^s$

∴ Sidereal Time Interval = Mean time interval + acceleration since L.M.N.

$= 6^h\,30^m + 1^m\,4.065^s = 6^h\,31^m\,4.065^s$

Now L.S.T. at L.M.N. $= 6^h\,31^m\,15.931^s$

Add S.I. since L.M.N. $= 6^h\,31^m\,4.065^s$

∴ L.S.T. at L.M.T. $= 13^h\,02^m\,19.996^s = 1^h\,02^m\,19.996^s$ P.M.

13.9.6. GIVEN THE LOCAL SIDEREAL TIME, TO DETERMINE THE LOCAL MEAN TIME

If the G.S.T. at G.M.M. is given, the L.S.T. at L.M.M. can be calculated as discussed earlier. The L.S.T. at L.M.M. can then be subtracted from L.S.T. to get the number of sidereal hours, minutes and seconds past midnight. This sidereal time interval can then converted into the mean time interval by subtracting the retardation at the rate of 9.8296^s per hour of S.I. thus obtaining the L.M.T. *The rules are, therefore :*

(*a*) Find the L.S.T. at L.M.M. from the known G.S.T. at G.M.M.

(*b*) Subtract L.S.T. at L.M.M. from the L.S.T. at get the S.I.

(*c*) Convert the S.I. into mean time interval, thus getting L.M.T.

Example 13.29. *The local sidereal time at a place (Longitude 112° 20′ 15″ W) is $18^h\,28^m\,12^s$.*

Calculate the corresponding L.M.T. given that G.S.T. at G.M.M. is $8^h\,10^m\,28^s$ on that day.

Solution : Let us first convert the longitude into time units :

	h	m	s
$112° = \frac{112}{15}$ h =	7	28	0
$20' = \frac{20}{15}$ m =	0	1	20
$15'' = \frac{15}{15}$ s =	0	0	1
∴ Longitude =	7	29	21

Since the place has west longitude,

L.S.T. at L.M.M.= G.S.T. at G.M.M. + acceleration.

Let us calculate the acceleration at the rate of 9.8565^s per hour.

$7^h \times 9.8565^s = 68.996$ seconds

$29^m \times 0.1642^s = 4.762$ seconds

$21^s \times 0.0027^s = 0.057$ second

Total $= 73.815^s = 1^m\ 13.815^s$

		h	m	s
G.S.T. at G.M.M.		= 8	10	28
Add acceleration		=	1	13.815
∴ L.S.T. at L.M.M.		= 8	11	41.815 ...(1)

	h	m	s
Now local sidereal time	= 18	28	12
Subtract L.S.T. at L.M.M.	= 8	11	41.815
∴ S.I. since L.M.M.	= 10	16	30.185

Let us now convert this sidereal interval into mean time interval by subtracting the retardation at the rate of 9.8296^s per hour.

Thus, $10^h \times 9.8296 = 98.296$ seconds

$16^m \times 0.1638 = 2.621$ seconds

$30.185^s \times 0.0027 = 0.081$ second

Total retardation $= 100.998^s = 1^m\ 40.998^s$

∴ Mean time interval = S.I. – retardation

$= 10^h\ 16^m\ 30.185 - 1^m\ 40.998^s = 10^h\ 14^m\ 49.187^s$ since L.M.M.

∴ L.M.T. $= 10^h\ 14^m\ 49.187^s$.

13.9.7. ALTERNATIVE METHOD OF FINDING L.S.T. FROM THE GIVEN VALUE OF L.M.T.

In the method discussed in § 13.9.5 to convert L.M.T. to L.S.T., double computation of time interval was involved. In this alternative method only one transformation of the time interval is necessary. *The steps for the computation are as follows* :

(*a*) Convert the given L.M.T. to the corresponding G.M.T., allowing for the difference of longitude. This gives the interval in mean solar time that has elapsed since G.M.M.

(*b*) Convert this mean time interval to sideral interval that has elapsed since G.M.M., by adding the acceleration at the rate of 9.8565 seconds per hour of mean time interval.

(*c*) Add the S.I. to the G.S.T. at G.M.M. to get the G.S.T. at the instant under consideration.

(*d*) Convert this G.S.T. to the corresponding L.S.T., allowing for the difference of longitude.

Thus, in the above method, though the theory is a little more complex, there is only one transformation of a time interval so that the actual computation is a little shorter. We shall work out example 13.28 by this method.

Example 13.30. *Solve example 13.28 by the alternative method.*

Solution.

Longitude $= 85° 20' E = 5^h\ 41^m\ 20^s$ E, as found earlier

L.M.T., $= 6^h\ 30^m$ P.M.

	h	m	s
L.M.T.	= 18	30	0
Subtract longitude	= 5	41	20
∴ G.M.T.	= 12	48	40

∴ M.T. interval since G.M.N. $= 12^h\ 48^m\ 40^s - 12^h = 48^m\ 40^s$.

Convert this mean time interval to sidereal time interval by adding the acceleration.

$48^m \times 0.1642^s = 7.882$ seconds

$40^s \times 0.0027^s = 0.108$ seconds

Total acceleration $= 7.990$ seconds

∴ Sidereal time interval = mean time interval + acceleration

$= 48^m\ 40^s + 7.990^s = 48^m\ 47.99^s$ since G.M.N.

	h	m	s
G.S.T. at G.M.N.	= 6	32	12
Add S.I.	= 0	48	47.99
∴ G.S.T. at the given instant	= 7	20	59.99
Add longitude	= 5	41	20.0
∴ L.S.T. at L.M.T.	= 13	02	19.99
	$= 1^h$	02^m	19.99^s P.M.

13.9.8. ALTERNATIVE METHOD OF FINDING L.M.T. FROM THE GIVEN VALUE OF L.S.T.

In the method discussed in § 13.9.6 to convert L.S.T. to L.M.T., double computation of time interval was involved. In this method, only one transformation of the interval is necessary. *The steps for the computation are as follows* :

(*a*) From the known L.S.T., compute the corresponding G.S.T. by allowing for the difference of longitude.

(*b*) From this G.S.T. calculated above, subtract the G.S.T. of G.M.M. to get the sidereal interval that has elapsed since G.M.M.

(*c*) Convert this sidereal interval into mean time interval by subtracting the retardation at the rate of 9.8296^s per hour of sidereal interval.

(*d*) The mean time interval obtained in (*c*) is thus the G.M.T. at the instant under consideration. Compute the L.M.T. by allowing for the difference of longitude.

We shall work out example 13.29 by the alternative method.

Example 13.31. *Solve example 13.29 by the alternative method.*

Solution

Longitude $= 112° 20' 15'' W = 7^h 29^m 21^s W$

	h	m	s
L.S.T.	= 18	28	12
Add longitude	= 7	29	21
∴ G.S.T. at the instant	= 25	57	33
G.S.T. at G.M.M.	= 8	10	28
∴ S.I. since G.M.M.	= 17	47	05

Let us now convert this S.I. in mean time interval by subtracting the retardation.

$17^h \times 9.8296 = 167.103$ seconds

$47^m \times 0.1638 = 7.699$ seconds

$5^s \times 0.0027 = 0.014$ seconds

Total retardation $= 174.816$ seconds $= 2^m 54.816^s$

∴ Mean time interval = S.I. – retardation

$= 17^h 47^m 05^m - 2^m 54.816^s$

∴ G.M.T. $= 17^h 44^m 10.184^s$

Subtract longitude $= 7^h 29^m 21^s$

L.M.T. $= 10^h 4^m 49.184^s$

13.9.9. TO DETERMINE THE L.M.T. OF TRANSIT OF A KNOWN STAR ACROSS THE MERIDIAN, GIVEN G.S.T. OF G.M.N.

We have already seen that when a star transits or culminates across the meridian, the R.A. of the star, expressed in time, is the sidereal time. In the Nautical Almanac, the astronomical co-ordinates of all the stars in terms of Right Ascension and declination are given. Thus, knowing the R.A., the L.S.T. at the time of transit of the star is known. The problem is now to convert the L.S.T. into the L.M.T. by the method described in § 13.9.6 or in §13.9.8. The following are the steps :

(*a*) Find the R.A. of the star from the N.A. This is then the L.S.T. at the time of the transit of the star.

(*b*) From the known value of G.S.T. of G.M.M. or (G.M.N.), calculate the L.S.T. of L.M.M. (or L.M.N.).

(*c*) Subtract this L.S.T. of L.M.M. from the L.S.T. of the transit of the star to get the S.I. that has elapsed since L.M.M.

(*d*) Convert this S.I. to mean time interval which, then, gives the L.M.T. at the transit of the star.

Example 13.32. *What will be the L.M.T.'s of upper and following lower transit at a place in longitude 162° 30′ 15″ W of a star whose R.A. is $22^h\ 11^m\ 30^s$, if the G.S.T. of previous G.M.N. is $10^h\ 30^m\ 15^s$.*

Solution.

		h	m	s
Longitude :	$162° = \frac{162}{15}$ h =	10	48	0
	$30' = \frac{30}{15}$ m =	0	2	0
	$15'' = \frac{15}{15}$ s =	0	0	1
		10	50	1

Since the place is to the west, we will have to add the acceleration to get the L.S.T. at L.M.N.

$10^h \times 9.8565^s = 98.565$ seconds

$50^m \times 0.1642^s = 8.210$ seconds

$1^s \times 0.0027^s = 0.003$ second

Total acceleration $= 106.778$ seconds $= 1^m 46.778^s$

	h	m	s
G.S.T. of G.M.N.	= 10^h	30^m	1.5^s
Add acceleration	=	1	46.778
∴ L.S.T. of L.M.N.	= 10	32	01.778
Now R.A. of star = L.S.T.	= 22	11	30
Subtract L.S.T. of L.M.N.	= 10	32	1.778
∴ S.I. since L.M.N.	= 11	39	28.222

Let us now convert this S.I. into mean time interval by subtracting retardation.

$11^h \times 9.8296^s = 108.126$ seconds
$39^m \times 0.1638^s = 6.388$ seconds
$28.222^s \times 0.0027^s = 0.076$ second

Total retardation $= 114.590$ seconds $= 1^m\ 54.59^s$

$\therefore$ Mean time interval $=$ S.I. $-$ retardation $= 11^h\ 39^m\ 28.222^s - 1^m\ 54.59^s$
$= 11^h\ 37^m\ 33.632^s$ since L.M.N.

$\therefore$ L.M.T. of upper transit $= \mathbf{11^h\ 37^m\ 33.632^s}$ **P.M.**

The lower transit of the star will take place at 12 sidereal hours later. To know the corresponding mean time, let us first convert the 12 sidereal hours into mean time hours.

Retardation for 12 hours $= 12 \times 9.8296^s = 1^m\ 57.955^s$

$\therefore$ Mean time interval $= 12^h - 1^m\ 57.955^s = 11^h\ 58^m\ 2.045^s$

Thus the lower transit occurs at a mean time interval of $11^h\ 58^m\ 2.045^s$ after the upper transit.

L.M.T. of upper transit $= 11^h\ 37^m\ 33.632^s$
Add the mean time interval $= 11^h\ 58^m\ 2.045^s$

L.M.T. of lower transit $= 23\ 35\ 35.677$ Since L.M.N.
$= \mathbf{11^h\ 35^m\ 35.677}$ **A.M.** (following day).

Example 13.33. *Calculate the L.M.T. and G.M.T. of transit of β Draconis (R.A. $17^h\ 28^m\ 40^s$) at a place in longitude 60° 30′E given G.S.T. of G.M.T. $= 7^h\ 30^m\ 48.6^s$.*

Solution.

		h	m	s
Longitude	$60° = \frac{60}{15}$ h	4	0	0
	$30' = \frac{30}{15}$ h	0	2	0
		4^h	2^m	0^s

Since the place has east longitude, let us calculate the retardation at the rate of 9.8565^s per hour.

$4^h \times 9.8565^s = 39.426$ seconds
$2^m \times 0.1642^s = 0.328$ second

Total retardation $= 39.754$ seconds

$\therefore$ L.S.T. at L.M.N. = G.S.T. at G.M.N. – Retardation
$= 7^h\ 30^m\ 48.6^s - 39.754^s = 7^h\ 30^m\ 8.846^s$

		h	m	s
L.S.T. = R.A. of star	=	17	28	40
Subtract L.S.T. of L.M.N.	=	7	30	8.846
$\therefore$ S.I. since L.M.N.	=	9	58	31.154

Let us convert it to the mean time interval by subtracting the retardation.

$9^h \times 9.8296 = 88.466$ seconds
$58^m \times 0.1638 = 9.500$ seconds
$31.154 \times 0.0027 = 0.084$ second

Total retardation $= 98.050$ seconds $= 1^m\ 38.05^s$

$\therefore$ Mean time interval since L.M.N. = S. I. – retardation $= 9^h\ 58^m\ 31.154^s - 1^m\ 38.05^s$

or L.M.T. transit $= 9^h\ 56^m\ 53.104^s$
Subtract the longitude $= 4^h\ 2^m\ 0^s$

$\therefore$ G.M.T. of transit $= 5^h\ 54^m\ 53.104^s$

13.9.10. GIVEN THE G.M.T. OF TRANSIT OF THE FIRST POINT OF ARIES, TO DETERMINE THE L.M.T. OF TRANSIT AT A PLACE IN ANY OTHER LONGITUDE

We have already seen that the sidereal clock gains over the mean time clock at the rate of 9.8565 seconds per mean solar hour or at the rate of 9.8296 seconds for each sidereal hour. When the first point of Aries transits over the Greenwich, the sidereal clock shows 0^h while the mean clock gives the mean time of the transit of the first point of Aries. It is the difference between the readings of the two clocks at the time of the transit. Now consider a place in west longitude where the transit of ϒ will take place after certain sidereal interval of time (obtained by dividing the longitude by 15). Since the sidereal clock continually gains over the mean time clock, the difference between mean time clock and the sidereal clock will continuously go on decreasing. *When the transit of ϒ occurs at the given meridian, the mean time clock will not be as far ahead of the sidereal clock as it was at Greenwich,* and the Greenwich reading of the mean time clock will be diminished by subtracting 9.8296 seconds for each hour of longitude. *Hence, if the meridian is to the west of Greenwich, the mean time must be corrected by the subtraction of 9.8296 seconds per hour of longitude, and if the place is to the east, it must be added.* The rule thus becomes:

$$\textit{L.M.T. of transit of } \Upsilon = \textit{G.M.T. of transit of } \Upsilon \pm \frac{E}{W}\left(9.8296 \times \frac{\text{Longitude in degrees}}{15}\right)$$

It must be noted that the difference between the readings of sidereal and mean time clocks at any place is the same all over the World at the same instant. At the time of transit of ϒ, the L.S.T is zero and hence L.M.T. is the difference between the two clocks at the time of transit.

Example 13.34. *The G.M.T. of transit of the first point of Aries (♈) on March 2 is $13^h\ 21^m\ 54^s$. Find the L.M.T. of transit of the first point of Aries on the same day at a place (a) Longitude 40° 30′ E (b) 40° 30′ W.*

Solution

Longitude = 40° 30′ E

	h	m	s
$40° = \frac{40}{15}$ h	= 2	40	0
$30' = \frac{30}{15}$ m	= 0	2	0
	= 2	42	0

Gain of sidereal clock at the rate of 9.8296^s per hour of longitude :

$2^h \times 9.8296^s = 19.659$ seconds

$42 \times 0.1638^s = 6.880$ seconds

Total = 26.539 seconds

	h	m	s
(a) G.M.T. of transit of ♈	= 13	21	54
Add the correction for eastern longitude	= 0	0	26.539
∴ L.M.T. of transit of ♈	= 13^h	22^m	20.539^s

	h	m	s
(b) G.M.T. of transit of ♈	= 13	21	54
Subtract the correction for the western longitude	= 0	0	26.539
∴ L.M.T. of transit of ♈	= 13	21	27.461

13.9.11. GIVEN THE L.S.T. AT ANY PLACE, TO DETERMINE THE CORRESPONDING L.M.T. IF THE G.M.T. OF TRANSIT OF THE FIRST POINT OF ARIES ON THE SAME DAY IS ALSO GIVEN

We know that L.S.T. at any instant is the time interval that has elapsed since the transit of ♈ on the meridian. This L.S.T. can be converted into equivalent number of mean hours by subtracting the retardation at the rate of 9.8296^s per sidereal hour. Also, from the known G.M.T. of transit of ♈, the L.M.T. of transit of ♈ can be calculated. This L.M.T. is nothing but the time shown by the mean clock when the sidereal clock shows 0^h. Therefore, the L.M.T. at the instant under consideration can be obtained by adding the mean hours (corresponding to the given L.S.T.) to the L.M.T. at the time of transit of ♈. *The steps therefore are* :

(1) From the known G.M.T. of transit ♈, calculate the L.M.T. of transit of ♈ by method discussed in §13.9.10.

(2) Convert the given L.S.T. to mean hours.

(3) Add (1) and (2) to get the L.M.T. corresponding to the given L.S.T.

Example 13.35. *The local sidereal time at a place (longitude 50° 30′ E) on 17th May, 1948 is $11^h\ 30^m\ 12^s$. Find the corresponding L.M.T. given that the G.M.T. of transit of ♈ on the 17th May, 1948 is $7^h\ 12^m\ 28^s$.*

Solution

Longitude = 50° 30′ E

	h	m	s
$50° = \frac{50}{15}$ h	= 3	20	0
$30' = \frac{30}{15}$ m	= 0	2	0
Total	= 3	22	0

The correction at the rate 9.8296 per hour of longitude is

$3^h \times 9.8296 = 29.489$ seconds

$22^m \times 0.1638 = 3.604$ seconds

Total correction = 33.093 seconds

	h	m	s
G.M.T. at transit of ♈	= 7^h	12^m	28^s
Add the correction	= 0	0	33.093
∴ L.M.T. at transit at ♈	= 7	13	1.093 ...(1)

L.M.T. = $11^h\ 30^m\ 12^s$, and may be converted to mean hours by subtracting the retardation.

$11^h \times 9.8296 = 108.126$ seconds

$30^m \times 0.1638 = 4.914$ seconds

$12^s \times 0.0027 = 0.032$ seconds

Total retardation = 113.072 seconds = $1^m\ 53.072^s$

Mean hours = Sidereal hours − Retardation = $11^h\ 30^m\ 12^s - 1^m\ 53.072^s = 11^h\ 28^m\ 18.928^s$ ...(2)

Adding (1) and (2), we get

L.M.T. = $7^h\ 13^m\ 1.093^s + 11^h\ 28^m\ 18.928^s$ = **$18^h\ 41^m\ 20.021^s$**.

13.9.12. GIVEN THE SIDEREAL TIME AT G.M.M., TO COMPUTE THE G.M.T. AT THE NEXT TRANSIT OF THE FIRST POINT OF ARIES

The given sidereal time at 0^h G.M.T. shows the number of sidereal hours that have elapsed since the transit of ♈. The next transit of ♈ will evidently take place 24 sidereal hours later than the previous transit. Let the G.S.T. at G.M.M. be s sidereal hours. Then the next transit will take place at $(24 - s)$ sidereal hours after the G.M.M. These $(24 - s)$ sidereal hours can be converted into the mean time hours which will give the G.M.T. at next transit of ♈.

Example 13.36. *On July 12, the G.S.T. at 0^h G.M.T. is $8^h 25^m 25^s$. Find the G.M.T. of the next transit of* ♈.

Solution

G.S.T. at G.M.M. $= 8^h 25^m 25^s$

$\therefore$ Time of previous transit $= 8^h 25^m 25^s$ sidereal interval before G.M.M.

$\therefore$ Time of next transit $= (24^h - 8^h 25^m 25^s)$ sidereal interval after G.M.M.

$= 15^h 34^m 35^s$ sidereal interval of time.

To convert it into the mean time interval, subtract the retardation

$15^h \times 9.8296 = 147.444$ seconds

$34^m \times 0.1638 = 5.569$ seconds

$35^s \times 0.0027 = 0.095$ second

Total retardation $= 153.108$ seconds $= 2^m 33.108^s$

Mean time interval $= 15^h 34^m 35^s - 2^m 33.108^s = 15^h 32^m 1.892^s$ since G.M.M.

$\therefore$ G.M.T. of next transit $= \mathbf{15^h 32^m 1.892^s}$.

13.9.13. GIVEN THE G.M.T. OF G.A.N. ON A CERTAIN DATE, TO FIND THE L.M.T OF L.A.N. ON THE SAME DATE

The local apparent noon will occur before or after the G.A.N. depending upon whether the longitude of the place is to the east or to the west of the Greenwich meridian. *The apparent time at the apparent noon is zero and hence G.M.T. of G.A.N. is the equation of time at Greenwich at noon.* Since the local apparent noon occurs either before or after the G.A.N., the equation of time will change and interpolation will have to be done. For example, if the place is to the east of Greenwich, the L.A.N. will occur earlier and we must know the difference between the given G.M.T. of G.A.N. and the G.M.T. of G.A.N. on the day *before,* in order to do the interpolation. Similarly, if the place is to the west of Greenwich, the L.A.N. will occur later and we must know the difference between the given G.M.T. of G.A.N. and the G.M.T. of G.A.N. on the day *after,* in order to do the interpolation. Once the correct equation of time is known, L.M.T. at L.A.N. can be computed as illustrated in example 13.37.

Example 13.37. *Given the following data from the N.A. for 1951:*

Sun at Transit at Greenwich

Date	G.M.T. h	m	s	
June 30	*12*	*02*	*22.44*	
				+ 11.94
July 1	*12*	*03*	*34.38*	
				+ 11.71
July 2	*12*	*03*	*46.09*	
				+ 11.45
July 3	*12*	*03*	*57.54*	
				+ 11.16

Find the L.M.T. of L.A.N. on July 2 at a place (a) in longitude 130° E (b) in longitude 49° W.

Solution. (*a*) Longitude $130° E = \frac{30}{15} h = 8^h 40^m E$

Since the place is to the east of Greenwich, the L.M.T. is $8^h 40^m$ *ahead* of the G.M.T. From the table, the difference between G.M.T. of G.A.N. on July 1, and July 2, is 11.71^s (for 24 hours).

$\therefore$ Difference for $8^h 40^m = (8^h 40^m) \frac{(11.71)}{24} = 4.23$ seconds

By the inspection of the table, it is clear that the values of G.M.T. are decreasing as we go back from July 2. Hence this difference of 4.23 seconds should be subtracted from the G.M.T. of G.A.N. on July 2 to get L.M.T. of L.A.N. on the same date.

Thus, G.M.T. of G.A.N. on July 2 $= 12^h\ 03^m\ 46.09^s$

Subtract difference due to east longitude $= 4.23^s$

$\therefore$ L.M.T. of L.A.N. on July 2 $= 12^h\ 03^m\ 41.86^s$

(*b*) Longitude $49° W = \frac{49}{15} h = 3^h 16^m$

Since the place is to the west of Greenwich, the L.M.T. is $3^h 16^m$ behind G.M.T. From the table, the difference between G.M.T. of G.A.N. on July 2 and July 3 is $+ 11.45^s$ (for 24 hours).

$\therefore$ Difference for $3^h 16^m = (3^h 16^m)\left(\frac{11.45}{24}\right) = 1.56$ seconds.

Since the values of G.M.T. are increasing as the dates increase, the difference of 1.56 seconds should be added to the G.M.T. of G.A.N. on July 2 to get L.M.T. of L.A.N. on the same date.

Thus, G.M.T. of G.A.N. on July 2 $= 12^h 03^m 46.09^s$

Add difference due to west longitude $= 1.56^s$

$\therefore$ L.M.T. of L.A.N. on July 2 $= 12^h 03^m 47.65^s$

13.9.14. TO FIND THE LOCAL SIDEREAL TIME OF ELONGATION OF A STAR

We have already seen in § 13.8 (Fig. 13.30) that

Star's hour angle + star's right ascension = Local Sidereal Time.

Thus, to get the L.S.T. of elongation of the star, add the westerly hour angle (or subtract the easterly hour angle) to the R.A. of the star at its elongation. If the result is more than 24^h, 24^h are deducted, while if the result is negative, 24 hours are added to it.

Example 13.38. *Find the L.S.T. at which β Ursae Minor is will elongate on the evening at a place in latitude 50° 30′ N given that the R.A. of the star is $14^h\ 50^m\ 52^s$ and its declination is + 74° 22′.*

Solution

The right ascension and the declination of the star are given. Let us first calculate its hour angle at elongation. When the star is at elongation, we have, from Eq. 13.19,

$$\cos H = \frac{\tan\theta}{\tan\delta} = \frac{\tan 50° 30'}{\tan 74° 22'}$$

$$\log \tan 50° 30' = \bar{1}.0838955$$

$$\log \tan 74° 22' = \bar{1}.5531022$$

∴ $\log \cos H = \bar{1}.5307933$

∴ $H = 70°\ 9'\ 18''$

or $H = 4^h\ 40^m\ 37.2^s$

Add R.A. = 14 50 52.0

∴ L.S.T. = $19^h\ 31^m\ 29.2^s$

Example 13.39. *If the G.S.T. of G.M.N. is $14^h\ 30^m\ 28.25^s$, what will be the H.A. of a star of R.A. $23^h\ 20^m\ 20^s$ at a place in longitude 120° 30′ W at 2.05 A.M. G.M.T. the same day ?*

Solution

We know that, L.S.T. = R.A. of star + Hour angle of the star.

From the above relation, the hour angle of the star can very easily be found out by subtracting R.A. of the star from the L.S.T. of the event. The only problem, therefore, is to calculate the L.S.T. corresponding to the given L.M.T., given the G.S.T. of G.M.N.

Let us first calculate the L.S.T. of L.M.N.

Longitude = 120° 30′ W = $8^h\ 2^m$ W.

Since the place is to the west, we have to add the acceleration at the rate of 9.8565 per hour of longitude to the G.S.T. of G.M.N. to get the L.S.T. of L.M.N.

Now $8^h \times 9.8565 = 78.85$ seconds

$2^m \times 0.1642 = 0.33$ second

Total acceleration = 79.18 seconds

G.S.T. of G.M.N. = $14^h\ 30^m\ 28.25^s$

Add acceleration = 79.18^s

∴ L.S.T. of L.M.N. = $14^h\ 31^m\ 47.43^s$ ...(1)

Now G.M.T. = $2^h\ 5^m\ 0^s$

Subtract longitude = $8^h\ 2^m\ 0^s$

∴ L.M.T. of the event = $18^h\ 3^m\ 0^s$ (previous day).

L.M.N. (day of given G.S.T. of G.M.N.) = $12^h\ 0^m\ 0^s$

Subtract L.M.T. of event (previous day) = $18^h\ 3^m\ 0^s$

∴ Mean time interval between the event = $17^h\ 57^m\ 0^s$ and the L.M.N.

Let us convert this mean time interval to the sidereal time interval by adding acceleration at the rate of 9.8565^s per mean hour.

Thus $17^h \times 9.8565 = 167.56$ seconds

$57^m \times 0.1642 = 9.36$ seconds

Total acceleration = 177.92 seconds = $2^m\ 57.92^s$

∴ S.I. between the event and L.M.N. = $17^h\ 57^m\ 0^s + 2^m\ 57.92^s = 17^h\ 59^m\ 57.92^s$ (before L.M.N.)

Now L.S.T. of L.M.N. = $14^h\ 31^m\ 47.43^s$

Subtract S.I. = $17^h\ 59^m\ 57.92^s$

L.S.T. of event = $20^h\ 31^m\ 49.51^s$ ...(2)

Now H.A. = L. S. T. − R. A.

$= (20^h\ 31^m\ 49.51^s) - (23^h\ 20^m\ 20^s) + 24^h = 21^h\ 11^m\ 29.51^s$

(**Note.** 24^h have been added to make the hour angle positive).

Example 13.40. *Find the R.A. of the mean sun at 5.30 A.M. on July 28, 1964 in a place in longitude 75° 28′ W, and also the R.A. of the meridian of the place, given that G.S.T. at G.M.M on the given date is $20^h\ 15^m\ 32.58^s$.*

Solution.

We know that, L.S.T. = R.A. of the star + hour angle of the star.

Here, the mean sun is fictitious star.

Hence L.S.T. = R.A.M.S. + hour angle of the mean sun.

But hour angle of mean sun = L.M.T. + 12 hours

(since L.M.T. is measured from the lower transit).

Hence, we have L.S.T. = R.A.M.S. + L.M.T. + 12^h

In order to calculate the R.A. of the mean sun, we must know L.S.T. at the time of event L.S.T. can be very easily found from the given L.M.T. and the given value of G.S.T. of G.M.M.

Now longitude = 75° 28′ $W = 5^h\ 1^m\ 52^s$

Since the place is having west longitude, we will have to add an acceleration at the rate of 9.8565^s per hour of longitude to the G.S.T. of G.M.M. to the get the L.S.T. of L.M.N.

$5^h \times 9.8565 = 49.28$ seconds

$1^m \times 0.1642 = 0.16$ second

$52^s \times 0.0027 = 0.14$ second

Total acceleration = 49.58 seconds

G.S.T. of G.M.M. = $20^h\ 15^m\ 32.58^s$

Add Acceleration = 49.58^s

∴ L.S.T. of L.M.M. = $20^h\ 16^m\ 22.16^s$

Now L.M.T. of event = $5^h\ 30^m$ A.M. = $5^h\ 30^m$ mean time after mid-night.

To convert this time interval to sidereal interval, add the acceleration at the rate of 9.8565^s per hour of mean time.

Thus, $5^h \times 9.8565^s = 49.28$ seconds

∴ $30^m \times 0.1642 = 4.93$ seconds

Total acceleration = 54.21^s

∴ S.I. since L.M.M. = $5^h\ 30^m + 54.21^s = 5^h\ 30^m\ 54.21^s$

∴ L.S.T. = L.S.T. of L.M.M.+ S.I.

$= 20^h\ 16^m\ 22.16^s + 5^h\ 30^m\ 54.21^s = 25^h\ 47^m\ 16.37^s$

Now, by definition, the R.A. of the meridian is equal to the L.S.T.

Hence R.A. of meridian = $\mathbf{25^h\ 47^m\ 16.37^s}$

Again R.A.M.S. = L. S. T. – L. M. T. – 12^h

$= (25^h\ 47^m\ 16.37^s) - (5^h\ 30^m) - (12^h) = \mathbf{8^h\ 17^m\ 16.37^s}$.

13.10. INTERPOLATION OF VALUES

The declination of a heavenly body is a constantly varying quantity and can be obtained from the nautical almanac which gives the values at Greenwich mean and apparent mid-night. The nautical almanac gives the values of declination both for mean sun and apparent sun at G.M.M. and G.A.M. for every day and also the rate of hourly variation at Greenwich mid-night. To find the declination at any given instant of Greenwich civil time, it is necessary to interpolate between the tabulated values. The required value may thus be obtained by :

(a) Simple linear interpolation between the successive tabulated values on the assumption that the rate of change is uniform and equal to its value at the middle of the interval.

(b) By interpolating strictly, taking higher order differences into account, by *Bessel's method*. The Bessel's interpolation formula is as follows :

$$f_n = f_0 + n\,\Delta'_{1/2} + \frac{n\,(n-1)}{4}\,(\Delta_0'' + \Delta_1'') \quad ...(13.26)$$

where f_n = the value of the function which is to be found, and which lies between f_0 and f_1.

n = Fractional value of the interval between two tabular values.

Δ' = First difference between the successive values of the function.

Δ'' = Second difference

Thus, $f_{-1} - f_0 = \Delta'_{-1/2}$ | $\Delta'_{1/2} - \Delta'_{-1/2} = \Delta''_0$

$f_1 - f_0 = \Delta'_{1/2}$ | $\Delta'_{3/2} - \Delta'_{1/2} = \Delta''_1$

$f_2 - f_1 = \Delta'_{3/2}$

where f_{-1}, f_0, f_1, f_2 ..... etc. are the successive values of the function to be interpolated.

The method of interpolation has been fully illustrated in the following example.

Example 13.41. *Find sun's declination at 10 A.M. on February 5, 1947 in longtiude 45° E.*

Solution

Let us first convert the local time to Greenwich mean time.

Longitude = 45° $E = 3^h$

∴ G.M.T. = 10 – 3 = 7 hours = 0.2917 day

∴ $n = 0.2917$

The following values of sun's declination are obtained from the N.A.

Date	*Sun's Declination at 0^h G.M.T.*	*Variation per day*
Feb. 4	– 16° 32′ 11″.2	
		+ 1067″.2
Feb. 5	– 16° 14′ 24″.0	
		+ 1083″.9
Feb. 6	– 15° 56′ 20″.1	
		+ 1100″.3
Feb. 7	– 15° 37′ 59″.8	

From the above table, f_0 = value at 0^h G.M.T. on Feb. 5 = – 16° 14′24″.0

f_{-1} = value on Feb. 4 = – 16° 32′11″.2

f_1 = value on Feb. 6 = – 15° 56′20″.1

f_2 = value on Feb. 7 = – 15° 37′59″.8

$\Delta'_{-1/2} = f_{-1} - f_0 = +\ 1067''.2$

$\Delta'_{1/2} = f_1 - f_0 = +\ 1083''.9$

$$\Delta_0'' = \Delta'_{1/2} - \Delta'_{-1/2} = 1083.9 - 1067.2 = +16''.7$$
$$\Delta'_{3/2} = f_2 - f_1 = +1100''.3$$
$$\Delta_1'' = \Delta'_{3/2} - \Delta'_{1/2} = 1100''.3 - 1083''.9 = +16''.4$$

Putting the values in the Bessel's formula, we get

$$f_n = f_0 + n\,\Delta'_{1/2} + \frac{n(n-1)}{4}(\Delta''_0 + \Delta_1'') = -16°\,14'\,24''.0 + 0.2917\,(+1083''.9)$$
$$-\frac{0.2917\,(0.2917-1)}{4} \times (16''.7 + 16''.4)$$
$$= -16°\,14'\,24''.0 + 316''.15 - 1''.71 = \mathbf{-16°\,9'\,9''.56}.$$

Note. (1) The four dates from which the interpolation is done should be so selected that the instant lies between the two middle dates.

(2) The value of the declination by the approximate method (linear interpolation) will be equal to $-16°\,14'\,24''.0 + 0.2917\ (1083''.9) = -16°\,9'\,7''.85$.

13.11. INSTRUMENTAL AND ASTRONOMICAL CORRECTIONS TO THE OBSERVED ALTITUDE AND AZIMUTH

(A) INSTRUMENTAL CORRECTIONS

The angle measuring instruments used in astronomical observations are theodolite and sextant. For precise work, a theodolite having a least count of 1″ (or less) is used. The theodolite should be in prefect adjustments. However, following are some of the instrumental corrections that are generally applied to the observed altitude and azimuth.

(*a*) Corrections for Altitudes

(1) Correction for Index Error. If the vertical circle verniers do not read zero when the line of sight is horizontal, the vertical angles measured will be incorrect. The error is known as the *index error*. The index error can be eliminated by taking both face observations. However, it may sometimes not be practicable to take both face observations when the altitude of a star or the sun is to be observed. In such a case, the correction for the index error is necessary.

The index error may be determined as follows :

(*i*) Set the theodolite on firm ground and level it accurately with reference to altitude bubble.

(*ii*) Bisect a well-defind object such as a church spire (or a chimney top) with the telescope normal (face left). Observe the vertical angle α_1.

(*iii*) Change the face and bisect the same object again with telescope reversed (face right). Observe the vertical angle α_2.

Let the index error be e.

∴ Correct vertical angle will be

$$\alpha = (\alpha_1 + e) \quad \text{and} \quad \alpha = (\alpha_2 - e)$$

$$\therefore \quad \alpha = \frac{(\alpha_1 + e) + (\alpha_2 - e)}{2} = \frac{\alpha_1 + \alpha_2}{2}$$

Thus, the correct vertical angle is the mean of the two observed angles.

Hence $e = (\alpha - \alpha_1)$

For example, let $\alpha_1 = 4°\,15'\,8''$ and $\alpha_2 = 4°\,15'\,16''$

∴ Mean vertical angle $= \alpha = 4°\,15'\,12''$

Hence, the index error correction for face left observation $= +4''$

Hence, the index error correction for face right observation $= -4''$

The index error correction is said to be + ve or − ve according as this amount is to be added to or subtracted from the observed altitude.

(2) Correction for Bubble Error. If the altitude bubble does not remain central while the observations are made, the correction for bubble error is essential. The correction for bubble error is given by

$$C = \frac{\Sigma O - \Sigma E}{n} \times v \text{ seconds} \qquad \ldots(13.27)$$

where C = correction for bubble error in seconds, to be applied to the mean altitude observed.

ΣO = the sum of readings of the object glass end of the bubble.

ΣE = the sum of readings of the eye-piece end of the bubble.

n = the number of bubble ends read (= 2 when single face observation is taken, and 4 when both face observations are made).

v = angular value of one division of the bubble in seconds.

If ΣO is greater than ΣE, the correction is positive, otherwise negative.

(*b*) Correction for Azimuths

Since most astronomical observations require the line of sight to be elevated through a large vertical angle, it is important that the horizontal axis shall be truly horizontal. To fulfill this, it is most important that (1) the instrument is accurately levelled so that the vertical axis is truly vertical and (2) the trunnion axis is exactly perpendicular to the vertical axis. If the vertical axis is not truly vertical (*i.e.* if the bubble does not preserve a central position through a series of observations), the trunnion axis will be inclined even though the instrument is in perfect adjustment. The error due to the inclination of the trunnion axis cannot be eliminated. However, its inclination can be determined by means of a striding level with a sensitive bubble tube.

Correction for Trunnion Axis Dislevelment. The bubble readings on the striding level will show whether the trunnion axis is truly horizontal or not. If not, each horizontal direction should be corrected for trunnion axis dislevelment. It can be shown that the correction to be applied to the azimuth of a low point with respect to a high point, caused by an inclination of the trunnion axis of the transit is given by

$$c = b \tan \alpha \text{ seconds}$$

where c = correction to the azimuth

b = inclination of the horizontal axis of the transit with respect to the horizontal, in seconds

α = vertical angle to the high point.

The value of b can be determined as under :

Let l_1 and r_1 be the left hand and right hand readings of the bubble ends in one position, and l_2 and r_2 be the left hand and right hand readings of the bubble ends in the second position.

Deviation of the centre of the bubble from the centre of the striding level in the first position $= \dfrac{l_1 - r_1}{2}$

Deviation of the centre of the bubble from the centre of the striding level in the second position $= \dfrac{l_2 - r_2}{2}$.

$\therefore$ The mean deviation of the centre of the bubble from the centre of the striding

$$\text{level} = \frac{1}{2}\left\{\frac{l_1 - r_1}{2} + \frac{l_2 - r_2}{2}\right\} = \frac{(l_1 + l_2) - (r_1 + r_2)}{4} = \frac{\Sigma l - \Sigma r}{4}$$

$\therefore$ Inclination of trunnion axis in seconds $= b = \dfrac{\Sigma l - \Sigma r}{4} \times d$ ...(13.28)

where d = angular value of one division of the striding level

Σl = the sum of the readings of the *left hand end* of the bubble in the direct and reversed positions of the striding level on the trunnion axis.

Σr = the sum of the readings of the *right hand end* of the bubble in the direct and reversed positions of the striding level on the trunnion axis.

The left-hand end of the axis will be higher if Σl is greater than Σr, and lower if Σl is less than Σr.

If the observed angle is the angle of elevation, the correction will be positive when the left-hand end of the axis is higher and negative when the left-hand end is higher.

If the observed angle is the angle of depression, the correction will be positive when the right-hand end of the axis is higher and negative when the left-hand end is higher.

The horizontal circle reading for each direction should be corrected separately and then the horizontal angle should be obtained by subtraction.

(B) ASTRONOMICAL CORRECTIONS

The observed or *apparent altitudes* of the celestial bodies like the sun or stars should be subjected to the following corrections:

1. Correction for parallax
2. Correction for refraction
3. Correction for dip of the horizon
4. Correction for semi-diameter.

1. Correction for Parallax. Parallax is the apparent change in the direction of a body when viewed from different points. The parallax in altitude, or *diurnal parallax*, is due to the difference in direction of a heavenly body as seen from the centre of the earth and from the place of observation on the surface of the earth. The stars are very far and the parallax is insignificant since the direction of rays as seen from the earth's surface and as seen from the centre of the earth are practically parallel. However, in the case of sun or moon, the parallax is significant and proper correction should be applied for the same.

Fig. 13.35 illustrates the sun's parallax.

O = Centre of the earth ; A = Place of observation

S = Position of the sun during observation; S' = Position of the sun at horizon.

OC = True horizon

AB = Sensible horizon

$\alpha' = \angle SAB$ = Observed altitude

$\alpha = \angle SOC$ = True altitude, corrected for parallax;

$p_a = \angle ASB$ = Parallax correction

$p_h = AS'O$ = Sun's horizontal parallax.

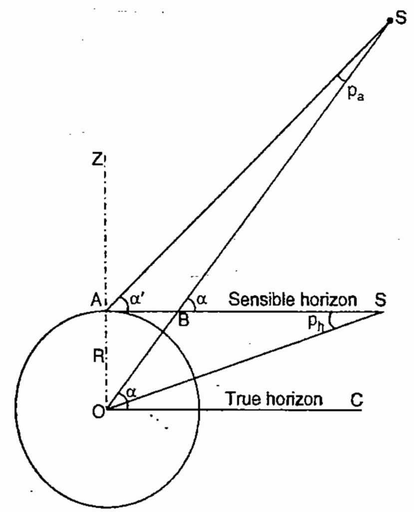


FIG. 13.35 SUN'S PARALLAX.

When the sun is on the horizon, its apparent or observed altitude is zero, and the angle (p_h) subtended at the centre of the sun is known as sun's *horizontal parallax.*

Evidently, $\sin p_h = \dfrac{R}{OS'}$

Thus, the sun's horizontal parallax varies *inversely* with its distance from the centre of the earth. It varies from 8.95″ early in January to 8.66″ early in July, and is given in the Nautical Almanac for every tenth day of the year. The mean value of the sun's horizontal parallax is 8.8″.

Now true altitude $\therefore$ $\alpha = SOC = SBS' = SAB + ASB = \alpha' + p_a$

Hence parallax correction $= (\alpha - \alpha') = p_a$

From triangle AOS, $\sin ASO = \sin OAS \cdot \dfrac{OA}{OS}$

or $\sin p_a = \sin (90° + \alpha') \dfrac{OA}{OS} = \cos \alpha' \cdot \dfrac{OA}{OS}$

But $\dfrac{OA}{OS} = \dfrac{OA}{OS'} = \sin p_h$

$\therefore$ $\sin p_a = \sin p_h \cos \alpha'$ ...[13.29 (*a*)]

Since p_a and p_h are very small, we have

$p_a = p_h \cos \alpha'$ ...[13.29 (*a*)]

or ***correction for parallax = horizontal parallax × cos apparent altitude*** $= + 8''.8 \cos \alpha'$ ...(13.29)

The correction for parallax is always additive. The correction is maximum when the sun is at horizon.

2 Correction for Refraction. The earth is surrounded by the layers of atmospheric air. The layers get thinner and thinner as its distance from the surface increases. When a ray of light emanating from a celestial body passes through the atmosphere of the earth, the ray is bent downward, as shown in Fig. 13.36 and the body appears to be nearer to the zenith than it actually is.

The angle of deviation of the ray from its direction on entering the earth's atmosphere to its direction at the surface of the earth is called the *refraction angle* of correction. The refraction correction is always subtractive to the observed altitude. The magnitude of refraction depends upon the following:

(*i*) the density of air
(*ii*) the temperature
(*iii*) the barometric pressure
and (*iv*) the altitude.

It is constant for all bodies and does not depend upon the distance of the body from the observer.

At a pressure of 29.6 inches of mercury and a temperature of 50° F, the correction for refraction can be calculated from the following formula :

Correction for refraction (in seconds)

$$= 58'' \cot \alpha = 58'' \tan z \qquad \ldots(13.30)$$

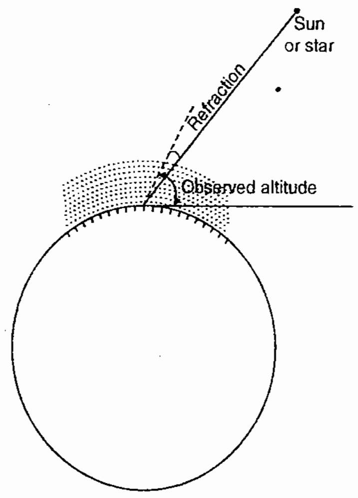


FIG. 13.36. REFRACTION.

where α = the apparent altitude of the heavenly body
z = the apparent zenith distance of the heavenly body.

The correction for refraction is always subtractive.

The values of mean refraction for different altitudes are given in Chamber's Mathematical Tables corresponding to barometer pressure, temperature of external air and temperature of thermometer attached to barometer.

The refraction correction for low altitudes is uncertain and hence observation for precise determination should never be taken on a celestial body which is nearer the horizon. The refraction, however, does not affect the azimuth.

3. Correction for Dip of the Horizon. The angle of the dip is the angle between the true and visible horizon. When the observations are taken with the help of a sextant at the sea, the altitude of the star or sun is measured from the visible horizon of the sea. Owing to the curvature of the earth, the visible horizon is below the true horizon. Hence, the angle of dip (*i.e.* the angle between the two horizons) must be subtracted from the observed altitude of the body.

In Fig. 13.37,

A = position of the observer
$AB = h$ = Height of the observer above sea level
S = position of the sun or star
AD = visible horizon
AC = true horizon
$\angle SAD = \alpha'$ = observed altitude of the sun or star
$\angle SAC = \alpha$ = true altitude of the sun or star
$\angle CAD = \beta$ = angle of dip

R = radius of the earth

Now, $BO = R$; $AO = (R + h)$

$\therefore \quad AD = \sqrt{(R+h)^2 - R^2}$

$\angle CAD = \angle AOD = \beta$

$$\therefore \quad \tan \beta = \frac{AD}{OD} = \frac{\sqrt{(R+h)^2 - R^2}}{R} = \sqrt{\frac{h\,(2R+h)}{R^2}} \quad \ldots\text{(exact)} \ldots[13.31\ (a)]$$

or
$$\tan \beta = \sqrt{\frac{2h}{R}} \ldots\text{(approximately)}\ldots[13.31\ (b)]$$

If β is small, we many have

$$\tan \beta = \beta \text{ (radians)} = \sqrt{\frac{2h}{R}} \qquad \ldots(13.31)$$

The correction for dip is always subtractive.

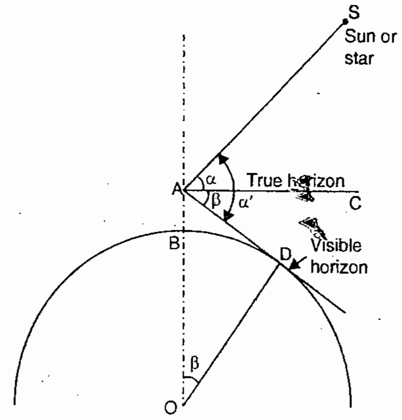


FIG. 13.37. DIP OF THE HORIZON.

4. Correction for Semi-diameter. The semi-diameter of the sun or star is half the angle subtended at the centre of the earth, by the diameter of the sun or the star. Since the distance of the sun from the earth is not constant throughout the year, the semi-diameter varies from 15′ 46″ in July to 16′ 18″ in January. Its value at its mean distance from the earth is 16′ 1″.18. The Nautical Almanac gives the values of sun's semi-diameter for every day in the year.

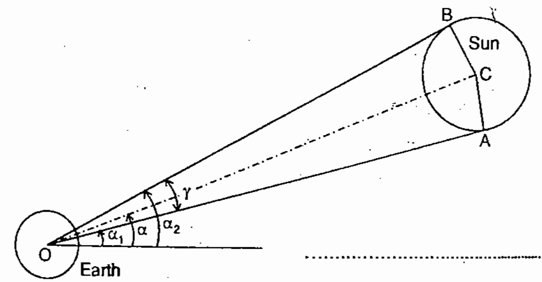


FIG. 13.37. (*a*) CORRECTION FOR SEMI-DIAMETER.

As the sun is large, its centre cannot be sighted precisely, and it is customary to bring the cross-hairs tangent to the sun's image. When the horizontal cross-hair is brought tangent to the lower edge of the sun, the sight is said to be taken at sun's lower limb [Fig. 13.38 (*a*)]. Similarly, when the horizontal cross-hair is brought tangent to the upper edge of the sun, the sight is said to be taken at sun's upper limb [Fig. 13.38 (*b*)]. Figs. 13.38 (*c*) and 13.38 (*d*) illustrate the observations taken to sun's right limb and left limb respectively.

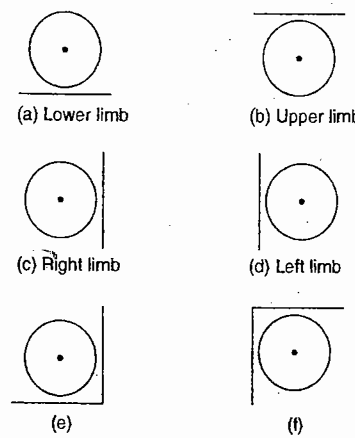


FIG. 13.38. OBSERVATION TO SUN.

In Fig. 13.37 (*a*), OA is the ray corresponding to the lower limb of the sun. The observed altitude α_1 is evidently lesser than the correct altitude α. Similarly, OB is the ray corresponding to the upper limb of the sun. The observed altitude α_2 is evidently more then the correct altitude α. If $\frac{\gamma}{2}$ is the semi-diameter, we have

$$\alpha = \alpha_1 + \frac{\gamma}{2} = \alpha_2 - \frac{\gamma}{2}$$

When a horizontal angle is measured to the sun's right or left limb, a correction equal to the sun's semi-diameter times the secant of the altitude is applied.

Thus, ***correction for semi-diameter in azimuth = semi-diameter × secant α.***

Example 13.42. *Determine the value of horizontal angle between two points A and B, the observations for which were made with a theodolite in which one division of the striding level corresponds to 20".*

Object	*Azimuth*	*Vertical angle*	*Striding level 1st Position l*	*r*	*Readings After reversal l*	*r*
A	*32° 41' 30"*	*+ 10° 21' 12"*	*11*	*7.5*	*10.5*	*8*
B	*110° 28' 42"*	*− 2° 18' 30"*	*11.5*	*7.0*	*10.0*	*7.5*

Except for the adjustment of transverse axis not being perpendicular to the vertical axis, all other adjustments were correct.

Solution.

Let us first find the value of b.

(a) *Observations of A* : $\Sigma l = 11 + 10.5 = 21.5$; $\Sigma r = 7.5 + 8 = 15.5$

$$\therefore \quad b = \frac{\Sigma l - \Sigma r}{4} \cdot d = \frac{21.5 - 15.5}{4} \times 20 = +30''$$

Thus, the left end of the axis is higher.

∴ The correction $c = b \tan \alpha = 30 \tan 10° 21' 12'' = 5''.48$ seconds. Since the vertical angle is the angle of elevation and the left-hand end of the bubble tube is higher, the correction is positive.

∴ Corrected azimuth = 32° 41' 30" + 5".48 = 32° 41' 35".48.

(b) *Observation to B* : $\Sigma l = 11.5 + 10 = 21.5$; $\Sigma r = 7.0 + 7.5 = 14.5$

$$\therefore \quad b = \frac{\Sigma l - \Sigma r}{4} d = \frac{21.5 - 14.5}{4} \times 20 = +35''$$

Thus, the left end of the axis is higher.

The correction $c = b \tan \alpha = 35 \tan 2° 18' 30'' = 1.41$ seconds.

Since the vertical angle is the angle of depression and the left-hand end of the bubble tube is higher, the correction is negative.

∴ Corrected azimuth = 110° 28' 42"−1".41 = 110° 28' 40".59

Hence horizontal angle between A and B = 110° 28' 40".59 − 32° 41' 35".48 = 77° 47' 5".11.

Example 13.43. *To determine the index error of a theodolite, a church spire was sighted and the face left and face right observations were 18° 36' 48" and 18° 35' 56" respectively. A face right observation on the sun's lower limb was then made and the altitude was found to be 28° 36' 20". The semi-diameter of the sun at the time of observation was 15' 59".35. Find the true altitude of the sun.*

Solution

The observed altitude of the sun is to be corrected for

(*i*) index error (*ii*) semi-diameter (*iii*) refraction (*iv*) parallax.

(*i*) *Corrections for index error*

Mean of the vertical angle readings = $\frac{1}{2}$(18° 36' 48" + 13° 35' 56") = 18° 36' 22"

∴ Index error for the face right reading = 18° 36' 22" − 18° 35' 56" = + 26".

The observed altitude of the sun = 28° 36' 20"

Add index correction = 26"

∴ Altitude of sun corrected for index error = 28° 36' 46".

(*ii*) *Correction for semi-diameter*

Since the lower limb of the sun was observed, the correction is positive.

Altitude of sun corrected for index error = 28° 36' 46"

Add semi-diameter = 15' 59".35

∴ Altitude of sun corrected for index eeeor and semi-diameter = 28° 52' 45".35

(*iii*) *Correction for refraction*

The correction for refraction is always subtractive and is equal to − 57" cot 28° 28' 46"
= − 1'44".48.

(*iv*) *Correction for parallax*

The correction for parallax is positive and is equal to 8".8 cos 28° 36' 46" = + 7".80

Altitude of sun corrected for index error and semi-diameter	= 28° 52' 45".35
Subtract refraction correction	= 1' 44". 48
	= 28° 51' 0".87
Add parallex correction	= 7".80
∴ Correct altitude of the sun	**= 82° 51' 8".67**

13.12. OBSERVATIONS FOR TIME

The observations for determining the local time consists mainly in finding the error of watch or chronometer which is read at the instant the observations are made. If the chronometer* keeps the sidereal time, it is required to determine the hour angle of the Vernal Equinox (or a star) at the time of observation. Similarly, if the chronometer keeps the solar time, it is required to determine the hour angle of the centre of the sun at the instant the observations are taken. Determinations are made from meridian or ex-meridian observations. The difference between the chronometer time and the time determined from the observation gives *chronometer correction* and should be added algebraically to the reading of the watch to give the true time at the instant. *The correction is positive when the chronometer is slow and negative when it is fast.*

The following are some of the methods usually employed for the determination of time :

(1) By meridian observation of a star or the sun. (By transit of a star or sun)

(2) By ex-meridian altitude of a star or the sun.

(3) By equal altitudes of star or the sun.

1 (*a*) TIME BY MERIDIAN TRANSIT OF A STAR

The application of this method requires a knowledge of the local longitude and a previous determination of the direction of the meridian. This forms the most direct method of obtaining local time and is used for primary field determinations. The basis of the method is the fact that when a star transits the meridian, its hour angle is zero and local sidereal time is equal to the right ascension of the star.

In Fig. 13.39, ZP is the observer's meridian and M is the position (in general) of a star.

$\angle SP\Upsilon$ = Local sidereal time

$\angle SPM$ = Hour angle (H) of the star (measured westward)

$\angle \Upsilon PM$ = R.A. of the star.

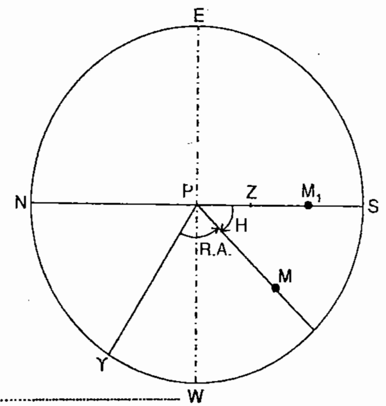


FIG. 13.39

Evidently, $\angle SP\Upsilon = \angle SPM + \angle \Upsilon PM$

or L.S.T. = Hours angle + R.A.

M_1 is the position of the star when it crosses the meridian, and its hour angle (H) is zero. Thus,

$$\text{L.S.T.} = \text{R.A.}$$

The right ascensions of various stars are given in the Ephemeris for the date.

The star is observed with a theodolite, the line of sight being directed along the known direction of the meridian. The chronometer is read at the instant the star transits across the vertical wire. The chronometer error is then determined by comparing the true sidereal time (equal to the right ascension) of the star with the sidereal time kept by the watch or chronometer. If the chronometer is keeping Greenwich sidereal time, it is necessary to apply only the local longitude to the right ascension of the star to obtain the true Greenwich sidereal time. If the chronometer keeps the local mean time, the local sidereal time determined above is converted into local mean time by method discussed earlier and the error of the chronometer is determined. Generally, the chronometer error is found in this way on two different days and average daily rate of error during the period is found by dividing the change in the error by the number of days elapsed.

1. (*b*) TIME BY MERIDIAN TRANSIT OF THE SUN

When the sun is observed on the meridian of the place at upper transit, its hour angle is zero and the L.A.T. is 12 hours. The transit of the sun is observed with a theodolite and the times at which the east and west limbs of the sun pass the vertical hair are noted by means of the chronometer. The mean of the two readings gives the mean time at the local apparent noon. If only one limb is observed, allowance must be made for the time that the semi-diameter takes to cross the meridian. From the Nautical Almanac, we can find the G.M.T. of G.A.N. for the given date, from which the L.M.T. of L.A.N. may be found. This L.M.T. of L.A.N. can then be compared with the chronometer time at the instant of the observation to give the error of the chronometer.

Error in the Observations of the Meridian Transit of Star or Sun

The method of meridian transit of a star or the sun, though simple, is not very much used because it is impracticable to secure that the instrumental line of sight lies exactly in the plane of the meridian. The observed times are subject to the following three principal corrections :

(*i*) The Azimuth Correction

If the instrument is in accurate adustment, but the direction of the meridian is in error, the line of sight set out along the meridian will pass through the zenith of the observer and not through the celestial pole. The correction is given by

$$\textit{Azimuth correction} = e \sin z \sec \delta$$

where e = error of azimuth in *seconds of time*

z = zenith distance

δ = declination of the star.

e is considered positive if the line of sight it too far east when the telescope is pointed south, and is negative if the line of sight is too far west. It can be shown that if the latitude of the place is 30° and the polar distance of a star is 40°, *an error of 1 minute of arc in the direction of the meridian will make the time of transit wrong by two seconds.* The method, therefore, requires the meridian to be set out very accurately.

The error is very great if the polar distance of the star is small, and is least for those that transit near the zenith.

(*ii*) The Level Correction

If the horizontal axis is not perfectly horizontal, the line of sight may depart considerably at high altitudes. Due to this, the transit will be observed either too soon or too late according to the direction of tilt of the transverse axis. The correction is given by :

$$\textit{Level correction} = b \cos z \sec \delta$$

where b = inclination of the horizontal axis in seconds of arc (determined by the readings of the striding level) and is positive when the left (or west) end of the axis is higher

z = zenith distance

δ = declination of the star.

(iii) **The Collimation Correction**

The collimation correction is necessary when the line of sight is not perpendicular to the horizontal axis. The correction is given by :

Collimation correction $= c \sec \delta$

where c = error of collimation in seconds of time taken positive when the line of sight is to east of the meridian, and negative when it is to the west).

δ = declination of the star.

2. (*a*) TIME BY EX-MERIDIAN OBSERVATION OF A STAR

The determination of time by ex-meridian observation of a star or sun is the most convenient and suitable method for surveyor. The method, in its simplest form consists in observing the altitude of the star when it is out of the meridian and at the same time observing the chronometer of the star and its altitude ; the hour angle can be computed by the solution of the astronomical triangle. The local sidereal time can then be known by adding the westerly hour angle to the R.A. of the star. The local sidereal time can be converted into local mean time and the error of chronometer (observing mean solar time) can be found.

In the astronomical triangle *ZPM* (Fig. 13.15), we know the following three sides:

ZP = co-latitude = $(90° - \theta) = c$ (say)

$MP = p$ = polar distance = $(90° - \delta)$

$ZM = z$ = zenith distance = $(90° - \alpha)$

$\angle MPZ = H$ = hour angle which can be computed from any one of the following formulae :

$$\tan \frac{H}{2} = \sqrt{\frac{\sin (s-c) \cdot \sin (s-p)}{\sin s \cdot \sin (s-z)}} \quad \ldots(1) \quad ; \quad \sin \frac{H}{2} = \sqrt{\frac{\sin (s-c) \cdot \sin (s-p)}{\sin c \cdot \sin p}} \quad \ldots(2)$$

$$\cos \frac{H}{2} = \sqrt{\frac{\sin s \cdot \sin (s-z)}{\sin c \cdot \sin p}} \quad \ldots(3) \quad ; \quad \cos H = \frac{\sin \alpha - \sin \theta \sin \delta}{\cos \theta \cos \delta} \quad \ldots(4)$$

where $s = \frac{1}{2}(z + c + p)$

It should be noted that if H is near to 0° or 90°, the tangent formula is the best one to adopt since it gives more precise result.

In the field observation, the altitude has to be observed and refraction correction must be applied. Due to uncertainties in the refraction for low altitudes, the star observed should have an altitude of at least 15°.

When the star is in or near the prime vertical, its altitude changes more rapidly and the star should be observed at this time since it gives more accurate results. The influence of error in observed altitude as well as in the value of the altitude, is a minimum when the star is actually on the prime vertical. To minimise the errors of observation, several altitudes of the star are observed in quick succession and the chornometer time of such observation is recorded. Half of the observations are taken with face left and half with the face right. If the observations are completed within a few minutes (say 10^m) it will suffice for most ordinary work if the mean of the chronometer times is taken as

the time for the mean altitude. *The motion of the star in altitude is not however, exactly proportional to time.* More accurate results are obtained when two stars are observed, one east and the other west of the meridian, thus eliminating the instrumental errors.

When the star is observed on its prime vertical, the hour angle is given by

$$\cos H = \frac{\tan \text{declination}}{\tan \text{latitude}} = \frac{\tan \delta}{\tan \theta}$$

Knowing the hour angle (in degrees), the L.S.T. is calculated from the formula :

$$L.S.T. = R.\ A. \pm \frac{H}{15}$$

Plus sign is used when the star is to the west of the meridian and minus when it is to the east. Knowing the G.S.T. of G.M.M. (for G.M.N.), the L.S.T. can converted to L.M.T. and the error of the chronometer keeping the mean solar time can be computed.

2. (*b*) TIME BY EX-MERIDIAN OBSERVATION OF THE SUN

The procedure of observation of the sun is the same as in the previous case. The altitude of the lower limb is observed with the telescope normal, and then the altitude to the upper limb is observed with the telescope inverted. The watch time at the instant of each observation is noted. The balancing is affected by measuring a succession of altitudes both in the morning and afternoon, the most suitable timings being between 8 and 9 A.M. and between 3 and 4 P.M. In each set, a minimum number of four observations are taken — both face observations of upper limb and both face observations lower limb. If the sun is not very near the meridian and if the observations extend over only a few minutes of time (say 10^m), the mean of the observed altitudes may be assumed to correspond to the mean of the observed times, thus neglecting the curvature of the path of the sun. The mean of the altitudes must be corrected for index error, refraction, and parallex, and for the semi-diameter if only one limb is observed. The hour angle of the sun can be calculated from the formula :

$$\tan \frac{H}{2} = \sqrt{\frac{\sin (s-c) \sin (s-p)}{\sin s \cdot \sin (s-z)}}$$

The above formula is more convenient for logarithmic computations. Then, if the sun is to the west of meridian,

$$\text{L.A.T. of observation} = \frac{H}{15} \text{ since local apparent noon.}$$

When the sun is to the east of meridian,

$$\text{L.A.T. of observation} = \left(24^h - \frac{H}{15}\right) \text{ since local apparent noon}$$

$$= \left(12^h - \frac{H}{15}\right) \text{ since local apparent midnight.}$$

The L.A.T. can then be converted into L.M.T. by methods discussed earlier.

In the above computations, a correct knowledge of sun's declination (δ) is required. For the computation of sun's declination for the mean instants of observation, a knowledge of local time is necessary. Since the local time is being determined, the computation of H should be performed by successive approximation. However, if the watch is not more than 2^m or 3^m in the error, the resulting error in computing the declination will not exceed 2″ or 3″, and recalculations are not necessary if observations are made with small instrument. If greater discrepancy is found between the correct and the chronometer time, the former is used for a better interpolation of δ and the computation of H is repeated with the new value. Also, a knowledge of the latitude of the place is essential for the computation of H. The precision in the knowledge of the latitude of the place depends upon the precision in the observation of altitude and also upon the time at which observation is made. When the sun is near the prime vertical, the effect of an error in latitude is small.

The error of the watch on local mean time is then equal to the difference between the time of observation by watch and the time of observation as determined by calculations. The observation is often combined with the observation of the sun for azimuth, the watch readings and altitude readings being common to both.

Booking of Field Observations

The field observations are usually entered in the field book in the following form: (Table 13.1).

TABLE 13.1

Star observed	Face	Vertical Angle												Time			Mean of time		
		A			B		Mean			Mean vertical Angle									
		°	′	″	′	″	°	′	″	°	′	″		h	m	s	h	m	s
α -ophiuchi W	L	38	30	20	30	40	38	30	30					7	21	11			
	R	37	26	30	26	10	37	26	20					7	27	30			
	R	36	30	40	30	20	36	30	30					7	32	20			
	L	35	50	10	50	00	35	50	5	37	4	21		7	38	05	7	29	46.5

3. (a) TIME BY EQUAL ALTITUDE OF A STAR

In this method, a star is observed at the same altitude on opposite sides of the meridian. The mean of the two chronometer times at which a star attains equal altitudes east and west of the meridian is evidently the chronometer time of transit, since the two observations are clearly made at equal intervals of time before and after the Star's meridian transit. The method is, therefore, very simple and accurate and is used when the direction of the meridian is not accurately known. The altitude of the star need not be determined and, therefore, no correction is required for refraction. The observations must be made when the star is near the prime vertical so that its altitude changes rapidly. When the star crosses the meridian, its hour angle is equal to zero and its right ascension

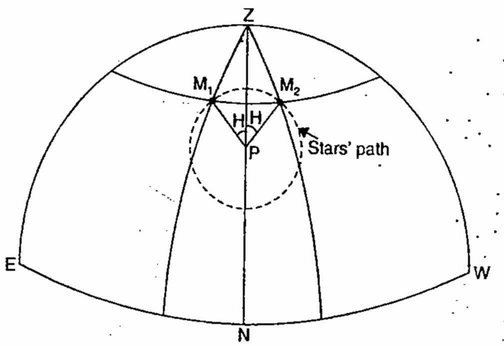


FIG. 13.40. TIME BY EQUAL ALTITUDE.

is therefore the local sidereal time. The local sidereal time so obtained may be converted to local mean time which can then be compared with the mean time of the chronometer during the observations, and the error of the chronometer can be known.

To make the observations, the following steps are necessary :

(1) Set up the instrument on firm ground and level it accurately.
(2) Compute the approximate altitude of the star and set it on the vertical circle.
(3) Follow the motion of the star in azimuth with the vertical cross-hair by means of horizontal tangent screw.
(4) Note the chronometer time (T_1) when the star crosses the horizontal hair.
(5) Turn the instrument in azimuth and again follow the star when the star approaches the same altitude to the other side of the meridian.
(6) Note the chronometer time (T_2) when the star crosses the horizontal hair.

Mean time of transit of the star $= \frac{1}{2}(T_1 + T_2)$

It is very important to note that during the above observations the face of the theodolite is not changed. However, the altitude bubble must be accurately centred by means of clip screws prior to each observation. For accurate results, a series of observations are made on the same star.

In Fig. 13.40, the dotted circle shows the daily path of the star round the pole. M_1 is the position of the star of the east of the meridian ZP and M_2 is its position to the west of the meridian when it attains the same altitude as at M_1.

The method has the following advantages :

(1) Since the actual altitude of the star is not required the instrumental errors– such as index error, collimation error, errors due to graduations etc. are not involved.
(2) No knowledge is required of latitude, declination, or even azimuth.

The method has, however, the following disadvantages :

(1) A long interval of time elapses between the two observations – sometimes several hours.
(2) The precision of the result depends upon the refraction having the same value for both observations. Due to long interval of time, the refraction may change appreciably, thus affecting the result.

However, the time between the two observations can be reduced if the declination of the selected star is nearly equal to the latitude. To eliminate the uncertainties of refraction near the horizon, the star should have an altitude of something more than 45°.

The Error due to Slight Inequality in the Altitudes of Two Corresponding Observations: In Fig. 13.40,

ZM_1 = zenith distance of first observation = z
ZP = co-latitude = c
PM_1 = polar distance = p
ZPM_1 = hour angle = H
M_1ZP = A = azimuth of the star.

Now, we have $\cos z = \cos c \cos p + \sin c \sin p \cos H$ ...(1)

When the star is at M_2, let

ZM_2 = zenith distance of seconds observation = $(z + y)$

where y is the small error due to inequality of the altitudes.

ZPM_2 = hour angle of $M_2 = (H + x)$

where x is the small error in the hour angle.

Hence we have $\cos(z + y) = \cos c \cos p + \sin c \sin p \cos (H + x)$ ...(2)

Subtracting (2) from (1) and treating x and y as small quantities, we get

$$y \sin z = x \sin c \,.\, \sin p \,.\, \sin H.$$

But $$\frac{\sin z}{\sin H} = \frac{\sin p}{\sin A}$$

Hence $$x = \frac{y \sin z}{\sin c \sin p \sin H} = \frac{y}{\sin c \sin A} \quad ...(3) \quad ...(13.32)$$

In order that x should be least for a given value of y, we must have $\sin A = 1$ or $A = 90°$. The error will evidently be greater for smaller value of A. *Hence we conclude that the error in the hour angle due to some error in altitude is minimum when the star is near the prime vertical.*

3. (*b*) TIME BY EQUAL ALTITUDES OF TWO STARS

The two disadvantages of the method of equal altitudes mentioned above (*i.e.* the long interval of time and the uncertainties in the value of refraction) can be reduced by making the equal altitude observations on two stars, one east and the other west of the meridian. In such observations, two stars having the same declination are selected. When they attain the same altitudes, one to the west and other to the east of the meridian, the mean of their right ascension will give the local sidereral time of transit. The local sidereal time can be converted into L.M.T. and can be compared with the mean of the chronometer readings for the determination of the chronometer error. If the two stars have some different declinations, a correction must be applied to the mean of their right ascensions. However, the difference in the declination of the two stars should not be more than 2° to 5°. The observations of a pair of stars generally takes few minutes. Several pairs should be used for good determination. The stars selected to form a pair should have a difference in right ascension of at least 6^h.

3. (*c*) TIME BY EQUAL ALTITUDS OF THE SUN

If the equal altitude observations are made on the sun, the same edge of the sun's image (*i.e.*, the upper limb or lower limb) should be brought to the horizontal hair and the image bisected by the vertical hair of the diaphragm. A series of altitudes is taken about 9 AM. and the same series is repeated in reverse order about 3 P.M. The mean of the times of the forenoon and afternoon equal altitudes does not exactly represent the instant of transit (L.A.N.) due to the rapid change of sun's declination. *The theory becomes complicated due to the fact that allowance must be made for the alteration of declination in the interval between the observations.* In order to apply the correction for the change

in the declination, the approximate value of the latitude and Greenwich mean time must be known.

Let y be the alteration in the sun's declination in *half* the time interval between the two observations.

In Fig. 13.40, M_1 = First position of the sun having polar distance $(p + y)$ say, when the sun is approaching the pole.

M_2 = Second position of the sun having the polar distance $(p - y)$, say,

If p were constant, we have, as earlier. $\cos z = \cos p \cos c + \sin p \sin c \cos H$ ...(1)

But the polar distance is $(p + y)$ and the hour angle is $(H + x)$. We have, thus

$$\cos z = \cos(p + y) \cos c + \sin(p + y) \sin c \cos(H + x) \quad ...(2)$$

Subtracting (1) from (2), and treating x and y to be small quantities, we have

$$x = y(\cot p \cot h - c \cot \operatorname{cosec} H) \quad ...(3) \quad ...(13.33)$$

For a given value of y, therefore, the value of x can be computed from the given equation.

The first observation will thus be made when the sun's hour angle is $(H + x)$ *before the apparent noon.* Similarly, the second observation will be made when the sun's hour angle is $(H - x)$ *after the apparent noon. The mean of these two observed times will therefore be when the sun is at an hour angle x before apparent noon.*

For example, let $H = 3$ hours ; and $x = 1$ min. (calculated from Eqn. 3)

Then, the hour angle of sun at first observation $= (H + x)$

= 3 hour 1 min. before apparent noon.

∴ Time of observation $= 12^h - 3^h 1^m = 8^h 59^m$ apparent time.

Similarly, the hour angle of sun at second observation $= H - x$

$= 2^h 59^m$ after apparent noon.

∴ Time of observation $= 12^h + 2^h 59^m = 14^h 59^m$ apparent time.

∴ Mean time of observation $= \frac{1}{2}(8^h 59^m + 14^h 59^m) = 11^h 59^m$

$= 1^m$ before the apparent noon

$= x$ before the apparent noon.

Hence we get the following rule :

True time of transit (*i.e.*, apparent noon)

$$= \text{Mean of observed time} \mp \frac{x}{15} \text{ (When } x \text{ is in angular measure).}$$

Minus sign is used when the sun is approaching the elevated pole (*i.e.*, the case discussed above) and plus sign when the sun is leaving the pole.

Example 13.44. *The time of transit of a star (R.A. $7^h 36^m 21.24^s$) recorded with a chronometer keeping standard time of $5^h 30^m$ E was $5^h 56^m 8.86^s$ P.M. The longitude of the place of observation is $4^h 30^m$ E. Determine the error of the chronometer if G.S.T. at G.M.M. on the day is $14^h 18^m 12^s$.*

Solution

Let us first convert the G.S.T. of G.M.M. into L.S.T. of L.M.M.

Longitude = $4^h\ 30^m\ E$

Loss in the sidereal time at the rate of 9.8565^s per hour of longitude is :

$$4^h \times 9.8565 = 39.43 \text{ seconds}$$
$$30^m \times 0.1642 = 4.93 \text{ seconds}$$

Total retardation $= 44.36$ seconds

$\therefore$ L.S.T. of L.M.M. = G.S.T. of G.M.M. – Retardation

$= 14^h\ 38^m\ 12^s - 44.36^s = 14^h\ 37^m\ 27.64^s$

Now L.S.T. of observation = R.A. of the star = $7^h\ 36^m\ 21.24^s$

$\therefore$ S.I. = L.S.T. of observation – L.S.T. of L.M.M.

$= (7^h\ 36^m\ 21.24^s - 14^h\ 37^m\ 27.64^s) + 24^h = 16^h\ 58^m\ 53.6^s$.

Let us now convert the S.I. into mean time interval by subtracting the retardation at the rate of 9.8296 seconds per hour of sidereal time.

$$16^h \times 9.8296 = 157.27 \text{ seconds}$$
$$58^m \times 0.1638 = 9.49 \text{ seconds}$$
$$53.6^s \times 0.0027 = 0.14 \text{ second}$$

Total retardation $= 166.90$ seconds $= 2^m\ 46.90^s$

$\therefore$ Mean time interval since L.M.M. = S.I. – Retardation

$= 16^h\ 58^m 53.6^s - 2^m\ 46.90^s = 16^h\ 56^m\ 6.7^s$

Standard time shown by chronometer

$= 5^h\ 56^m\ 8.86^s$ P.M. $= 17^h\ 56^m\ 8.86^s$ since L.M.M.

$\therefore$ Local time of chronometer

$= 17^h\ 56^m\ 8.86^s$ – Difference of longitude

$= 17^h\ 56^m\ 8.86^s - 1^h = 16^h\ 56^m\ 8.86^s$

(Since the place of observation is at longitude 1^h to the west of standard meridian).

$\therefore$ Chronometer error = **2.16 seconds (Fast).**

Example 13.45. *The following notes refer to an observation for time made on a star on Feb. 18, 1965 :*

Latitude of the place	$= 36° 30' 30''$ *N*
Mean observed altitude of the star	$= 30° 12' 10''$
R.A. of star	$= 5^h\ 18^m\ 12.45^s$
Declination of the star	$= 16° 12' 18''.4$

This star is to the east of the meridian.

Mean sidereal time observed by sidereal chronometer $= 1^h\ 2^m\ 5.25^s$

Find the error of the chronometer.

Solution. The hour angle of the star is determined from the following formula :

$$\tan \frac{H}{2} = \sqrt{\frac{\sin (s-c) \sin (s-p)}{\sin s \cdot \sin (s-z)}} \quad ; \quad \text{where} \quad s = \frac{1}{2}(z + c + p)$$

$$z = 90° - \alpha = 90° - 30° 12' 10'' = 59° 47' 50''$$
$$p = 90° - \delta = 90° - 16° 12' 18''.4 = 73° 47' 41''.6$$
$$c = 90° - \theta = 90° - 36° 30' 30'' = 53° 29' 30''$$

$$2s = 187° 05' 01''.6$$
$$s = 93° 32' 30''.8$$

$(s - c) = 40° 3' 0''.8$; $(s - p) = 19° 44' 49''.2$; $(s - z) = 33° 44' 40''.8$

$$\log \sin (s - c) = \bar{1}.8085208$$
$$\log \sin (s - p) = \bar{1}.5287565$$
$$\log \text{cosec}\ s = 0.0008302$$
$$\log \text{cosec}\ (s - z) = 0.2553212$$

$\log \tan^2 \frac{H}{2} = \bar{1}.5934287$; $\log \tan \frac{H}{2} = \bar{1}.7967144$

$\therefore$ $\frac{H}{2} = 32° 3' 17''.6$ or $H = 64° 6' 35''.2 = 4^h\ 16^m\ 26.3^s$

Since the star is to the east of the meridian, the westerly hour angle

$= 24^h - 4^h\ 16^m\ 26.3^s = 19^h\ 43^m\ 33.7^s$

R.A. of the star $= 5^h\ 18^m\ 12.45^s$

Add hour angle $= 19^h\ 43^m\ 33.70^s$

$\therefore$ L.S.T. of observation $= 25^h\ 01^m\ 46.15^s = 1^h\ 01^m\ 46.15^s$

Sidereal time by chronometer $= 1^h\ 2^m\ 5.25^s$

$\therefore$ Error of chronometer $=$ **19.1^s** (fast).

Example 13.46. *The mean observed altitude of the sun, corrected for refraction, parallax and level was 36° 14′16″.8 at a place in latitude 36° 40′30″ N and longitude 56° 24′12″ E. The mean watch time of observation was* $15^h\ 49^m\ 12.6^s$ *the watch being known to be about* 3^m *fast on L.M.T. Find the watch error given the following :*

Declination of the sun at the instant of observation $= +17° 26' 42''.1$.

G.M.T. of G.A.N. $= 11^h\ 56^m\ 22.8^s$.

Solution

The hour angle of the sun is given by the formula

$$\tan \frac{H}{2} = \sqrt{\frac{\sin (s-c) \sin (s-p)}{\sin s \,.\, \sin (s-z)}} \quad \text{where } s = \frac{1}{2}(z+c+p)$$

Here
$z = 90° - \alpha = 90° - 36° 14' 16''.8 = 53° 45' 43''.2$
$p = 90° - \delta = 90° - 17° 26' 42''.1 = 72° 33' 17''.9$
$c = 90° - \theta = 90° - 36° 40' 30'' = 53° 19' 30''.0$

$2s = 179° 38' 31''.1$; $s = 89° 49' 15''.6$

$(s-c) = 36° 29' 45''.6$; $(s-p) = 17° 15' 57''.7$; $(s-z) = 36° 03' 32''.4$

$\log \sin (s-c) = \bar{1}.7743468$
$\log \sin (s-p) = \bar{1}.4724776$
$\log \text{cosec}\, s = 0.0000919$
$\log \text{cosec}\, (s-z) = 0.2301672$

$\log \tan^2 \frac{H}{2} = \bar{1}.4770835$; $\log \tan \frac{H}{2} = \bar{1}.7385417$

$\frac{H}{2} = 28° 42' 34''.1$ or $H = 57° 25' 08''.2 = 3^h 49^m 40.6^s$

∴ L.A.T. = $15^h 49^m 40.6^s$

Let us convert this to L.M.T.

Longitude = 56° 24′ 12″ = $3^h 45^m 36.8^s$
L.A.T. = $15^h 49^m 40.6^s$
Subtract longitude = $3^h 45^m 36.8^s$

∴ G.A.T. = $12^h 04^m 03.8^s$

Now G.M.T. of G.A.N. = $11^h 56^m 22.8^s$

or G.M.T. of 12^h apparent time = $11^h 56^m 22.8^s$

Now Greenwich apparent time = Greenwich mean time + E.T.

∴ $12^h = 11^h 56^m 22.8^s$ + E. T.

∴ E.T. = $12^h - 11^h 56^m 22.8^s = 3^h 37.2^s$

Subtractive from the apparent time.

G.M.T.= G.A.T. – E.T. = $12^h 04^m 03.8^s - 3^m 37.2^s = 12^h 0^m 26.6^s$

∴ L.M.T.= G.M.T + longitude = $12^h 0^m 26.6^s + 3^h 45^m 36.8^s = 15^h 46^m 03.4^s$

∴ Error of chronometer = $15^h 49^m 12.6^s - 15^h 46^m 03.4^s$ = **3^m 8.8^s** (Fast)

Example 13.47. *At a certain place in longitude 138° 45′ East, the star is observed East of the meridian at $6^h 45^m 21^s$ P.M. with a watch keeping local mean time. It was*

again observed at the same altitude to the west of meridian at $8^h 48^m 43^s$ P.M. Find the error of the watch given that

G.S.T. at G.M.N. on that day = $9^h 26^m 12^s$; R.A. of the star = $17^h 12^m 48^s$

Solution

L.S.T. of transit of star across the meridian = R.A. of the star = $17^h 12^m 48^s$

Let us convert sidereal time into mean time.

Longitude = 138° 45′ $E = 9^h 15^m E$. Since the place has east longitude,

L.S.T. at L.M.N. = G.S.T. at G.M.N. – retardation

$9^h \times 9.8565^s = 88.71$ seconds
$15^m \times 0.1642^s = 2.46$ seconds

Total retardation = $91.17^h = 1^m 31.17^s$

G.S.T. at G.M.N. = $9^h 26^m 12^s$
Subtract retardation = $1^m 31.17^s$

L.S.T. at L.M.N. = $9^h 24^m 40.83^s$

Now local sidereal time = $17^h 12^m 48^s$
Subtract L.S.T. at L.M.N. = $9^h 24^m 40.83^s$

∴ S.I. since L.M.N. = $7^h 48^m 07.17^s$

Let us convert this S.I. into mean time interval by subtracting the retardation at the rate of 9.8296^s per sidereal hour.

$7^h \times 9.8296 = 68.81$ seconds
$48^m \times 0.1638 = 7.86$ seconds
$7.17^s \times 0.0027 = 0.02$ second

Total retardation = 76.69 seconds = $1^m 16.69^s$

S.I. = $7^h 48^m 07.17^s$
Subtract retardation = $1^m 16.69^s$

∴ M.I. since L.M.N. = $7^h 46^m 50.48^s$

∴ Local mean time of transit of star $7^h 46^m 50.48^s$ P.M. ...(1)

Now L.M.T. of watch for east observation = $6^h 45^m 21^s$ P.M.

L.M.T. of watch for west observation = $8^h 48^m 43^s$ P.M.

= $15^h 34^m 04^s$

∴ L.M.T. of transit of the star as shown by the chronometer = $7^h 47^m 02^s$ P.M. ...(2)

Chronometer error = **11.52 seconds (Fast)**

13.13. TIME OF RISING OR SETTING OF A HEAVENLY BODY

In Fig. 13.41, *SEN* is the horizon and *M* is the position of a star when it is rising. It is required to find the time of rising and setting of the star.

The spherical triangle *PMN* is right-angled at *N*, since the plane of the observer's meridian is perpendicular to the horizon.

∴ $\cos MPN = \cos MP \cdot \tan PN$

Now $\angle ZPM = H$ = hour angle of the star at its rising

$MP = \delta$ = declination of the star

$PN = \theta$ = altitude of the pole = latitude of the observer

$\angle MPN = 180° - H$

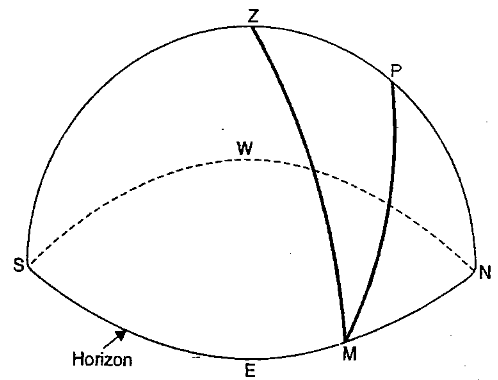


FIG. 13.41. RISING AND SETTING OF STAR.

Hence $$\cos H = -\tan\delta \tan\theta$$

Knowing the declination of the star and the latitude of the place, its hour angle can be known. Then,

L.S.T. of rising of star = R.A. of the star + Hour angle.

Thus, the local sidereal time of the rising of the star can be known, and this can be converted into L.M.T., if desired.

The hour angle of setting will obviously be the same as that of rising. In the above treatment, we have neglected the effect of refraction, which amounting as it does to about 36′ on the horizon, will cause stars to be just visible when they are really 36′ below the horizon.

Length of Day and Night :

The hour angle *H* of the sunrise or sun-set is given by

$\cos H = -\tan\delta \tan\theta$ where δ is the declination of the sun.

If the change in the declination δ of the sun is ignored

Length of the day = twice hour angle in time units = $\frac{2H}{15}$

Similarly, length of the night = $2\left(\frac{180° - H}{15}\right)$

The equation $\cos H = -\tan\delta\tan\theta$ can be used to determine the length of the day at different places and at different times.

(1) At a place at equator, $\theta = 0$

∴ $\cos H = 0$ or $H = 0°$ and $H = 90°$

∴ Length of day (or night) = $\frac{2H}{15} = 12^h$

Hence for all values of δ, the days are always equal to the nights at equator.

(2) At the time of equinox, the sun is at equator and hence $\delta = 0$

∴ $\cos H = 0$ or $H = 0°$ and $H = 90°$

∴ Length of day (or night) = $\frac{2H}{15} = 12^h$

Hence for all values of θ (i.e., at all the places on the earth) the day is equal to the night.

(3) If $\delta = 90° - \theta$; $\cos H = -1$ or $H = 180°$

∴ Length of day = $\frac{2 \times 180°}{15} = 24°$ (*i.e.* the sun does not set).

(4) If $\delta = -(90° - \theta)$; $\cos H = 1$ and $H = 0$

∴ Length of the day = 0^h

Hence the sun does not rise at all.

The Duration of Twilight

Twilight is the subdued light which separates night from day. When the sun sets below the horizon, the darkness does not come instantaneously because the sun's rays still illuminate the atmosphere above us. The particles of vapour etc. in the atmosphere reflect the light and scatter it in all directions. As the sun sinks down, the intensity of the diffused light diminishes. Observations have shown that the diffused light is received so long as the sun does not sink 108° below the horizon. To find the duration of twilight at particular place, we must, therefore, find the time the sun takes to alter its zenith distance from 90° to 108° in the evening, or from 108° to 90° in the morning.

With our previous notations, we have

$$\cos 108° = \sin\delta\sin\theta + \cos\delta\cos\theta\cos H' \quad ...(1)$$

where H' = hour angle of the end of twilight.

If *H* is the hour angle of the sunset we have $\cos H = -\tan\delta\tan\theta$ ...(2)

From the above two equations, *H* and *H′* can be calculated for given values of δ and θ.

Hence duration of twilight = $H' - H$.

13.14. THE SUN DIALS

The sun dial enables the time to be fixed accurately enough for ordinary purposes of life, though the precision obtained is much less than that obtained by the methods already discussed. The sun dial gives apparent solar time from which mean time may be obtained. It is useful particularly in places where there are no means available for checking watch or clock times.

A sun dial essentially consists of :

(*i*) a straight edge, called the *stile* or *gnomon* of the dial and

(*ii*) the graduated circle on which the shadow of the gnomon falls.

When the sun shines, the shadow of the gnomon falls on the graduated circle, and intersects it at some point. The reading against the intersection line gives the local apparent time.

A sun dial may be classified under the following heads :

(i) *The Horizontal Dial* : in which the graduated circle is horizontal.

(ii) *The Prime Vertical Dial*: in which the graduated circle is kept in prime vertical.

and (iii) *The Oblique dial* : in which the plane of the graduated circle is kept inclined to the horizontal.

In each case, the stile is always kept parallel to the earth's axis, and, therefore, always points north.

We shall discuss here the principle of graduating a horizontal sun-dial.

In Fig. 13.42, *BXAY* is the plane of the dial, in the horizontal plane. *CP* is the direction of the rod, stile or gnomon which, if produced indefinitely, will interseet the celestial sphere in the celestial pole *P*. *BPA* is the plane of the meridian. *M* is the position of the sun at any instant and *CY* is the shadow of the gnomon on the horizontal plane intersecting the latter at *Y*.

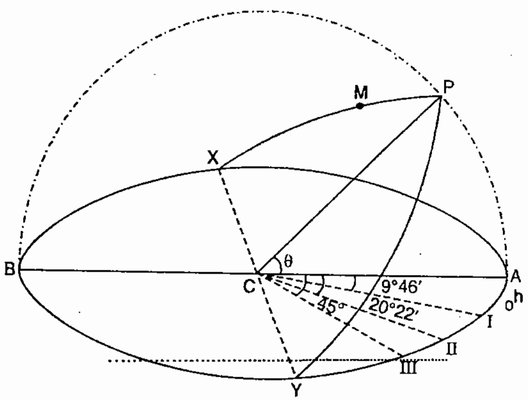


FIG. 13.42. THE HORIZONTAL SUN-DIAL.

Since *CP* is the direction of the meridian also, its shadow will fall on the line *CA* at apparent noon. At one hour after the noon, the shadow will fall on CI, at two hours after the noon, it will fall on CII, and so on. The problem is now to mark the points I, II etc. on the dial so as to correspond to the times of 1^h, 2^h etc. after the apparent noon. At any instant, for the position *M* of the sun, the shadow of the gnomon *CP* will fall on the line *CY* which is the line of intersection of the plane of the dial with the plane containing *CP* and *M*. *XPY* represents such a plane passing through *CP* and *M*.

If the small variation in the declination of the sun is neglected, the diurnal path of the sun (*M*) will describe a circle uniformly on the celestial sphere about *P* as the centre. The projections of the equal angular divisions of the diurnal circle of the sun's path will give unequal angular divisions on the dial. The angle *MPB* is the hour angle of the sun at the instant.

The triangle *YPA* is right angled at *A*.

AP = altitude of the pole = latitude of the place = θ.

$\angle APY$ = hour angle of the sun = H

$AY = x$ = required angular division along the dial corresponding to the hour angle H.

Hence, from the right angled triangle *PYA*, we get

$$\sin\theta = \cot H \tan x \qquad \text{or} \quad \tan x = \sin\theta \tan H$$

or
$$x = \tan^{-1}(\sin\theta \tan H)$$

The above equation gives the values of x corresponding to the different values of H.

To graduate the dial hourly intervals, put $H = 15°, 30°, 45°$, etc., and compute the corresponding values of x for a place of known latitude θ.

For example, let $\theta = 40°$; Then $x = \tan^{-1}(\sin 40° \tan H)$

When $H = 15° = 1^h$; $x_1 = \tan^{-1}(\sin 40° \tan 15°) = 9° 46'$

When $H = 30° = 2^h$; $x_2 = \tan^{-1}(\sin 40° \tan 30°) = 20° 22'$

When $H = 45° = 3^h$; $x_3 = \tan^{-1}(\sin 40° \tan 45°) = 45°$ and so on.

The points I, II, III corresponding to the angles $x_1, x_2 . x_3$ etc., from *CA* can then be marked on the dial.

It should be noted that the sun-dial gives only the local apparent time. To covert it into local mean time, approximate value of equation of time must be known.

13.15. THE CALENDAR

The calendars of historical times were lunar in origin, the year consisting of twelve lunar months. Since the return of the seasons depends upon the tropical year, these calendars resulted in a continual change in the dates at which the seasons occurred. The calendar was, therefore, frequently changed in an arbitrary manner, to keep the seasons in their places. In the year 45 B.C., Julius Caesar introduced the Julian Calendar based on a year of $365\frac{1}{4}$ days. The Julian Calendar has January 1 as the commencement of the year. The calendar has ordinary year of 365 days, and was regulated by introducing one extra day on every fourth year which is known as the *leap year*. However, the year actually contains 365.2422 days (or $365^d\, 05^h\, 48^m\, 46^s$) while the Julian Calendar assumed the year to contain 365.25 days (or $365^d\, 06^h\, 0^m$). Thus the Julian Calendar made the year too long by $11^m\, 14^s$, and this created one day excess in 128 years. After many centuries, this difference accumulated to the tune of 10 days and it was observed that the Vernal Equinox in 1582, occurred on 11th March instead of 21^{st} March. Pope Gregory XIII, in 1582, therefore, adjusted the whole calendar in such a way that the Vernal Equinox occurred more or less on 21 st March, by dropping 10 days. In the future, the dates are to be computed by omitting leap year in those century years not divisible by 400 (*i.e.* years as 1700, 1800 and 1900). This will result in omission of 3 days in every 400 years, thus making the mean calendar year of 365.2425 days (or $365^d\, 05^h\, 49^m\, 12^s$). It has also been suggested to omit leap year in the year 4000, and all even multiples thereof, so as to make the mean calendar year of 365.2422 days (or $365^d\, 05^h\, 48^m\, 46^s$).

13.16. DETERMINATION OF AZIMUTH

An *azimuth* is the horizontal angle a celestial body makes with pole. The determination of azimuth, or the direction of the meridian at survey station consists in obtaining the azimuth or true bearing of any line from the station, so that the azimuths of all the survey lines meeting there may be derived. The determination of the direction of the true meridian or of the azimuth of a line is most important to the surveyor. There are several methods of determining the direction of the true meridian, but preference is given to such methods as will allow a set of observations to be taken so that (*i*) instrumental errors may be eliminated, by taking face left and face right observations and (*ii*) interval or time between the observations may not be too great.

Reference mark

In order to determine the azimuth of a star or other celestial body, it is frequently necessary to have a *reference mark* (R.M.) or *referring object* (R.O.). When steller observations are taken, the reference mark should be made to imitate the light of a star as nearly as possible. The reference mark may be a triangulation station or it may consist of a lantern or an electric light placed in a box or behind a screen, through which a small circular hole is cut to admit the light to the observer. The diameter of the hole should not be more than 1 cm. The mark should preferable be so far from the instrument that the focus of the telescope will not have to be altered when changing from the star to the mark. A distance of about a mile is quite satisfactory.

The following are some of the principal methods of determining the azimuth or the direction of the true meridian :

1. By observations on star at equal altitudes.
2. By observations on a circumpolar star at elongation.
3. By hour angle of star or the sun.
4. By observation of Polaris.
5. By ex-meridian observations on sun or star.

...1 (*a*) OBSERVATIONS ON THE STARS AT EQUAL ALTITUDES

The simplest method of determining the direction of the celestial pole is probably that observing at star at equal altitudes. In this method, the knowledge of the latitude or local time is not necessary, and no calculations are involved. However, the duration of the work is a great inconvenience, extending from four to six hours at night. Also the effects of atmospheric refraction may vary considerably during the interval, affecting the vertical angles to an unknown extent.

The method is based on the fact that if the angle subtended between the reference mark and a star is measured in two positions of equal altitude, the angle between the mark and the meridian is given by half the algebraic sum of the two observed angles.

The dotted circle in Fig. 13.40 represents the circular path of a star round the pole, and it is required to determine the direction of the centre P of this circle. M_1 and M_2 are the two positions of the star at equal altitude, and all that the observer has to do to get his true meridian is to bisect the angle between M_1 and M_2

Thus, in Fig. 13.43, R is the reference mark (R.M.) and O is the position of the instrument station through which the direction of the true meridian is to be established. M_1 and M_2 are two positions of a star at equal altitudes. The field observations are taken in the following steps :

(1) Set the instrument at O and level it accurately.

(2) Sight the R.M. with the reading 0° 0′ 0″ on the horizontal circle.

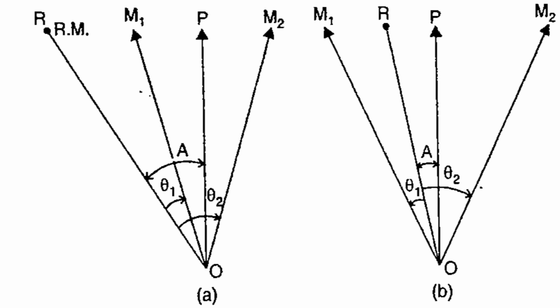


FIG. 13.43. AZIMUTH BY EQUAL ALTITUDES.

(3) Open the upper clamp and turn the telescope clockwise to bisect accurately the star at position M_1. Clamp both horizontal as well as vertical circle.

(4) Read the horizontal angle θ_1 as well as the altitudes α of the star.

(5) When the star reaches the other side of the meridian, follow it through the telescope, by unclamping the upper clamp, and bisect it when it attains the same altitude. In this observation, the telescope is turned in azimuth and the vertical circle reading remains unchanged. Read the angle θ_2.

Let A be the azimuth of the line OR, *i.e.* the angle between the true meridian and the reference object. Since the direction of the meridian is midway between the two positions of the star, the azimuth of the line may be determined according as both the positions of the star are to the same side of R or to the different sides of R.

Case I : Both positions of the star to the same side [Fig. 13.43 (a)].

$\theta_1 = \angle ROM_1$; $\theta_2 = \angle ROM_2$

A = azimuth = $\angle ROP$, (where P is the position of the pole)

$$= \theta_1 + \frac{\theta_2 - \theta_1}{2} = \frac{\theta_1 + \theta_2}{2}$$

Hence the azimuth of the line is equal to half the sum of the two observed angles.

Knowing the azimuth of the line OR, the azimuth of any other line through O can be determined by measuring the horizontal angle between OR and that line. Also if it is required to set out the direction of the true meridian, and angle equal to $\frac{\theta_1 + \theta_2}{2}$ can be set out from the line OR.

***Case II. Both positions of the star are on opposite sides of the line.* [Fig. 13.43 (*b*)].**

$$\text{Azimuth} = A = \angle M_1OP - \angle M_1OR = \tfrac{1}{2}\angle M_1OM_2 - \angle M_1OR = \tfrac{1}{2}(\theta_1 + \theta_2) - \theta_1 = \frac{\theta_2 - \theta_1}{2}$$

Hence the azimuth of the line is equal to half the difference of the two observed angles.

In the observations taken above, it is assumed that the instrument is in perfect adjustment. If it is not so, it is necessary to take at least four observations (two with face left and two with face right) to eliminate the instrumental errors. The position M_1 of the star is observed with both the faces, and the position M_2 is also observed with both the faces, and the mean is taken. However, in the duration that elapses between two face observations of M_1, the position and altitude of the star slightly changes and this should be properly accounted for. In Fig. 13.44, M_1 and M_2 are the two positions of the star to one side of the meridian when both face observations are taken, and M_3 and M_4 are the two positions of the star to the other side of the meridian, in such a way that M_1 and M_2 have equal altitude, and M_2 and M_3 have equal altitude.

The angles θ_1, θ_2, θ_3 and θ_4 with the R.M. corresponding to the positions M_1, M_2, M_3 and M_4 are measured as follows :

(1) The instrument is set at O and, with both plates clamped to zero, bisect R with the vertical circle to the left.

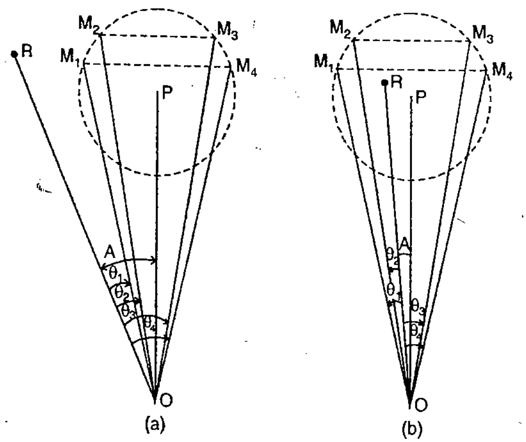


FIG. 13.44

(2) Unclamp the upper clamp, turn the telescope in azimuth and bisect the star at M_1. Note the horizontal angle θ_1 and the vertical angle (*i.e.* the altitude) α.

(3) Change the face of the instrument and again bisect R with both plates clamped to zero. During this time, the star goes to the position M_2. Unclamp the upper clamp and turn the telescope in azimuth to bisect the star at M_2. Clamp the vertical circle. Read the horizontal angle θ_2 and the vertical angle α'.

(4) Leave the instrument undisturbed with the vertical circle clamped to the angle α'. When the star reaches the other side of the meridian, unclamp the upper clamp and turn the telescope in azimuth to bisect the star in position M_3 when it attains the altitude α' (*i.e.* an altitude equal to that at M_2). Read the horizontal angle θ_3.

(5) Change the face of the instrument and again bisect the R.O. with both the plates clamped to zero. Set the angle α (*i.e.* the altitude of the star at the position M_1). Unclamp the upper clamp and turn the telescope in azimuth to bisect the star at the position M_4 when it attains the altitude α. Read the horizontal angle θ_4 in this position.

Thus, we have got four horizontal angles, *i.e.* $\theta_1, \theta_2, \theta_3$ and θ_4. The mean of θ_1 and θ_2 gives the position of the star to one side of the meridian when it has an average altitude equal to $\frac{\alpha + \alpha'}{2}$. Similarly the mean of θ_3 and θ_4 gives the position of the star to the other side of the meridian when it has the *same* average altitude, *i.e.* $\left(\frac{\alpha + \alpha'}{2}\right)$.

When the average positions of the star are to the same side of the R.M., we have

$$\text{Azimuth of } OR = A = \frac{\theta_1 + \theta_2}{2} + \frac{1}{2}\left(\frac{\theta_3 + \theta_4}{2} - \frac{\theta_2 + \theta_1}{2}\right) = \frac{(\theta_1 + \theta_2) + (\theta_3 + \theta_4)}{4}.$$

Similarly, if both the average positions of the star are to the opposite sides, we have

$$A = \frac{1}{2}\left(\frac{\theta_1 + \theta_2}{2} + \frac{\theta_3 + \theta_4}{2}\right) - \frac{1}{2}(\theta_1 + \theta_2) = \frac{(\theta_3 + \theta_4) - (\theta_1 + \theta_2)}{4}.$$

1. (*b*) OBSERVATION ON SUN AT EQUAL ALTITUDES

When the sun is observed for equal altitudes, the sequence of observations is the same as that for a star. Since the actual altitude of the sun is not required, its upper limb or lower limb may be observed *throughout*. A series of horizontal angles is measured between the reference mark and the sun in the forenoon, and a similar series is observed with the sun at the same altitudes in the afternoon. Since the sun's centre cannot be bisected, observations should be made on the right-hand and left-hand limbs of the sun with the telescope normal and inverted in both the morning as well as afternoon observations. However, in the interval between the forenoon and the afternoon observations of equal altitudes, the declination of the sun changes, and hence the mean of the horizontal angles requires a suitable correction to determine the azimuth of the survey line from it. To apply the correction, the watch-time of each observation should also be recorded. The correction is given by

$$c = \frac{1}{2}(\delta_W - \delta_E) \sec \theta \, . \, \operatorname{cosec} t \qquad \ldots(13.34)$$

where c = angular correction to be applied to the algebraic mean of the observed horizontal angles to give the azimuth of the reference line

t = half the interval between the times of equal altitude.

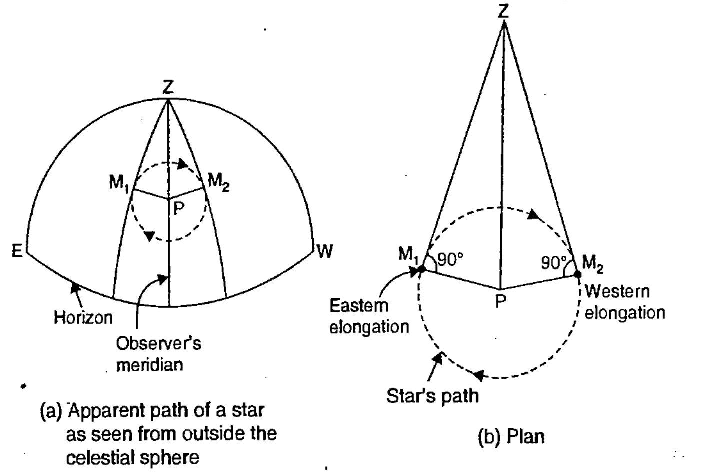
θ = latitude of the observer's place.

δ_E = sun's average declination of morning observations.

δ_W = sun's average declination of evening observations.

(2) OBSERVATIONS ON A CIRCUMPOLAR STAR AT ELONGATION

A circumpolar star is that which is always above the horizon, and which does not, therefore, set. Such a star appears to the observer to describe a circle above the pole (see Fig. 13.19). A circumpolar star is said to be at elongation when it is at its greatest distance east or west of the meridian. When the star is at its greatest distance to the



(a) Apparent path of a star as seen from outside the celestial sphere

(b) Plan

FIG. 13.45. STAR AT ELONGATION.

east of the meridian, it is said to be in eastern elongation. When it is at its greatest distance to the west of the meridian, it is said to be in western elongation. In this position, the star's diurnal circle is tangent to the vertical circle to the star.

Figs. 13.45 (*a*) and 13.45 (*b*) show two views of the stars at elongation. M_1 is the position of the star at its eastern elongation, and M_2 is the position of the star as its western elongation. In this position, the vertical circle of star makes its greatest angle with the plane of the meridian. The vertical through M_1 (or M_2) is tangential to the diurnal path of the star shown by dotted circle. Evidently, therefore, $\angle ZM_1P$ is a right angle. Also, when the star is at western elongation (position M_2), $\angle ZM_2P$ is a right angle.

At the instant of elongation of the star, its motion is vertical and it is in a favourable position for observations upon its azimuth because its horizontal movement is very slight for some time before and some time after it arrives M_1 (or M_2). When the star is in eastern elongation (M_1), it appears to move vertically downwards, and when it is in western elongation, it appears to move vertically upwards at the instant of elongation. It is clear from the figure that the points M_1 and M_2 will always be at a greater altitude than the celestial pole P. However, greater the declination of the star, more nearly will be the altitude of M_1 and M_2 approach that of P.

Prior to making the field observations, it is necessary to *calculate* the time at which the star will elongate. This can be done as follows:

(*i*) The hour angle (H) of the star can be calculated from equation 13.19

$$\cos H = \frac{\tan \theta}{\tan \delta} = \tan \theta \cot \delta$$

(*ii*) Calculate the local sidereal time of elongation :

L.S.T. (of elongation) = R. A. $\pm H$

Use plus sign for western elongation and minus sign for eastern elongation.

(*iii*) Convert this L.S.T. to mean time by method discussed earlier.

Thus, the mean time of elongation of the star is known. At least 15 to 20 minutes before the time of elongation, the instrument is set up and carefully levelled. Five minutes before the time of elongation, a pointing is made on the reference mark. The upper clamp is then unclamped and the star is sighted. The star is then followed in azimuth. At the time of elongation, the star stops moving horizontally, and appears to move vertically along the vertical hair. This will take place exactly at the time calculated above. The horizontal circle reading gives the angle that the star makes with the reference line. To this, if we add the azimuth of the star, the azimuth of the survey line can very easily be known.

The azimuth of the star at its elongation can be calculated from Eq. 13.21:

$$\sin A = \frac{\cos \delta}{\cos \theta} = \cos \delta \, . \, \sec \theta.$$

However, in order to eliminate the error, at least two observations should be made — one with face left a few minutes before the elongation and other with the face right a few minutes after the elongation. If more time is taken between these two sets of readings, the azimuth will not be correct. In general, the observations should not be extended beyond five minutes on either side of the time of elongation and during this time as many readings between the R.M. and the star as are possible should be taken.

The following table gives the time after the moment of elongation when the aizmuth changes by 5″ for a place in latitude 30°:

Polar distance of the star	*Time after moment of elongation before azimuth changes by 5″*
10°	3 min. 33 sec.
15°	3 min. 7 sec.
20°	2 min. 35 sec.
30°	2 min. 11 sec.

As there will be a corresponding and nearly equal period before elongation, it follows that for stars having 20° polar distance, 5 min. and 10 seconds can be the maximum time to the observer before the azimuth can change by 5″ in that period. For a star whose polar distance is 10°, the corresponding time is 7 min. 6 sec. *The nearer the star is to the pole the greater the length of time available for the observations.* In ordinary observations, a surveyor uses a 20″ theodolite so as to determine the azimuth within 20″. Hence, it will be sufficiently accurate if he takes two observations of the star, one with the face left and the other with the face right, not exactly at the time of elongation, but one just before and the other just after the elongation.

However, for very accurate results, it is better to apply the following correction to the value of azimuth (A) of the star from the formula for elongation.

$$\textit{correction (in seconds)} = 1.96 \tan A \sin^2 \delta \, (t_E - t)^2 \quad \text{...(13.35)}$$

where $t_E - t$ is the sidereal interval in minutes between the time of observation and that of elongation. The above formula is applied only when $(t_E - t)$ does not exceed 30 minutes.

The Effect of an Error in the Latitude

For the calculation of the azimuth, the declination (δ) of the star and the latitude (θ) of the place of observation must be accurately known. The declination is taken from the star almanac. Let us now study the effect of an error in the latitude on the determination of the azimuth

Let y = error in the latitude and x = corresponding error in the azimuth.

We have $\sin A = \dfrac{\cos \delta}{\cos \theta}$ or $\sin A \, . \cos \theta = \cos \delta$ ...(1)

Putting the actual values of A and θ in the above expression, we get

$$\sin(A + x) \cos (\theta + y) = \cos\delta$$

Expanding $\sin (A + x)$ and $\cos (\theta + y)$, and replacing $\sin x$, $\sin y$ by x and y respectively, and $\cos x$, $\cos y$ by unity, we get

$$(\sin A + x \cos A) (\cos \theta - y \sin \theta) = \cos \delta \quad \text{...(2)}$$

Subtracting (1) from (2), and neglecting the term having the product of small quantities x and y, we get

$$x \cos A \cos \theta - y \sin A \sin \theta = 0$$

or $$x = y \tan\theta \tan A = y \tan\theta \frac{\sin A}{\sqrt{1-\sin^2 A}}$$

Substituting $\sin A = \frac{\cos\delta}{\cos\theta}$ and $\sqrt{1-\sin^2 A} = \sqrt{1 - \frac{\cos^2\delta}{\cos^2\theta}} = \sqrt{\frac{\cos^2\theta - \cos^2\delta}{\cos^2\theta}}$

we get $$x = y \tan\theta \cdot \frac{\cos\delta}{\sqrt{\cos^2\theta - \cos^2\delta}} \quad \text{...(13.36)}$$

From the above expression,

If $\theta = 0$, $x = 0$

If $\theta = \delta$, $x = \infty$

Also, if $\delta = 90°$, $x = 0$.

Hence, in any given latitude, the error is least when the star selected is nearest to the pole.

The following table gives the ratio $\left(\frac{x}{y}\right)$ of error in azimuth to small error in latitude.

Declination	Latitude = 20°	Latitude = 30°	Latitude = 40°
(δ)	x/y	x/y	x/y
60°	0.22	0.40	0.70
70°	0.14	0.24	0.40
80°	0.06	0.10	0.19

An error in latitude of say 5″ will produce an error in azimuth of less than 5″ if the value of declination is less than the value of latitude. The error in azimuth will, however, be greater than the error is latitude if the value or the declination of the star approaches the value of the latitude :

(3) AZIMUTH BY HOUR ANGLE OF THE STAR OR THE SUN

In this method, the azimuth of a star or sun is determined by observing the hour angle when it is on or near its prime vertical. In the field, the angle between the star and the R.M. is measured, and the chronometer time at the instant of observation is observed very accurately. The altitude of the star is not necessary in this method and hence there is no effect of the errors of refraction. The field work is carried out in the following:

(*i*) Set up the theodolite over the station point and level it accurately.

(*ii*) Select a suitable star as near the prime vertical as possible.

(*iii*) Bisect the R.M. with both the plates clamped to zero, and with the vertical circle to the left.

(*iv*) Unclamp the upper clamp, rotate the telescope in the azimuth and sight the star. When the star is exactly at the intersection of the cross-wires, give the signal to the chronometer observer to observe the chronometer time *very accurately*. Take the reading of the horizontal circle.

(*v*) Repeat the observations with face right.

The mean of the above readings will give the chronometer time and the angle between the star and the R.M.

From the observed mean time of the chronometer, the local sidereal time can be easily calculated by the method discussed earlier. The hour angle of the star can be computed from the expression.

$$\text{L.S.T.} = \text{R. A.} \pm \text{Hour angle.}$$

The R.A. of the star can be known from the star almanac.

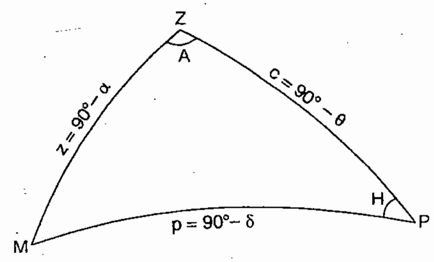


FIG. 13.46

Thus, the hour angle of the star (or the sun) is known from the observed chronometer time. In case the chronometer is fast or slow, its correction should be known before hand, and the same should be applied to the observed time *before* hour angle is calculated.

Knowing the hour angle, the declination and the latitude of the place, the azimuth can be calculated by the solution of the astronomical triangle.

Thus, in Fig. 13.46, M is the position of the star at the instant of observation when its hour angle is H.

$$ZP = \text{co-latitude} = (90° - \theta) = c \quad \text{(known)}$$
$$MP = \text{co-declination} = (90° - \delta) = p \quad \text{(known)}$$
$$\angle ZPM = \text{hour angle} = H \quad \text{(known)}.$$

The value of the azimuth (A) can be calculated from the following expression :

$$\tan A = \tan H \cdot \cos B \cdot \text{cosec}\,(B - \theta) \quad \text{...(13.37)}$$

where $$B = \tan^{-1}(\tan\delta \sec H) \quad \text{...(13.38)}$$

Knowing the azimuth of the star, the azimuth of the survey line can be known.

The above method, though simple and straight forward, is not very much used since separate observations for determining the chronometer error are required. However, if the chronometer error is known, the method is much better than ex-meridian altitudes. *However, if the star is observed near its prime vertical, the errors of time have very little effect on the result.*

While computing the value of H from the chronometer time, a linear relationship between the chronometer timings and the motion of the star in the azimuth was assumed. However, for more precise work, a correction for the curvature of the path of the star must be applied to the mean of the face left and face right observations. The correction (ΔA) in seconds to be applied to the azimuth is given by

$$\Delta A'' = \pm \frac{1}{8} \sin A \cos\theta \sec^2\alpha (\cos\alpha \sin\delta - 2\cos A \cos\theta) \times (\Delta t)^2 \times \sin 1'' \quad \text{...(13.39)}$$

where Δt = difference in time, expressed in seconds of arc, between the face right and face left observation.

The correction is evidently zero at culmination.

(4) AZIMUTH BY OBSERVATIONS ON POLARIS OR CLOSE CIRCUMPOLAR STAR

The *most precise* determination of azimuth may be made by measuring the horizontal angle between the R.M. and a close circumpolar star. The chronometer time of each observation is noted very precisely. From the corrected chronometer times the hour angle of the circumpolar star can then be obtained as discussed earlier. The azimuth of the star can then be calculated by the solution of the astronomical triangle. Since the close circumpolar stars move very slowly in azimuth and errors in the observed times will thus have a small effect upon the computed azimuths, it is evident that only such stars should be chosen for primary or precise work.

Since Polaris (α Ursae Minoris) is the brightest circumpolar star, it is used in preference to others whenever practicable. In general, however, the observations on close circumpolar stars have the following advantages :

(1) Since the motion in the azimuth is very slow, the number of observations may be increased materially and greater accuracy may be secured.

(2) Observations may be made at any convenient time, without calculating the time of elongation or waiting for the time of elongation.

(3) If observations are made on the bright pole star, it is usually possible to sight the star during the twilight when no artificial illumination for the R.M. and for the instrument is necessary.

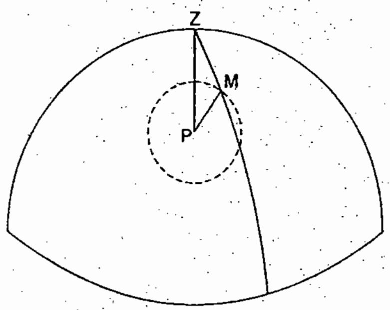


FIG. 13.47. OBSERVATIONS TO POLARIS.

In Fig. 13.47, P is the pole, Z is the zenith of the observer and M is the position of the close circumpolar star. The dotted circle shows the diurnal path of the polar star.

The hour angle H ($\angle ZPM$) is known from the observed chronometer time.

$\angle MZP = A$ = azimuth of the pole star (to be computed)

PM = polar distance = co-declination (known)

ZP = co-latitude $= c = 90° - \theta$ (known)

The azimuth (A) is given by

$$\tan A = \frac{\sin H}{\cos\theta \tan\delta - \sin\theta\cos H}$$

or

$$\tan A = \sec\theta \,.\, \cot\delta \sin t \,.\, \left(\frac{1}{1-a}\right) \qquad \text{...(13.40)}$$

where

$$a = \tan\theta \cot\delta \cos H \qquad \text{...(13.41)}$$

The values of $\log \frac{1}{1-a}$ are tabulated for different values of A in the *Special Publication 14*, United States Coast and Geodetic Survey.

The value to be taken for the hour angle is that corresponding to the mean of corrected chronometer timings of n observations. However, for the accurate results, the curvature of the path of the star should be taken into consideration, and the calculated azimuth should be corrected by the following amount :

$$\text{Curvature correction for one set} = \frac{\tan A \sin^2\delta}{n}\,\Sigma\,\frac{2\sin^2\frac{1}{2}\Delta t}{\sin 1''} \qquad \text{...(13.42)}$$

where n = number of the observations in one set

Δt = angular equivalent of the sidereal time interval (in seconds) between the individual observation and the mean of the set.

For the most accurate work, the striding level should also be observed. If the horizontal axis is inclined during a pointing on the star or the R.M., the horizontal circle reading should be corrected by :

$$\text{Level correction} = \frac{d}{2n}(\Sigma W - \Sigma E)\tan\alpha \qquad \text{...(13.43)}$$

where d = value of one division of the striding level

ΣW and ΣE = sum of west and east reading of the bubble end, reckoned from centre of bubble in direct and reversed position

α = altitude of star or R.M.

Programme of observations

The field observations are arranged in the following steps :

(1) With the face left, point twice the R.M. Read both the verniers of the horizontal circle at each pointing.

(2) With the face left, point twice the star and read both the verniers of the horizontal circle at each pointing. Note the timing of each pointing.

(3) Change the face. Read twice on the star with face right and note the time and the angles.

(4) Read twice upon the R.M. with face right.

Alternative programme of field observations

1. Set the instrument over the instrument mark. With both the plates clamped to zero, sight the R.M.
2. Turn the telescope in azimuth and bisect the star. Note the chronometer time.
3. Read the striding level and reverse it.
4. Read the circle.
5. Intersect the star again and note the time.
6. Read the striding level.
7. Read the circle.
8. Point to R.M. and read the circle.

5. (a) AZIMUTH BY EX-MERIDIAN OBSERVATIONS ON STAR

The determination of azimuth by ex-meridian observation of a star or sun is the method most commonly used by a surveyor except for the determination of primary standard. The observations are the same as that for the determination of time, and the two determinations may be combined if the watch times of the altitudes are also recorded. Knowing the latitude of the place and the declination of the star, the astronomical triangle can be observed for azimuth.

Since the mean refraction for objects at an altitude of 45° is 57″, it necessary to correct for refraction in the measurements of the altitude. The refraction correction is almost uncertain for stars very near to horizon. The stars should be observed when it is changing rapidly in altitude and slowly in azimuth. A favourable position occurs when the star is on the prime vertical when the influence of errors of observed altitude is small.

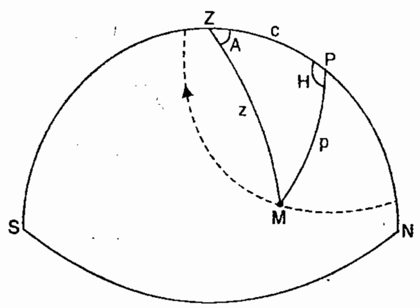


FIG. 13.48. EX-MERIDIAN OBSERVATION OF STAR

In Fig. 13.48, M is the position of the star when its altitude (α) is observed.

In the Astronomical triangle,

ZP = co-latitude = $90° - \theta = c$ (known)

MP = co-declination of star = $90° - \delta = p$ (known)

ZM = corrected co-altitude of star = $90° - \alpha = z$ (observed)

The azimuth (A) can be calculated by one of the following expressions :

$$\sin \frac{1}{2} A = \sqrt{\frac{\sin (s-z) \cdot \sin (s-c)}{\sin z \cdot \sin c}} \; ; \quad \cos \frac{1}{2} A = \sqrt{\frac{\sin s \cdot \sin (s-p)}{\sin z \cdot \sin c}}$$

$$\tan \frac{1}{2} A = \sqrt{\frac{\sin (s-z) \sin (s-c)}{\sin s \cdot \sin (s-p)}} \; ; \quad \text{where} \quad s = \frac{1}{2}(p + c + z).$$

At least two measurements of the altitude and the horizontal angle with the R.M. should be taken, one with face left and the other with face right. In the interval between the face left and face right observations, the star moves considerably in altitude. If the azimuth is calculated from any one of the above formulae by using mean value of the altitude, it will not be exactly the same thing as the mean of the azimuth in the two observed positions. The error will be negligible if the difference in altitude of the star at the two observations is not more than 1° or 2°. However, if the change in altitude is more and if the mean value of the altitude is taken to compute the azimuth, the correction to be applied to the latter is given by

$$\Delta A'' = \frac{1}{8} \cot M \cdot \sec^2 \alpha \, (\sin \alpha - 2 \cot A \operatorname{cosec} 2M) \, (\Delta \alpha)^2 \sin 1'' \qquad \ldots(13.44)$$

where M = the parallactic angle $ZMP = \sin^{-1} (\cos \theta \cdot \sin A \cdot \sec \delta)$.

The value of the correction may be computed by using a four figure log table using the values of the various angles to the nearest minute.

Programme of field observations

1. Set the instrument over the station mark and level it very accurately.
2. Clamp both the plates to zero and sight the R.M. with face left.
3. Unclamp the upper clamp, and bisect the star. Note the horizontal and vertical angles.

4. Change the face of the theodolite and bisect the star again. Obtain the vertical angle and the horizontal angle to the reference mark as before.
5. Observe a second set in the same manner with a new zero.

5. (*b*) AZIMUTH BY EX-MERIDIAN OBSERVATION ON THE SUN

The general procedure of observations are the same as for a star. However, since the declination of the sun changes very rapidly, an exact knowledge about the time of observation is very essential. Also apart from the correction due to refraction, the parallax correction is also to be applied to the observed altitude, since the sun is very near to the earth than the star.

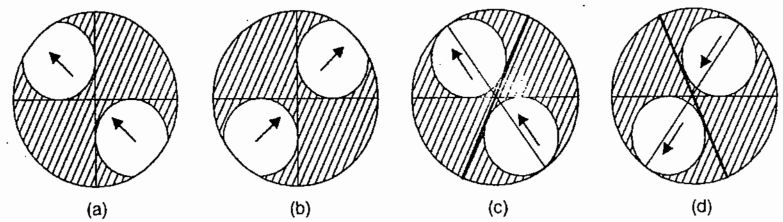


FIG. 13.49. OBSERVATIONS OF THE SUN

The required altitude and the horizontal angles are those to the sun's centre. Hence the hairs should be set tangential to the two limbs simultaneously. The opposite limbs are then observed by changing the face, as shown in Fig. 13.49 (*a*) and (*b*). If however, the diaphragm has no vertical hair, the sun must be placed in opposite angles as shown in Fig. 13.49 (*c*) and (*d*).

Programme of field observations

1. Set the instrument over the station mark and level it very accurately.
2. Clamp both the plates to zero, and sight the R.M.
3. Turn to the sun and observe altitude and horizontal angle with the sun in quadrant 1 (Fig. 13.50) of the cross-wire system. The motion in the azimuth is slow, and the vertical hair is kept in contact by the upper slow motion screw, the sun being allowed to make contact with the horizontal hair. The time of observation is also noted.
4. Using the two tangent screws, as quickly as possible, bring the sun into quadrant 3 of the cross-wires, and again read the horizontal and vertical angle. Observe also the chronometer time.

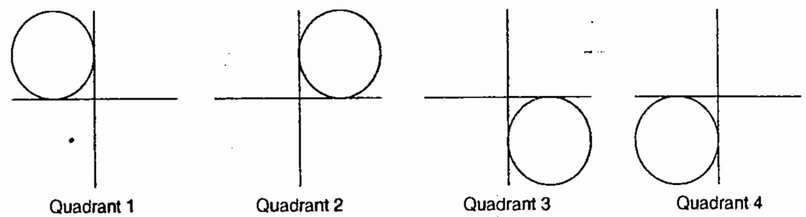


FIG. 13.50. SUN'S LIMB OBSERVED IN VARIOUS QUADRANTS

5. Turn to the R.M., reverse the face and take another sight on the R.M.

6. Take two more observations of the sun precisely in the same way as in steps (3) and (4) above, but this time with the sun in quadrants 2 and 4. Note the time of each observation.

7. Finally bisect the R.M. to see that the reading is zero.

During the above four observations (two with face left and two with face right), the sun changes its position considerably, and accurate results cannot be obtained by averaging the measured altitudes and the times. However, the time taken between the first two readings, with the sun in quadrants 1 and 3, is very little and hence the measured altitudes and the corresponding times can be averaged to get one value of the azimuth. Similarly, the altitudes and the timings of the last two readings, with the sun in quadrants 2 and 4, can be averaged to get another value of the azimuth. The two values of azimuths so obtained (one with face left and the other with face right) can be averaged to get the final value of the azimuth.

For very precise work, the altitude readings should be corrected for the inclination, if any, of the trunnion axis as discussed earlier.

The reduction is performed in the same manner as for the corresponding star observation. The correct value of sun's declination can be computed by knowing the time of observation, by the methods discussed earlier.

The Effect of an Error in Latitude upon the Calculated Azimuth

Let y = error in co-latitude (c)

and x = the corresponding error in the calculated value of azimuth.

We know that $\cos p = \cos c \cos z + \sin c \sin z \cdot \cos A$ ...(1)

Hence $\cos p = \cos (c + y) \cos z + \sin (c + y) \sin z \cdot \cos (A + x)$ ...(2)

Subtracting these two and making the approximations that

$\sin x = x$, $\sin y = y$, $\cos x = 1$ and $\cos y = 1$, we get

$\cos z \cdot y \sin c + \sin z \sin c \cos A - \sin z (\sin c + y \cos c) \times (\cos A - x \sin A) = 0$

or $\cos z \cdot y \sin c - y \sin z \cos c \cos A + x \sin z \sin c \sin A = 0$

(neglecting the terms having product of x and y)

$$\therefore \quad x = \frac{-\cos z \sin c + \sin z \cos c \cos A}{\sin z \cdot \sin c \cdot \sin A}$$

which gives on simplification, $x = \dfrac{-\cot H}{\sin c} \cdot y$ ...(13.45)

It is clear from the above formula that for a given value of y, x is maximum when $\cot H$ is maximum, *i.e.*, when H is minimum. *Hence at all times near noon, the error in azimuth produced by a defective knowledge of the latitude is very much increased.* The error is least at 6 A.M. or 6 P.M. The error also increases with increase in the value of θ, and is the greatest near the pole.

The Effect of an Error in the Sun's Declination upon the Calculated Azimuth

Let y = error in the co-declination (p) of the sun.

x = corresponding error in calculated value of A.

Then $x = (\text{cosec } c \cdot \text{cosec } H) \cdot y$ ...(13.46)

For a given value of y, x is maximum at times near to noon, and is least at 6 A.M. and at 6 P.M.

Also, x increases as the latitude of the place increases. *This method becomes unreliable in arctic or antarctic regions where the given value of y produces very great error in the azimuth.*

The Effect of an Error in the Measured Altitude

Let y = error in the co-altitude (z)

x = corresponding error in the calculated value of azimuth

Then $x = -(\cot M \cdot \text{cosec } z)\, y$; where M = parallactic angle ZMP. ...(13.47)

The value of x is infinitely great when $M = 0°$ or $180°$, *i.e.* when the sun is on the meridian. *Hence, in this case also, it is concluded that the resulting error in azimuth is very great if the observations are made near noon.* The error is however, small if angle M is near 90°.

Example 13.48. *A star was observed at western elongation at a station A in latitude 54° 30′ N and longitude 52° 30′ W. The declination of the star was 62° 12′ 21″ N and its right ascension $10^h\ 58^m\ 36^s$, the G.S.T. of G.M.N. being $4^h\ 38^m\ 32^s$. The mean observed horizontal angle between the referring object B and the star was 65° 18′ 42″. Find (a) the altitude of star at elongation, (b) the azimuth of the line AP and (c) the local mean time of elongation.*

Solution

(a) Altitude of the star, its hour angle and azimuth.

Since the star is observed at elongation, the angle ZMP of the astronomical triangle ZMP is a right angle. Hence, from Napier's rule for circular parts.

$$\sin \alpha = \frac{\sin \theta}{\sin \delta} = \frac{\sin 54° 30'}{\sin 62° 12' 21''} \quad ...(1)$$

or $\alpha = 66° 58' 6''.7$

$\therefore$ Hence the altitude of the star $= \mathbf{66° 58' 6''.7}$.

Also, $$\sin A = \frac{\cos \delta}{\cos \theta} = \frac{\cos 62° 12' 21''}{\cos 54° 30'} \quad ...(2)$$

or $A = \mathbf{53° 25'}$

and $$\cos H = \frac{\tan \theta}{\tan \delta} = \frac{\tan 54° 30'}{\tan 62° 12' 21''}$$

or $H = 42° 21' 20'' = \mathbf{2^h\ 49^m\ 25.3^s}$ ...(3)

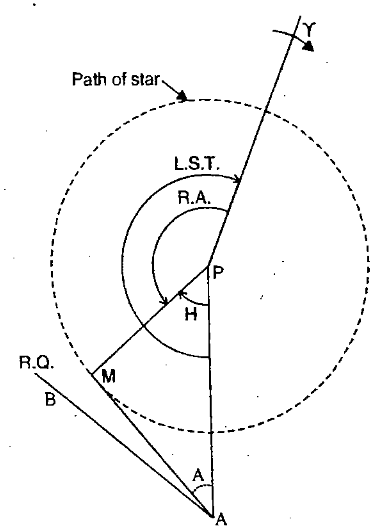


FIG. 13.51. STAR AT WESTERN ELONGATION.

(*b*) ***Azimuth of the line.***

Since the star was at western elongation, it is to the west of the meridian.

∴ Azimuth of the line AB = azimuth of the star + horizontal angle between the line and the star = 53° 25′ + 65° 18′ 42″ = 118° 43′ 42″

∴ Azimuth of line AB in clockwise from north = 360° – 118° 43′ 42″ = **241° 16′ 18″**.

(*c*) ***Local mean time of observation.***

In order to calculate the local mean time of observation, let us first calculate the L.S.T. of L.M.N, from the given value of G.S.T. of G.M.N.

$$\text{Longitude} = 52° 30' W = 3^h 30^m \text{ west.}$$

Acceleration at the rate of 9.8565 per hour :

$$3^h \times 9.8565 = 29.57 \text{ seconds}$$
$$30^m \times 0.1642 = 4.93 \text{ seconds}$$

$$\text{Total acceleration} = 34.50 \text{ seconds}$$
$$\text{G.S.T. of G.M.N.} = 4^h 38^m 32^s$$
$$\text{Add acceleration} = 34.5^s$$

∴ L.S.T. of L.M.N. = $4^h 39^m 06.5^s$

Now L.S.T. of observation = R.A. of star + H.A. of the star

$$= 10^h 58^m 36^s + 2^h 49^m 25.3^s = 13^h 48^m 01.3^s$$

Thus L.S.T. = $13^h 48^m 01.3^s$

Subtract L.S.T. of L.M.N. = $4^h 39^m 06.5^s$

∴ S.I. from L.M.N. = $9^h 8^m 54.8^s$

Let us now convert the S.I. into the mean time interval by subtracting at the retardation at the rate of 9.8296 per sidereal hour.

$$9^h \times 9.8296 = 88.47 \text{ seconds}$$
$$8^m \times 0.1638 = 1.31 \text{ seconds}$$
$$54.8^s \times 0.0027 = 0.15 \text{ second}$$

$$\text{Total retardation} = 89.93 \text{ seconds} = 1^m 29.93^s$$

∴ Mean time interval from L.M.N.

$$= \text{S.I.} - \text{retardation} = 9^h 8^m 54.8^s - 1^m 29.93^s$$

∴ L.M.T. of observation = $\mathbf{9^h 7^m 24.87^s}$

Fig. 13.51 shows the relative positions of observer (A), the star (M), the pole (P), the ♈ and R.O. at the instant of observation.

Example 13.49. *A star was observed at its eastern elongation in latitude 53° 32′ N and the mean angle between a line and the star was found to be 75° 18′ 20″, the star and the line being to the opposite sides of the meridian. Find (a) the azimuth of the line, (b) the altitude of the star at observation, (c) the L.M.T. of observation with the following data :*

Declination of the star	*56° 42′ 53″.2 N*
Longitude of the place	*$5^h 40^m 18^s$ W*
R.A. of the star	*$10^h 58^m 3.9^s$*
S.T. at G.M.M.	*$4^h 58^m 23.84^s$*

(P.U.)

Solution

Since the star was observed at its elongation, the astronomical triangle ZPM is right angled at M. The azimuth altitude and hour angle of the star can be calculated from the Napier's rule.

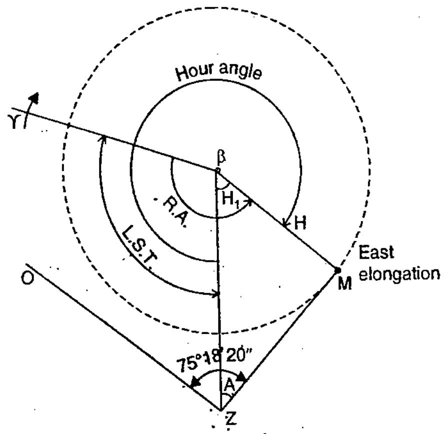


FIG. 13.52. STAR AT EASTERN ELONGATION.

(*a*) Thus, $\sin \alpha = \dfrac{\sin \theta}{\sin \delta} = \dfrac{\sin 53° 32'}{\sin 56° 42' 53''.2}$

$$\alpha = 74° 9' 32''.9$$

Hence altitude of the star = **74° 9′ 32″.9**

(*b*) $\sin A = \dfrac{\cos \delta}{\cos \theta} = \dfrac{\cos 56° 42' 53''.21}{\cos 53° 32'}$

∴ $A = 67° 25' 18''.2\ E$

Since the line and the star are to the opposite sides of the meridian, the azimuth of the line to the west of meridian

= Angle between the line and the star – Azimuth of the star

= 75° 18′ 20″ – 67° 25′ 18″.2 = 7° 53′ 1″.8 to the west of the meridian

∴ Azimuth of the line clockwise from the north

= 360° – 7° 53′ 1″.8 = **352° 6′ 58″.2**.

(*c*) $\angle ZPM = H_1$ = Easterly hour angle of the star.

Hence $\cos H_1 = \dfrac{\tan \theta}{\tan \delta} = \dfrac{\tan 53° 32'}{\tan 56° 42' 53''.2}$

From which $H_1 = 27° 20' 22''.4 = 1^h 49^m 21.5^s$

Hence westerly hour angle of the star = $H = 24^h - H_1$ (see Fig. 13.52)

$$= 24^h - 1^h 49^m 21.5^s$$
$$= 22^h 10^m 38.5^s$$

Add R.A. of the star = $10^h 58^m 3.9^s$

∴ L.S.T. of observation $= 33^h 8^m 42.4^s = 9^h 8^m 42.4^s$

To convert this L.S.T. to L.M.T., let us first find the L.S.T. of L.M.M. from the given value of G.S.T. at G.M.M.

Longitude = $5^h\ 40^m\ 18^s\ W$

Acceleration for this at the rate of 9.8565 seconds per hour of longitude is

$5^h \times 9.8565 = 49.28$ seconds

$40^m \times 0.1642 = 6.57$ seconds

$18^s \times 0.0027 = 0.05$ second

Total correction = 55.90 seconds

$\therefore$ L.S.T. at L.M.M. = G.S.T. at G.M.M. + acceleration

$= 4^h\ 58^m\ 23.84^s + 55.90^s = 4^h\ 59^m\ 19.74^s$

Now S.I. between the L.M.M. and elongation

= L.S.T. − L.S.T. at L.M.M.

$= 9^h\ 8^m\ 42.4^s - 4^h\ 59^m\ 19.74^s = 4^h\ 09^m\ 22.66^s$

This may be converted to mean time interval by subtracting the retardation at the rate of 9.8296 seconds per sidereal hour.

$\therefore$ $4^h \times 9.8296 = 39.32$ seconds

$9^m \times 0.1638 = 1.47$ seconds

$22.66^s \times 0.0027 = 0.06$ second

Total retardation = 40.85 seconds

Mean time interval = S.I. − retardation

$= 4^h\ 09^m\ 22.66^s - 40.85^s = \mathbf{4^h\ 8^m\ 41.81^s}$

Fig. 13.52 shows the relative positions, in plan, of the observer (Z), the pole (P), the star (M), the ♈, and referring object (R.O.).

Example 13.50. *At a place (Latitude 35° N, Longitude 15° 30′ E), the following observations were taken on a star :*

Observed angle between the R.M. and star = 36° 28′ 18″ (clockwise)

R.A. of star :	$10^h\ 12^m\ 6.3^s$
Declination of star :	20° 6′ 48″.4
G.M.T. of observation :	$19^h\ 12^m\ 28.6^s$
G.S.T. of G.M.M. :	$10^h\ 12^m\ 36.2^s$

Calculate the true bearing of the reference mark.

Solution

Here, the observations have been taken for the hour angle of the star to calculate the azimuth of the line. From the observed chronometer time (G.M.T.) let us first calculate the hour angle of the star.

G.S.T. of G.M.M. = $10^h\ 12^m\ 36.2^s$.

Since the place has western longitude, let us subtract the retardation from the given G.S.T. of G.M.M. to calculate the L.S.T. of L.M.M.

Longitude = 15° 30′ E = $1^h\ 2^m\ E$

$1^h \times 9.8656 = 9.87$ seconds

$30^m \times 0.1642 = 4.93$ seconds

Total = 14.80 seconds

$\therefore$ L.S.T. of L.M.M. = $10^h\ 12^m\ 36.2^s - 14.80^s = \mathbf{10^h\ 12^m\ 21.4^s}$

Now G.M.T. of observation = $19^h\ 12^m\ 28.6^s$

Add east longitude = $1^h\ 2^m$

$\therefore$ L.M.T. of observation = $20^h\ 14^m\ 28.6^s$

Convert this L.M.T. into S.I. by adding the acceleration at the rate of 9.8656 per hour.

$20^h \times 9.8656 = 197.13$ seconds

$14^m \times 0.1642 = 2.30$ seconds

$28.6^s \times 0.0027 = 0.79$ second

$\therefore$ Total = 200.22 seconds = $3^m\ 20.22^s$

$\therefore$ S.I. = Mean time + acceleration

$= 20^h\ 14^m\ 28.6^s + 3^m\ 20.22^s = 20^h\ 17^m\ 48.82^s$

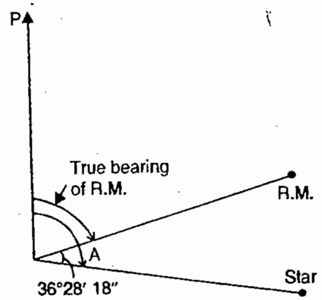


FIG. 13.53

$\therefore$ L.S.T. of observation = L.S.T. of L.M.M. + S.I.

$= 10^h\ 12^m\ 21.4^s + 20^h\ 17^m\ 48.82^s$

$= 30^h\ 30^m\ 10.22^s$

Subtract R.A. of star = $10^h\ 12^m\ 6.30^s$

$\therefore$ Hour angle of the star = $20^h\ 18^m\ 3.92^s$ = 304° 30′ 58″.8 (westerly)

$\therefore$ Smallest hour angle in arc (*i.e.* easterly hour angle)

$= H_1 = 360° - H = 360° - 304° 30' 58''.8 = 55° 29' 1''.2$ ...(1)

Thus the hour angle is known to us.

The value of the azimuth (A) of the star is calculated from the following expression:

$\tan A = \tan H \cdot \cos B \cdot \text{cosec}\,(B - \theta)$ (Eq. 13.37)

where $\tan B = \tan \delta \sec H$ (Eq. 13.38) = tan 20° 6′ 48″.4 . sec 55° 29′ 1″.2

$\therefore$ $B = 32° 52' 27''$

and $\quad B - \theta = 32° 52' 27'' - 35° = -2° 7' 33''$

Hence $\quad \tan A = \tan 55° 29' 1.2'' \cos 32° 52' 27'' \operatorname{cosec}(-2° 7' 33'')$

$\therefore \quad A = 91° 43' 48''$

Now clockwise angle from R.M. to the star = 36° 28′ 18″

∴ True bearing of the line = Azimuth of star − angle between the line and the star

= 91° 43′ 48″ − 36° 28′ 18″ = **45° 15′ 30″**.

Example 13.51. *The following observations of the sun were taken for azimuth of a line in connection with a survey :*

Mean time = $16^h\ 30^m$

Mean horizontal angle between the sun and the referring object = 18° 20′ 30″

Mean corrected altitude = 33° 35′ 10″

Declination of the sun from N.A. = + 22° 05′ 36″

Latitude of place = 52° 30′ 20″ (U.L.)

Determine azimuth of line.

Solution.

In the astronomical triangle *ZPM*,

ZM = zenith distance = $z = 90° - \alpha = 90° - 33° 35' 10'' = 56° 24' 50''$

PM = Polar distance = co-declination = $90° - \delta$

$= 90° - 22° 05' 36'' = 67° 54' 24''$

ZP = co-latitude = $90° - 52° 30' 20'' = 37° 29' 40''$

By cosine rule :

$$\cos PM = \cos ZP . \cos ZM + \sin ZP \sin ZM . \cos A$$

or
$$\cos A = \frac{\cos PM - \cos ZP . \cos ZM}{\sin ZP . \sin ZM} = \frac{\cos 67° 54' 24'' - \cos 37° 29' 40'' . \cos 56° 24' 50'}{\sin 37° 29' 40'' . \sin 56° 24' 50''}$$

From which $A = 97° 6' 48''$

∴ Azimuth of the sun = 97° 6′ 48″

Since the sun is to the west (or left) of the R.O., the true bearing of R.O.

= Azimuth of sun + horizontal angle

= 97° 6′ 48″ + 18° 20′ 30″ = **115° 27′ 18″** (Clockwise from North).

Example 13.52. *At a point in latitude 55° 46′ 12″ N, the altitude of sun's centre was found to be 23° 17′ 32″ at* $5^h\ 17^m$ *P.M. (G.M.T.). The horizontal angle of the R.M. and sun's centre was 68° 24′ 30″. Find the azimuth of the sun.*

Data:

(*a*) *Sun's declination of G.A.N. on day of observation* = 17° 46′ 52″ N

(*b*) *Variation of declination per hour* = − 37″

(*c*) *Refraction for altitude 23° 20′* = 0° 2′ 12″

(*d*) *Parallax for altitude* = 0′ 8″

(*e*) *Equation of time (App. − mean)* = $6^m\ 0^s$. (I.R.S.E.)

Solution

(1) *Calculation of declination*

G.M.T. of observation	= $5^h\ 17^m\ 0^s$ (P.M.)
Add Equation of time	= $0^h\ 6^m\ 0^s$
∴ G.A.T. of observation	= $5^h\ 23^m\ 0^s$ (P.M.)
Now declination at G.A.T.	= 17° 46′ 52″ N
Apparent time interval since G.A.N.	= $5^h\ 23^m\ 0^s$

∴ Variation in the declination in this time interval at the rate of 37″ per hour = 3′ 39″ (decrease).

∴ Declination at G.A.T. of observation = 17° 46′ 52″ − 3′ 39″

= 17° 43′ 13″

(2) *Calculation altitude*

Observed altitude of sun's centre	= 23° 17′ 32″
Subtract refraction correction	= 0° 2′ 12″
	= 23° 15′ 20″
Add parallax correction	= 0′ 8″
∴ Correct altitude	= 23° 15′ 28″

Now, co-altitude $= c = 90° - \theta = 90° - 55° 46' 12'' = 34° 13' 48''$

Co-declination $= p = 90° - \delta = 90° - 17° 43' 13'' = 72° 16' 47''$

Co-latitude $= z = 90° - \alpha = 90° - 23° 15' 28'' = 66° 44' 32''$

$2s = 173° 15' 07''$

$s = 86° 37' 34''$

$s - c = 52° 23' 46''$; $\quad s - p = 14° 20' 47''$; $\quad s - z = 19° 53' 02''$

Now, the azimuth of the sun is given by

$$\tan \frac{A}{2} = \sqrt{\frac{\sin (s-z) \sin (s-c)}{\sin s . \sin (s-p)}} = \sqrt{\frac{\sin 19° 53' 02'' \sin 52° 23' 46''}{\sin 86° 37' 34'' \sin 14° 20' 47''}}$$

$\therefore \quad \frac{A}{2} = 46° 15' 43''$ or $A = $ **92° 31′ 26″**.

Example 13.53. *At a station in latitude 52° 8′ N, longitude 19° 30′ E, the direction of the meridian is known approximately but in order to fix it more precisely it is decided to make an extra-meridian observation of bright-star* (δ = 29° 52′ N, R. A. = $16^h\ 23^m\ 30^s$) *in the late afternoon. It is considered that the most suitable time is* $17^h\ 5^m$ *G.M.T. on a* *date when G.S.T. of G.M.M. in $3^h\ 12^m\ 12^s$. Calculate the approximate direction, east or west of the meridian, and the altitude, at which the telescope should be pointed to locate the star so that exact observations may be made on it.*

Solution. In order to calculate the hour angle of the star, let us first compute the L.S.T. of observation of the star.

G.M.T. of observation $= 17^h\ 5^m\ 0^s$

To covert it into S.I., add the acceleration at the rate of 9.8656 seconds per hour.

$17^h \times 9.8656 = 167.56$ seconds

$5^m \times 0.1642 = \quad 0.82$ second

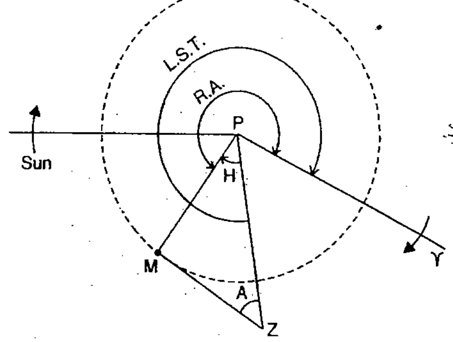


FIG. 13.54

Total $= 168.38$ seconds $= 2^m\ 48.38^s$

$\therefore$ S.I. = G.M.T. + acceleration

$= 17^h\ 5^m + 2^m\ 48.38^s = 17^h\ 7^m\ 48.38^s$

$\therefore$ G.S.T. of observation = G.S.T. of G.M.M.+ S.I.

$= 3^h\ 12^m\ 12^s + 17^h\ 7^m\ 48.38^s$

$= 20^h\ 20^m\ 0.38^s$

Add west longitude $= 1^h\ 18^m$

$\therefore$ L.S.T. of observation $= 21^h\ 38^m\ 0.38^s$

Subtract R.A. of star $= 16^h\ 23^m\ 30.0^s$

$\therefore$ H.A. of star $= 5^h\ 14^m\ 30.38^s = 78°\ 37'\ 36''$

In Fig. 13.54, M is the position of the star at the instant of observation, in relation to the sun and ♈. Z is zenith of the observer and P is the pole.

$\therefore$ PM = co-declination $= 90° - 29°\ 52' = 60°\ 08' = p$

PZ = co-latitude $= 90° - 52°\ 8' = 37°\ 52' = c$

Now, from the astronomical triangle ZPM,

$$\cos H = \frac{\sin \alpha - \sin \delta \sin \theta}{\cos \delta \cdot \cos \theta} = \frac{\cos z - \cos p \cos c}{\sin p \cdot \sin c}$$

or $\cos z = \cos H \cdot \sin p \sin c + \cos p \cos c$

$= \cos 78°\ 37'\ 36'' \cdot \sin 60°\ 08' \cdot \sin 37°\ 52' + \cos 60°\ 08' \cdot \cos 37°\ 52'$

From which $z = 60°\ 7'\ 32''$

$\therefore$ Altitude of star $= 90° - z = 29°\ 52'\ 28''$

Also by rule, $\dfrac{\sin A}{\sin p} = \dfrac{\sin H}{\sin z}$

$$\therefore \quad \sin A = \sin p \cdot \frac{\sin H}{\sin z} = \sin 60°\ 08' \cdot \frac{\sin 78°\ 37'\ 36''}{\sin 60°\ 7'\ 32''}$$

$A = \mathbf{78°\ 38'\ 56''}$ (west).

Example 13.54. *Find the azimuth of the line QR from the following ex-meridian observations for azimuth.*

	Object	*Face*	*Altitude Level O*	*Altitude Level E*
1	*Q*	*L*	...	...
2	*Sun*	*L*	*5.4*	*4.6*
3	*Sun*	*R*	*5.2*	*4.8*
4	*R*	*R*	...	...

	Horizontal Circle A	*Horizontal Circle B*	*Vertical Circle C*	*Vertical Circle D*
1.	*30° 12′ 20″*	*210° 12′ 10″*	...	...
2.	*112° 20′ 30″*	*292° 20′ 20″*	*24° 30′ 20″*	*24° 30′ 40″*
3.	*293° 40′ 40″*	*113° 40′ 30″*	*25° 00′ 00″*	*25° 1′ 00″*
4.	*211° 50′ 30″*	*31° 50′ 20″*	...	...

Latitude of station $Q = 36°\ 48'\ 30''$ *N ; Longitude of station* $Q = 4^h\ 12^m\ 32^s$ *E*

Declination of the sun at G.M.N. $= 1°\ 32'\ 16''.8$ *N decreasing 56″.2 per hour*

Mean of L.M.T. of two observations $= 4^h\ 15^m\ 30^s$ *P.M. by watch ; watch 4 seconds slow at noon, gaining 0.8 seconds per day.*

The value of level division = 15″

Correction for horizontal parallax = 8″.76

Correction for refraction = 57″ cot (apparent altitude).

Solution.

Mean horizontal angle $= \frac{1}{2}\,[(112°\ 20'\ 25'' - 30°\ 12'\ 15'') + (293°\ 40'\ 35'' - 211°\ 50'\ 25'')]$

$= \frac{1}{2}\,[(82°\ 8'\ 10'' + 81°\ 50'\ 10'')] = 81°\ 59'\ 10''$

Mean observed altitude = mean of the four vernier readings $= 24°\ 45'\ 30''$

Level correction $= +\dfrac{\Sigma O - \Sigma E}{4} \times$ value of the one level division

$= +\dfrac{10.6 - 9.4}{4} \times 15'' = +4''.5$

$\therefore$ Apparent altitude $= 24°\ 45'\ 30'' + 4''.5 = 24°\ 45'\ 34''.5$

Refraction correction $= -57'' \cot 24°\ 45'\ 34''.5 = 1'\ 6''.7$

Correction for parallax $= +8''.77 \cos 24°\ 45'\ 34''.5 = 7''.8$

$\therefore$ True altitude $= 24°\ 45'\ 34''.5 - 1'\ 6''.7 + 7''.8 = 24°\ 44'\ 35''.6$

Mean time of observation	$= 4^h\ 15^m\ 30^s$
Watch correction $= +\left(4 - \dfrac{0.8 \times 4.26}{24}\right)$	$= +\quad 3.86^s$
$\therefore$ Correct L.M.T.	$= 4^h\ 15^m\ 33.86^s$
Deduct East Longitude	$= 4^h\ 12^m\ 32.0^s$
$\therefore$ G.M.T. of observation	$= 0^h\ 3^m\ 1.86^s$
Sun's declination at G.M.N.	$= 1°\ 32'\ 16''.8\ N$
Variation for $3^m\ 1.86^s$	$= -\ 56''.2\ (0.0505^h) = -\ 2.8^s$

$\therefore$ Declination of sun at the instant of observation

$$= 1°\ 32'\ 16''.8 - 2.8^s = 1°\ 32'\ 14''$$

Now, in the astronomical triangle ZPM,

$ZP = c = 90° - \theta = 90° - 36°\ 48'\ 30'' = 53°\ 11'\ 30''$

$ZM = z = 90° - \alpha = 90° - 24°\ 44'\ 35''.6 = 65°\ 15'\ 24''.4$

$PM = p = 90° - \delta = 90° - 1°\ 32'\ 14'' = 88°\ 27'\ 46''$

$2s = 206°\ 54'\ 40''.4$

$s = 103°\ 27'\ 20''.2$

$\therefore \quad s - c = 50°\ 15'\ 50''.2 \ ; \ s - z = 38°\ 11'\ 55''.8 \ ; \ s - p = 14°\ 59'\ 33''.8$

The azimuth A is given by

$$\tan \frac{A}{2} = \sqrt{\frac{\sin (s - z) \sin (s - c)}{\sin s \ . \ \sin (s - p)}} = \sqrt{\frac{\sin 38°\ 11'\ 55''.8 \ . \ \sin 50°\ 15'\ 50''.2}{\sin 103°\ 27'\ 20''.2 \ . \ \sin 14°\ 59'\ 33''.8}}$$

$\therefore \quad \frac{A}{2} = 62°\ 7'4''.9 \quad$ or $\quad A = 124°\ 14'\ 9''.8$

$\therefore$ Azimuth of the sun $= 124°\ 14'\ 9''.8$

(west, since the sun was observed in the evening)

Clockwise angle from the R.M. to the sun $= 81°\ 59'\ 10''$

$\therefore$ Azimuth of line from north towards west

$$= 124°\ 14'\ 9''.8 + 81°\ 59'\ 10'' = 206°\ 13'\ 19''.8$$

$\therefore$ Azimuth of line from north (clockwise)

$$= 360° - 206°\ 13'\ 19''.8 = \mathbf{153°\ 46'\ 40''.2.}$$

13.17. THE DETERMINATION OF LATITUDE

The following are some of the most practicable and most generally used methods for determining the latitude of a place :

1. By meridian altitude of sun or star.
2. By zenith pair observation of stars.
3. By meridian altitude of star at lower and upper culmination.

4. By ex-meridian observation of star or sun.
5. By prime vertical transits.
6. By determining the altitude of the pole star.
7. By circum-meridian altitude of sun or star.

1. (*a*) LATITUDE BY MERIDIAN ALTITUDE OF STAR

In this method, the altitude of a heavenly body is measured when it is crossing the meridian. The method is based on the important fact that the latitude of the place is equal to the altitude of the pole. If we can measure the meridian altitude of the star whose declination (and hence polar distance) is known, the latitude can be easily computed. The observed altitude should be corrected for the refraction, as discussed earlier. The accuracy of determination may be increased if it is possible to take two observations for altitude upon the same star, the face of instrument being reversed after the first reading is taken. This is possible with close circumpolar stars, specially when observations are taken with an ordinary 20″ theodolite. The method is, therefore, used for less refined determinations. The direction of the meridian of the place must be known, or must be established before the observations are made.

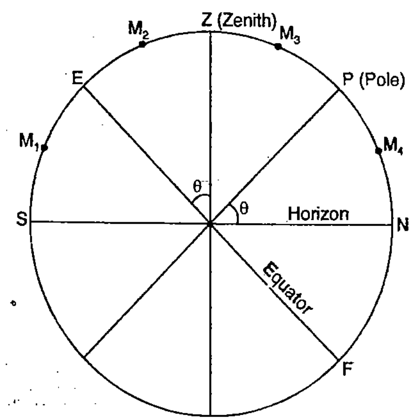


FIG. 13.55. MERIDIAN ALTITUDE OF STAR

To calculate the latitude (θ) of the place of observation from the known value of declination (δ) and the observed value of the altitude (α), we will consider the *four cases* that arise according to the position of the star (Fig. 13.55).

Case 1. When the star is between the horizon and the equator.

M_1 is the position of the star when it is between the horizon and the equator.

ZP = co-latitude $= 90° - \theta$

EZ = latitude $= \theta$

$SM_1 = \alpha_1$ = altitude of the star

$ZM_1 = 90° - \alpha_1 = z_1$ = zenith distance of the star

$EM_1 = \delta_1$ = declination of the star (south)

Now $\quad EZ = ZM_1 - EM_1 \quad$ or $\quad \theta = (90° - \alpha_1) - \delta_1 = z_1 - \delta_1$

Hence $\quad$ ***latitude = zenith distance − declination.***

Case 2. When the star is between the equator and the zenith.

M_2 is the position of the star when it is between the equator and the zenith.

$SM_2 = \alpha_2$ = altitude of the star

$ZM_2 = (90° - \alpha_2) = z_2$ = zenith distance of the star

$EM_2 = \delta_2$ = declination of the star.

Now $EZ = ZM_2 + EM_2$

or $\theta = (90° - \alpha_2) + \delta_2$ or $\theta = z + \delta_2$

Hence ***latitude = zenith distance + declination.***

Case 3. When the star is between the zenith and the pole.

M_3 is the position of the star when it is between the zenith and the pole.

$NM_3 = \alpha_3$ = altitude of the star

$ZM_3 = (90° - \alpha_3) = z_3$ = zenith distance of the star

$EM_3 = \delta_3$ = declination of the star

Now $EM = EM_3 - ZM_3$

or $\theta = \delta_3 - (90° - \alpha_3) = \delta_3 - z_3$

Hence ***latitude = declination – zenith distance.***

Case 4. When the star is between the pole and the horizon.

M_4 is the position of the star when it is between the pole and the horizon.

$NM_4 = \alpha_4$ = altitude of the star

$ZM_4 = (90° - \alpha_4) = z_4$ = zenith distance of the star

$FM_4 = \delta_4$ = declination of the star.

Now PN = altitude of the pole = latitude of the place = θ

$= NM_4 + PM_4 = \alpha_4 + (PF - FM_4)$

$= \alpha_4 + (90° - \delta_4) = (90° - z_4) + (90° - \delta_4) = 180° - (z_4 + \delta_4)$

Hence ***latitude = 180° –(zenith distance + declination).***

1. (*b*) LATITUDE BY MERIDIAN ALTITUDE OF THE SUN

The altitude of the sun at local apparent noon (meridian passage) may be measured by placing the line of sight of the transit in the plane of the meridian and observing the altitude of the upper or lower limb of the sun when it is on the vertical cross hair. The observed altitude is then corrected for instrumental errors, refraction, parallax and semi-diameter. The mean time of observation should also be noted. The declination of the sun continually changes, and hence a correct knowledge of mean time and longitude of the place of observation is essential in order to compute the value of declination at the instant of observation. Knowing the altitude and the declination of the sun at the instant of observation, the latitude can be computed as follows (Fig. 13.56).

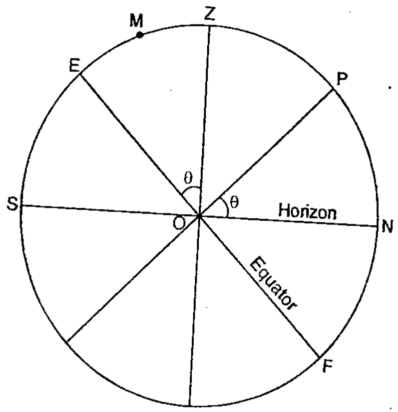


FIG. 13.56. MERIDIAN ALTITUDE OF THE SUN.

In Fig. 13.56, M is the position of the sun.

$SM = \alpha$ = meridian altitude of the sun (corrected).

$ZM = 90° - \alpha = z$ = meridian zenith distance of the sun.

$EM = \delta$ = declination of the sun.

Then latitude $= \theta = EZ = ZM + EM$

$= (90° - \alpha) + \delta = z + \delta$

or ***latitude = zenith distance + declination.***

In the above expression, δ is positive or negative according as the sun is to north or south of the equator.

If the direction of the meridian is not known, the maximum altitude of the sun is observed and may be taken as the meridian altitude. This is not strictly true, due to sun's changing declination. However, the difference between the maximum altitude and the meridian altitude is usually a fraction of a second, and may be entirely neglected for observations made with the engineer's transit or the sextant.

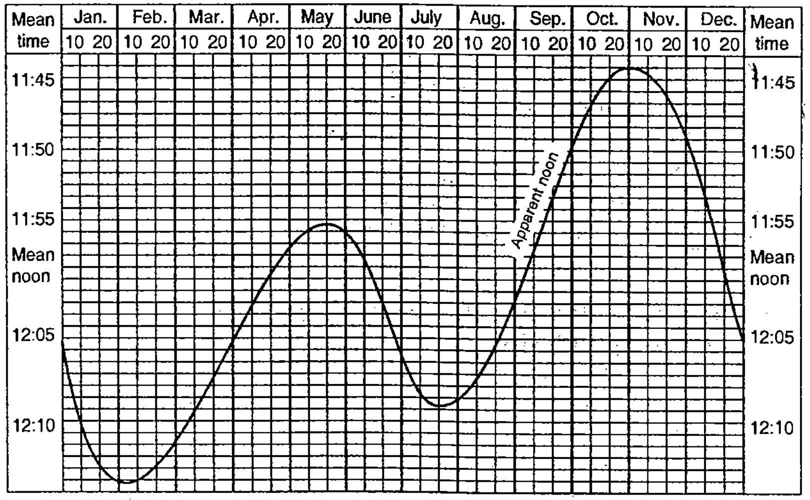


FIG. 13.57. MEAN TIME OF APPARENT NOON.

In order that the observer may be well ready for taking the observations at the meridian transit, standard time or the watch time of local apparent noon must be known. The standard time of local apparent noon varies throughout the year. Fig. 13.57 shows graphically the local mean time of the local apparent noon. The standard time can be known by applying a correction for the difference in longitude between the local meridian and the standard meridian. The observer should be ready to begin observing at this time.

(2) LATITUDE BY ZENITH PAIR OBSERVATIONS OF STARS

This method is an improvement over the previous method to get more precise results. The errors of observation, refraction and instrument can be effectively reduced by making observations upon two stars which culminate at approximately equal latitudes on opposite sides of observer's zenith. The altitude of one star at its culmination is observed first.

The telescope is then reversed in azimuth and the meridian altitude of the other star is then observed. The two stars chosen should be such that their right ascensions differ by 10 to 30 minutes. The time of culmination of these two stars will then differ by 10 to 30 minutes and the observer will have sufficient time in observing the second star after taking the reading of the first and reversing.

Thus, let M_2 and M_3 (Fig. 13.55) be the two stars having approximately equal altitudes to the north and south side of the observer's zenith, and having their time of culminations differing by 10 to 30 minutes.

As derived earlier,

For the position M_2, latitude $\theta = (90° - \alpha_2) + \delta_2$

For the position M_3, latitude $\theta = \delta_3 - (90° - \alpha_3)$ ...(1)

$\therefore$ Average latitude $= \frac{1}{2}\left[\{(90° - \alpha_2) + \delta_2\} + \{\delta_3 - (90° - \alpha_3)\}\right] = \frac{\alpha_3 - \alpha_2}{2} + \frac{\delta_2 + \delta_3}{2}$ ...(2)

From the above expression, it is clear that the average latitude depends upon the *difference* in latitudes of the two stars, and *not* on the individual latitude. Hence any error in the correction for the refraction will be common to both the latitudes (which are approximately equal) and will be eliminated by taking the difference of the two latitudes. Similarly, the instrumental errors are also largely eliminated because these will be practically the same with each observation.

It should be noted that the face of the instrument is *not* reversed while reading the altitude of the second star. To take the reading for the meridian altitude, the telescope is directed to the true meridian, and the altitude is measured when the star intersects the vertical wire.

(3) LATITUDE BY MERIDIAN ALTITUDE OF A CIRCUMPOLAR STAR AT UPPER AND LOWER CULMINATIONS

In this method, the altitude of a circumpolar star is measured both at its upper as well as the lower culmination. The mean of these two altitudes gives the altitude of the pole and hence the latitude of the place of observation. This is proved below (Fig. 13.58).

M is a circumpolar star. A is its position at the upper culmination when its altitude is maximum. B is its position at the lower culmination when its altitude is minimum. The dotted circle shows the path of the star round the pole.

$AN = \alpha_1$ = altitude of the star at its upper culmination.

$BN = \alpha_2$ = altitude of the star at its lower culmination.

Now latitude of place = altitude of the pole = $\theta = PN$

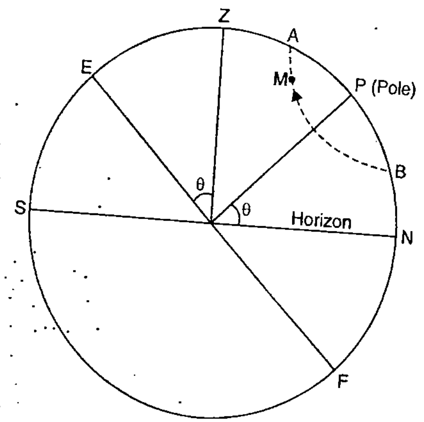


FIG. 13.58

$PN = BN + BP = \alpha_2 + BP$

Also $PN = AN - AP = \alpha_1 - AP$

Adding the two, we get

$2PN = (\alpha_1 - AP) + (\alpha_2 + BP)$

But $AP = BP$ = co-declination of the star

$\therefore$ $2PN = \alpha_1 + \alpha_2$ or $PN = \theta = \frac{\alpha_1 + \alpha_2}{2}$

Hence the latitude of the place of observation is equal to half the sum of the altitude observed at its upper and lower culminations. In this method, the knowledge of the declination of the star is not necessary. However, the method is open to the objection that 12 sidereal hours elapse between the two observations. The method is, therefore, not much used.

(4) LATITUDE BY EX-MERIDIAN OBSERVATION OF STAR OR SUN

In this method, the altitude of the star is observed in any position. The exact chronometer time is also noted at the instant the observation is taken. The known mean time of the chronometer is converted into the local sidereal time. The hour angle of the star can then be computed from the expression:

L.S.T. = R.A. of the star + H.A. of the star.

In the astronomical triangle MPZ in Fig. 13.59,

$ZM = 90° - \alpha = \frac{\pi}{2} - \alpha$ (known)

$PM = 90° - \delta = \frac{\pi}{2} - \delta$ (known)

$\angle MPZ = H$ (known)

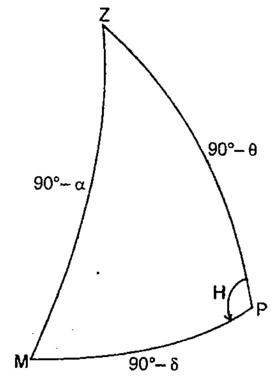


FIG. 13.59

Hence the side $ZP = (90° - \theta)$ can be calculated from the cosine formula

$$\cos\left(\frac{\pi}{2} - \alpha\right) = \cos\left(\frac{\pi}{2} - \theta\right)\cos\left(\frac{\pi}{2} - \delta\right) + \sin\left(\frac{\pi}{2} - \theta\right)\sin\left(\frac{\pi}{2} - \delta\right)\cos H$$

or $\sin\alpha = \sin\theta\sin\delta + \cos\theta\cos\delta\cos H$ ...(1)

In the above equation, there are two terms for θ, *i.e.*, sin θ and cos θ. The equation can be best solved by introducing two arbitrary unknowns m and n as follows :

Let $\sin\delta = m\sin n$ ...(*i*) and $\cos\delta\cos H = m\cos n$ ...(*ii*)

Dividing (*i*) by (*ii*), we get

$\frac{\sin\delta}{\cos\delta\cos H} = \frac{\sin n}{\cos n}$ or $\tan\delta\sec H = \tan n$ ...(*iii*) ...(13.48)

Substituting the value of equations (*i*) and (*ii*) in equation (1), we get

$\sin\alpha = \sin\theta \,.\, m\sin n + \cos\theta \,.\, m\cos n$

or $\sin\alpha = m(\sin\theta\sin n + \cos\theta\cos n)$

or $\sin\alpha = m\cos(\theta - n)$

or $m = \sin\alpha\sec(\theta - n)$ ...(13.49)

Substituting the value of m in equation (*i*), we get

$$\sin \delta = \sin \alpha \;.\; \sec (\theta - n) \;.\; \sin n$$

or

$$\cos (\theta - n) = \sin \alpha \;.\; \sin n \;.\; \text{cosec}\, \delta \qquad \ldots(iv) \quad \ldots(13.50)$$

Thus, the value of n is obtained from equation (*iii*), and then substituted in equation (*iv*) to get the value of θ. For the use of the method of computation of θ, see example 13.60.

(5) LATITUDE BY PRIME VERTICAL TRANSIT

As defined earlier, the prime vertical is a plane at right angles to the meridian, running truly east and west. A star, having polar distance less than 90° and greater than the co-latitude of the place, will cross the prime vertical twice in a sidereal day. The field work, therefore, consists in measuring the time interval between east and west transits of the star. The best stars for observations are those that cross the prime vertical near the zenith.

Thus in Fig. 13.60 (*a*), S, W, N and E are the south, west, north and east points on the horizon. Z is the zenith of the observer, and P the pole. The dotted circle shows the path of a circumpolar star, WZE is the plane of the prime vertical passing through the west-east points and hence perpendicular to the meridian at Z. M_1 and M_2 are the east and west transits of the star across the prime vertical. Half the time that elapses between the two transits M_1 and M_2 in sidereal hours represents the angle M_1PZ (H).

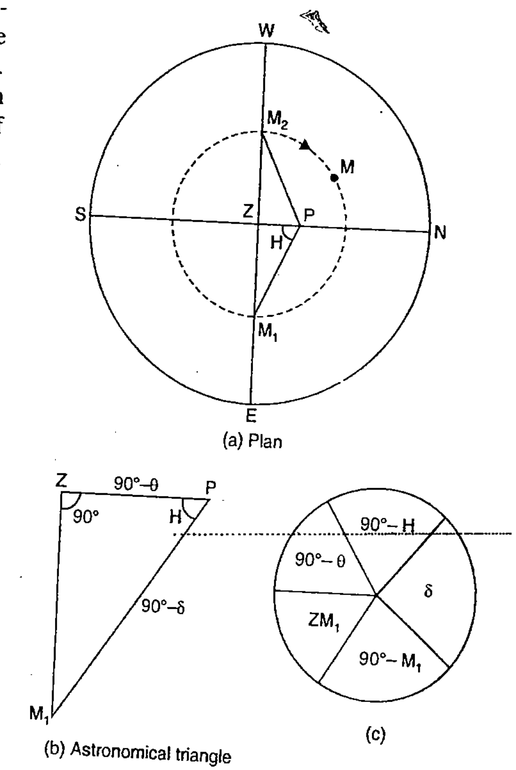


FIG. 13.60. LATITUDE BY PRIME VERTICAL TRANSIT

From the right angled triangle M_1PZ

$$M_1P = 90° - \delta \qquad \text{(known)}$$

$$\angle M_1PZ = H \qquad \text{(known)}$$

$$ZP = (90° - \theta), \qquad \text{to be computed.}$$

From the Napier's rule for the right-angled triangle, [Fig. 13.60 (*c*)].

sine of the middle part = product of tangents of adjacent parts

$$\therefore \quad \sin (90° - H) = \tan (90° - \theta) \tan \delta$$

or

$$\cos H = \cot \theta \tan \delta$$

or $\tan \theta = \tan \delta \;.\; \sec H$

where H = half the interval of time between the east and the west transits expressed in angular measure.

Since the altitude is not measured in this method, the errors due to uncertainty in the value of refraction is largely eliminated. Also, the exact knowledge of local time is not required since we have to simply measure the interval of sidereal hours that elapses between the two transits. However, the approximate local time of prime-vertical transits must be known. To take the time readings, the instrument has to be directed towards the direction of prime vertical, first to the east side and then to the west side, and measure the time when transit occurs, *i.e.*, where the star crosses the vertical cross-hair.

The effect of an Error in the Determination of the Time Interval

Let y = error in the determination of the *time interval*

and x = corresponding error in the latitude.

Then

$$x = y \frac{\sin 2\theta}{2} \sqrt{\frac{\tan^2 \theta}{\tan^2 \delta} - 1} \qquad \ldots(13.51)$$

From the above relationship between the two errors, we draw the following conclusions:

(1) If $\delta = \theta$, x is very small. However, the star would pass through the zenith and observations cannot be made.

(2) If $\delta = 0$, the star would pass through E and W points, the interval between the transits will be exactly 12 hours whatever may be the position of the observer and hence the determination cannot be made. The value of x will be great for very small value of δ.

Hence the stars observed should be as high up on the prime vertical as is consistent with an exact determination of the time of transit.

The effect of an Error in the Direction of Prime Vertical

The error in the setting out of the direction of the prime vertical has very little effect in the latitude of the place for ordinary engineering purposes. If the eastern transit occurs earlier due to the wrong direction of the prime vertical, the western transit will also take place correspondingly earlier, though not exactly by the same amount. In a latitude of 30°, even if the prime vertical is set out by 1° out of its true position, the resulting error in latitude determination will be less then 1″ for observations on a star having declination = 20°.

Striding Level Correction to Prime Vertical Determinations

For the prime vertical determinations, the instrument must be in perfect adjustment. If the transverse axis of the instrument is inclined by a certain value, the resulting error in the determination will be equal to this value. Hence striding level should always be used when taking the vertical observations.

Thus, in Fig. 13.61, if the transverse axis is inclined, ECW is the circle upon which observations are made instead of the true prime vertical EZW. The star is then observed to the transit at the point M on the inclined prime vertical. The observed angle $MPC = H$.

Then $\cot CP = \tan\delta \times \sec H$, if we take $\angle PCM = 90°$.

Then true co-latitude = $CP \pm ZC$

or true latitude = 90° − co-latitude = observed latitude $\mp$ ZC

where ZC = angular measure of the level correction $= \frac{N-S}{2} d$, ...(13.52)

where N = mean reading of north side of bubble.

S = mean reading of south side of bubble.

d = value of one division.

Use − sign if C and P are to the same side of Z and + sign if C and P are to the opposite side of Z.

Thus if the south end of the axis is higher, C and P will be to the same side of Z and the level correction ZC should be subtracted from the calculated value of the latitude to get the true value of the latitude.

However, if the north end of the axis is higher, C and P will be to the opposite sides of Z and the level correction ZC should be added to the calculated value of the latitude to get the true value of the latitude.

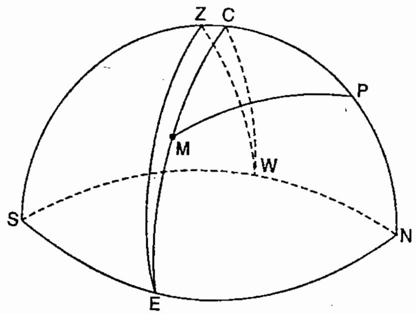


FIG. 13.61

(6) LATITUDE BY DETERMINING THE ALTITUDE OF THE POLE STAR AT ANY TIME

We know that the latitude of a place is equal to the altitude of the pole. If there were any star *at the pole*, we could have observed its altitude. However, the pole star is very near to the pole. The altitude observations on the pole star can, therefore, be made at any known time, and correction can be applied to the observed altitude to get the latitude of the place of observation.

Thus, in Fig. 13.62, M is the position of the pole star at the time of observation. Let α be the observed altitude. The mean time is also observed from the chronometer, and is converted into sidereal time. The hour angle H is then computed from the relation:

L.S.T. = R.A. of pole star + Hour angle.

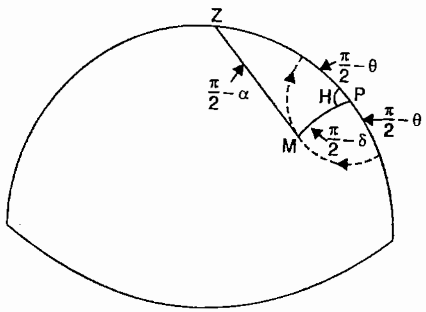


FIG. 13.62.

In the triangle ZPM, $ZM = \frac{\pi}{2} - \alpha$ (known) ; $PM = \left(\frac{\pi}{2} - \delta\right) = p$ (known)

$\angle ZPM = H$ (known)

The co-latitude ZP can be calculated from the cosine formula

$$\cos\left(\frac{\pi}{2} - \alpha\right) = \cos\left(\frac{\pi}{2} - \theta\right)\cos\left(\frac{\pi}{2} - \delta\right) + \sin\left(\frac{\pi}{2} - \theta\right)\sin\left(\frac{\pi}{2} - \delta\right)\cos H$$

or $$\sin\alpha = \sin\theta\sin\delta + \cos\theta\cos\delta\cos H$$

or $$\sin\alpha = \sin\theta\cos p + \cos\theta\cos\delta\cos H \quad ...(1)$$

Let α differ from θ by a small amount x, so that

$\alpha = \theta + x$, where x is the small correction

Substituting $\alpha = \theta + x$ in (1), we get

$$\sin\theta\cos x + \cos\theta\sin x = \sin\theta\cos p + \cos\theta\sin p\cos H$$

Expanding the terms having small quantities x and p, we get

$$\sin\theta\left(1 - \frac{x^2}{2} + ...\right) + \cos\theta\left(x - \frac{x^3}{6} + ...\right) = \sin\theta\left(1 - \frac{p^2}{2}...\right) + \cos\theta\cos H\left(p - \frac{p^3}{6} + ...\right) \quad ...(2)$$

Neglecting the square and higher values of x and p in the above, we get

$$x = p\cos H \quad ...(i)\ (13.53)$$

This gives the values of x to the first approximation.

Next, retaining the squares of x and p, and neglecting their higher powers in equation (2), we get

$$x\cos\theta = p\cos\theta\cos H - \frac{p^2}{2}\sin\theta + \frac{x^2}{2}\sin\theta$$

Putting $x^2 = p^2\cos^2 H$, we get $x = p\cos H - \frac{p^2}{2}\tan\theta\sin^2 H$ ...(ii) (13.54)

This gives the value of x to the second approximation.

The second term in this expression is very small, and becomes still small when multiplied by p^2. Hence we can approximately write $\tan\theta = \tan\alpha$ so that

$$x = p\cos H - \tfrac{1}{2}p^2 . \tan\alpha . \sin^2 H$$

where x and p are in circular measure.

If, however, x and p are measured in seconds, we get

$$x = p\cos H - \tfrac{1}{2}p^2\tan\alpha . \sin^2 H . \sin 1''$$

The correct latitude is, therefore, given by

$$\theta = \alpha - x$$

or $$\theta = \alpha - p\cos H + \tfrac{1}{2}p^2\tan\alpha . \sin^2 H . \sin 1'' \quad ...(13.55)$$

The above formula gives accurate results within 1″.

The field observations consist in observing four altitudes in quick succession – first with face right, two with face left and then again with face right – and the chronometer time of all the four determinations are observed. The mean values of the four altitudes and the four times are taken for the computation of θ. The declination and R.A. of the pole star are taken from the *nautical almanac*.

(7) LATITUDE BY CIRCUM-MERIDIAN ALTITUDE OF STAR OR THE SUN

The *circum-meridian observations* are the observations of stars or the sun taken near to the meridian. The method is used for very accurate determination of latitude by observing the circum-meridian altitudes at noted times of each of the several stars for a few minutes before and after transit and reducing them to the meridian altitude. The errors due to erroneous value of refraction, personal error and those due to instruments are very much reduced by observing an equal number of north and south stars in pairs of similar altitude. Accurate chronometer time and its error is also essential to calculate the hour angle of the individual stars. The observation of each star is commenced about 10^m before the computed time of transit and is continued for about 10^m after transit. Equal number of the face right and face left observations are necessary on a particular star. However, both face observations are not taken if observations are adequately paired on north and south stars.

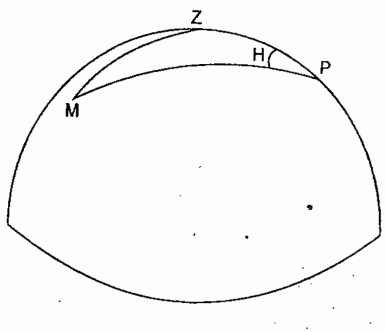


FIG. 13.63

In Fig. 13.63, let

$z = MZ$ = zenith distance of star M, corrected for refraction

$p = MP$ = polar distance

$c = PZ$ = co-latitude

$H = \angle MPZ$ = Hour angle

From the astronomical triangle MPZ, we get

$$\cos z = \cos c \cos p + \sin c \cdot \sin p \cos H \quad \text{...(1)}$$

Let x = correction to be applied to the observed z to get the meridian zenith distance when the star is on meridian.

Then meridian zenith distance $= z - x$.

Again, when the star is on the meridian, its zenith distance

$$= MZ = MP - ZP = p - c.$$

Hence
$$z - x = p - c \quad \text{...(2)}$$

Writing $\cos H = 1 - 2 \sin^2 \frac{H}{2}$ in (1), we get

$$\cos z = \cos c \cos p + \sin c \sin p - 2 \sin c \cdot \sin p \cdot \sin^2 \frac{H}{2}$$

or
$$\cos z = \cos (c - p) - 2 \sin c \cdot \sin p \cdot \sin^2 \frac{H}{2}$$

or
$$\cos z - \cos (p - c) = - 2 \sin c \cdot \sin p \cdot \sin^2 \frac{H}{2}$$

Substituting $p - c = z - x$ from (2), we get

$$\cos z - \cos (z - x) = - 2 \sin c \cdot \sin p \sin^2 \frac{H}{2}$$

$$\therefore \quad \sin \frac{x}{2} \sin \left(z - \frac{x}{2} \right) = \sin c \cdot \sin p \sin^2 \frac{H}{2}$$

From which
$$\sin \frac{x}{2} = \frac{\sin c \cdot \sin p \cdot \sin^2 \frac{H}{2}}{\sin \left(z - \frac{x}{2} \right)}$$

Since x is small, we can replace $\sin \frac{x}{2}$ by $\frac{x}{2} \sin 1''$, if x is measured in seconds of arc.

$$\therefore \quad x = \frac{\sin c \cdot \sin p}{\sin \left(z - \frac{x}{2} \right)} \cdot \frac{2 \sin^2 \frac{H}{2}}{\sin 1''}$$

Also, putting $\sin \left(z - \frac{x}{2} \right) = \sin (z - x) = \sin (p - c)$ (approximately)

we get
$$x = \frac{\sin c \cdot \sin p}{\sin (p - c)} \cdot \frac{2 \sin^2 \frac{H}{2}}{\sin 1''} \quad \text{...(3)}$$

But $\sin c = \cos \theta$; $\sin p = \cos \delta$

and $\sin (p - c) = \sin$ (meridian zenith distance) $= \cos$ (meridian altitude) $= \cos h$

where h = meridian altitude.

Then equation (3) reduces to

$$x = \frac{\cos \theta \cos \delta}{\cos h} \cdot \frac{2 \sin^2 \frac{H}{2}}{\sin 1''} \quad \text{...(4) (13.56)}$$

But $h = \alpha + x$, where α = observed altitude

Hence
$$h = \alpha + \frac{\cos \theta \cos \delta}{\cos h} \cdot \frac{2 \sin^2 \frac{H}{2}}{\sin 1''} \quad \text{...(6)}$$

or
$$h = \alpha + Bm \quad \text{...(13.57)}$$

where
$$B = \frac{\cos \theta \cdot \cos \delta}{\cos h} \quad \text{...[13.57 (a)]} \quad \text{and} \quad m = \frac{2 \sin^2 \frac{H}{2}}{\sin 1''} \quad \text{...[13.57 (b)]}$$

(*H is in arc measure*)

The factor m is usually taken from the tables.

If a series of observations are made upon the same star, the factor B is the same for each observation.

In the factor B, the value θ to be used is the approximate value deduced from the map or determined from the meridian observations. Similarly, h is the meridian altitude computed from the approximate latitude and the known declination of the star.

Let $\alpha_1, \alpha_2, \alpha_3 \ldots$ = circum-meridian altitudes of the same star

$m_1, m_2, m_3 \ldots$ = corresponding value of m.

Then $h_1 = \alpha_1 + Bm_1$; $h_2 = \alpha_2 + Bm_2$; $h_3 = \alpha_3 + Bm_3$, etc. etc.

Hence $h_0 = \alpha_0 + Bm_0$

where h_0 = mean of the deduced meridian altitudes
α_0 = mean of the actual observed altitudes
m_0 = mean of the computed factors m

Thus the meridian altitude of the star is known.

More exact formula

A more elaborate formula for getting the meridian altitude from the observed circum-meridian is as follows :

$$h = \alpha + Bm + Cm' \quad ...(13.58)$$

where $C = B^2 \tan h$ ...[13.58 (a)] and $m' = \frac{2 \sin^4 \frac{H}{2}}{\sin 1''}$ ...[13.58 (b)]

The term $C.m'$ is never more than 1″.

Knowing the meridian altitude (h), the latitude θ can be calculated by the formula developed in method 1 of determination of latitude.

If special tables are not available, m can be calculated as follows :

$$m = \frac{2 \sin^2 \frac{H}{2}}{\sin 1''}; \text{ (where } H \text{ is arc measure) ;}$$

But 1 sec. time (H) = 15″ arc

$$\therefore \quad m = \frac{2 \sin^2 \frac{1}{2} (15 H)}{\sin 1''} \quad \textit{(where H is in seconds of time)} \quad ...[13.59 (a)]$$

$$= \frac{225}{2} H^2 . \frac{(\sin 1'')^2}{\sin 1''} = \frac{225}{2} H^2 . \frac{1}{206265}$$

$$= \frac{H^2}{1834}, \; H \textit{ being in seconds of time.} \quad ...(13.59)$$

Example 13.55. *The meridian altitude of a star was observed to be 65° 40′ 18″ on a certain day, the star lying between the pole and the zenith. The declination of the star was 53° 12′ 10″ N. Find the altitude of the place of observation.*

Solution. (Fig. 13.55)

M_3 is the position of the star under observation. Let us first correct the altitude of the star for refraction.

Correction for refraction = 57″ cot 65° 40′ 18″ = 25″.78

∴ True altitude = observed altitude − refraction

= 65° 40′ 18″ − 25″.78 = 65° 39′ 42″.22

∴ zenith distance z_3 = 90° − 65° 39′ 42″.22 = 24°20′17″.78

Now latitude = declination − zenith distance

$= \delta_3 - z_3$ = 53° 12′ 10″ − 24° 20′ 17″.48 = **28° 51′ 52″.22 N.**

Example 13.56. *The meridian altitude of a star was observed to be 64° 36′ 20″ on a certain day, the star lying between the zenith and the equator. The declination of the star was 26° 12′ 10″ N. Find the latitude of the place of observation.*

Solution. (Fig. 13.55)

M_2 is the position of the star under observation. Let us first correct altitude of the star for refraction.

Refraction correction = 57″ cot 64° 36′ 20″ = 27″.06

True altitude = observed altitude − refraction

= 64° 36′ 20″ − 27″.06 = 64° 35′ 52″.94

∴ Zenith distance = z_3 = 90° − 64° 35′ 52″.94 = 25° 24′ 7″.06

∴ Latitude = $\delta_2 + z_2$ = 26° 12′ 10″ + 25° 24′ 7″.06 = **51° 36′ 17″.06 N.**

Example 13.57. *An observation for altitude was made at a place in longitude 75° 20′ 15″ W. The meridian altitude of the sun's lower limb was observed to be 44° 12′ 30″, the sun being to the south of the zenith. Sun's declination at G.A.N. on the day of observation was + 22° 18′ 12″.8, increasing 6″.82 per hour, and semi diameter 15′ 45″.86. Find the latitude of the place of observation.*

Solution. (Fig. 13.56)

In Fig. 13.56, M is the position of the sun, to the south of zenith.

The latitude of the place = corrected declination + corrected zenith distance.

Let us first correct the observed altitude for refraction, parallax and semi-diameter.

(i) correction for refraction = − 57″ cot 44° 12′ 30″ = − 59″.6

(ii) correction for parallax = + 8″.78 cos 44° 12′ 30″ = + 6″.29

(iii) correction for semi-diameter = + 15′ 45″.86. The correction is additive since the sun's lower limb was observed.

Now observed altitude of sun = 44° 12′ 30″
~~Add parallax~~ correction = 06″.29
Add semi-diameter = 15′ 45″.86

= 44° 28′ 22″.15

Subtract refraction correction = 59″.60

Correct altitude = 44° 27′ 22″.55

∴ Zenith distance z = 90° − 44° 27′ 22″.55 = 45° 32′ 37″.45 ...(1)

Now when the sun is over the meridian, the L.A.N. is zero.

Longitude = 75° 20′ 15″ W $\quad 5^h\,1^m\,21^s$ west

L.A.T. of observation = $0^h\,0^m\,0^s$

Add west longitude = $5^h\,1^m\,21^s$

∴ G.A.T. of observation = $5^h\,1^m\,21^s$

Declination of sun at G.A.N. = 22° 18′ 12″.8

Add increase = (6″.82 × 5.022) = 34″.25

∴ Declination of sun at L.A.N. = 22° 18′ 47″.05 ...(2)

Since the sun is to the south of the latitude,

$$\theta = \delta + z = 22° 18' 47''.05 + 45° 32' 37''.45 = 67° \mathbf{51' 25''.5}.$$

Example 13.58. *A star of declination 46° 45′ 33″ (south) is to be observed at lower and upper transit at a place in approximate latitude 80° south. Find the approximate apparent altitudes at which the star should be sighted in order that accurate observations may be made upon it.*

Solution

In Fig. 13.64, P' is the south pole and Z is the zenith of the observer. EO is the equator, and NS is the horizon, N and S being north and south points on it. M_1 is the position of the star at its upper transit and M_2 is the position at lower transit.

α_1 = apparent altitude at upper transit (north)

α_2 = apparent altitude at lower transit (south)

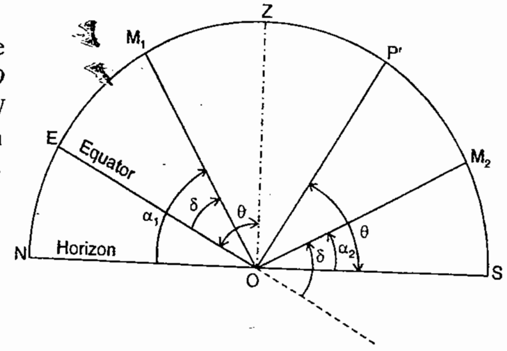


FIG. 13.64

Now $\alpha_1 = NOM_1 = NOZ - M_1OZ = 90° - (EOZ - EOM_1)$

$= 90° - (\theta - \delta) = 90° - \theta + \delta = 90° - 80° + 46° 45' 30'' = \mathbf{56° 45' 30'' N}.$

Similarly, $\alpha_2 = SOM_2 = P'OS - P'OM_2 = \theta - (90° - \delta) = \theta - 90° + \delta$

$= 80° - 90° + 46° 45' 30'' = \mathbf{36° 45' 30'' S}.$

Example 13.59. *The following data relate to an observation of latitude by zenith pair. Calculate the latitude.*

Star	*Declination*	*Observed altitude at transit*
M_1	20° 25′ 48″ S	48° 18′ 12″ N
M_2	79° 30′ 52″ S	47° 54′ 6″ S

Solution.

In Fig. 13.64, M_1 and M_2 denote the two stars ; P' is the south pole.

From the observations to star M_1 :

Latitude $= \theta = EOZ = NOZ - NOE = 90° - (NOM_1 - EOM_1) = 90° - \alpha_1 + \delta_1$ ...(1)

where α_1 = altitude of star M_1 and δ_1 = declination of the star M_1

From the observations to star M_2 :

Latitude $\theta = P'OS = 90° - (\delta_2 - \alpha_2) = 90° - \delta_2 + \alpha_2$ ...(2)

where α_2 = altitude of star M_2 and δ_2 = declination of star M_2

Hence average latitude $= \frac{1}{2}\left[(90° - \alpha_1 + \delta_1) + (90° - \delta_2 + \alpha_2)\right] = 90° - \frac{\alpha_1 - \alpha_2}{2} + \frac{\delta_1 - \delta_2}{2}$

In the above expression, α_1 and α_2 are the observed altitudes. These two altitudes are not exactly equal, and hence there will be little difference in the refraction correction for the two altitudes.

Taking into account the refraction correction, we have

α_1' (corrected) $= \alpha_1 - r_1$; and α_2' (corrected) $= \alpha_2 - r_2$

where r_1 and r_2 are the refraction corrections.

Hence average latitude $= 90° - \frac{\alpha_1 - \alpha_2}{2} + \frac{\delta_1 - \delta_2}{2} + \frac{r_1 - r_2}{2}$

Here $r_1 = 58'' \cot \alpha_1 = 58'' \cot 48° 18' 12'' = 51''.68$

$r_2 = 58'' \cot \alpha_2 = 58'' \cot 47° 54' 6'' = 52''.41$

Substituting the values, we get

$$\text{Average latitude} = 90° - \frac{48° 18' 12'' - 47° 54' 6''}{2} + \frac{20° 25' 48'' - 79° 30' 52''}{2} + \frac{51''.68 - 52''.41}{2}$$

$= 90° - 24' 6'' - 59° 5' 4'' - 0''.36 = \mathbf{30° 30' 49''.64}$

It will be seen here that if the effect of refraction is *assumed* to be cancelled, the latitude will be 30° 30′ 50″. The effect of refraction is thus extremely small, and may be almost neglected if latitude is required to an accuracy of nearest 1″.

Example 13.60. *The altitudes of a star were observed at its upper and lower culmination at a place in north latitude and corrected for refraction. The values obtained are as follows:*

Star : α Aldebaran

Altitude at lower culmination = 18° 36′ 40″

Altitude at upper culmination = 59° 48′ 20″

Find the latitude of the place and the declination of the star.

Solution. (Fig. 13.58)

The latitude $\theta = \frac{\alpha_1 + \alpha_2}{2} = \frac{18° 36' 40'' + 59° 48' 20''}{2} = 39° 12' 30''$

Declination of the star $= EA = EZ + ZA = EZ + (ZN - AN) = \theta + (90° - \alpha_1)$

$= 90° + \theta - \alpha_1 = 90° + 39° 12' 30'' - 59° 48' 20'' = \mathbf{69° 24' 10''}.$

Check : Declination $= EA = EP - AP = 90° - AP = 90° - BP = 90° - (\theta - \alpha_2)$

$= 90° - 39° 12' 30'' + 18° 36' 40'' = \mathbf{69° 24' 10''}.$

Example 13.61. *A star was observed for latitude determination, and its corrected altitude is 40° 36′ 30″. The declination of the star is 10° 36′ 40″ and hour angle is 46° 36′ 20″. Compute the latitude of the place of observation.*

Solution. (Fig. 13.59). The latitude of the place is computed from the formula

$$\sin \alpha = \sin \theta \sin \delta + \cos \theta \cdot \cos \delta \cdot \cos H \quad ...(1)$$

To solve this equation for θ, let $\sin \delta = m \sin n$ and $\cos \delta \cos H = m \cos n$.

Then, by reduction, the value of n is given by

$$\tan n = \tan \delta \sec H = \tan 10° 36' 40'' \sec 46° 36' 20''$$

or $n = 15° 15' 12''$

Then, the value of θ is given by

$$\cos(\theta - n) = \sin\alpha \cdot \sin n \cdot \text{cosec}\,\delta = \sin 40°\,36'\,30'' \cdot \sin 15°\,15'\,12''\ \text{cosec}\ 10°\,36'\,40''$$

$\therefore$ $\theta - n = 21°\,36'\,33''$

or $\theta = n + 21°\,36'\,33'' = 15°\,15'\,12'' + 21°\,36'\,33'' = \mathbf{36°\,51'\,45''}$.

Example 13.62. *Find the latitude of the place from the following data :*

Longitude of the place, 108° 30′ W

Altitude of sun's upper limb, 42° 12′ 40″

L.M.T. of observation $2^h\,50^m$ *P. M.*

Date of observation : Dec. 15, 1947

Sun's declination at 0 hour on Dec. 15, 1947 : 23° 12′ 18″.6 (South) increasing at 10″.6 per hour.

Equation of time at 0^h *on Dec. 15* $= +6^m\,18.5^s$, *decreasing at* 1.2^s *per hour.*

Sun's semi-diameter = 15′ 16″.4

Solution

(a) *Calculation of true altitude*

Correction for refraction $= 57''\cot\alpha = 57''\cot 42°\,12'\,40'' = 62''.84$ (subtractive)

Correction for parallax $= 8''.77\cos\alpha = 8''.77\cos 42°\,12'\,40'' = 6''.50$ (additive)

Correction for semi-diameter $= 15'\,16''.4$ (subtractive)

Net correction $= -62''.84 + 6''.50 - 15'\,16''.4 = -16'\,12''.74$

$\therefore$ True altitude $= 42°\,12'\,40'' - 16'\,12''.74 = 41°\,55'\,27''.26$.

(b) *Calculation of hour angle*

Longitude $= 108°\,30'\,W$ $= 7^h\,14^m\,W$

L.M.T. of observation $= 14^h\,50^m$ P.M.

$\therefore$ G.M.T. of observation $= 22^h\,04^m$

E.T. at $0^h = +6^m\,18.5^s$

Decrease at 1.2^s per hour for $22^h\,04^m = (1.2 \times 22^h\,4^m) = 26.48^s$

Now interval since L.M.N. = L.M.T. $- 12^h = 14^h\,50^m - 12^h$

$= 2^h\,50^m$

Add E.T. $=$ 26.48^s

$\therefore$ Interval since L.A.N. $= 2^h\,50^m\,26.48^s$

Hence hour angle (H) = interval since L.A.N. $= 2^h\,50^m\,26.48^s = 42°\,36'\,37''.20$

(c) *Calculation of declination*

G.M.T. of observation $= 22^h\,04^m$

Declination of sun at $0^h = 23°\,12'\,18''.6\,S$

Increase at $10''.6$ per hour for $22^h\,04^m = (10''.6 \times 22^h\,04^m) = 233.91^s = 3'\,53''.91$

$\therefore$ Sun's declination at the time of observation

$= 23°\,12'\,18''.6 + 3'\,53''.91 = 23°\,16'\,12''.51$ (south).

(d) *Calculation of the latitude*

The latitude can be calculated from the following formula :

$$\tan\frac{ZP}{2} = \frac{\sin\frac{1}{2}(A+H)}{\sin\frac{1}{2}(A-H)} \cdot \tan\frac{1}{2}(PM - ZM) \qquad \ldots(1)$$

Let us first calculate the value of the azimuth (A) of the sun.

In the astronomical triangle ZPM, we have

ZM = co-altitude $= 90° - 41°\,55'\,27''.26 = 48°\,4'\,32''.74$

PM = co-declination $= 90° + 23°\,16'\,12''.51 = 113°\,16'\,12''.51$

$\angle ZPM = H = 42°\,36'\,37''.2$

Using the sine rule, we get

$$\sin PZM = \frac{\sin PM}{\sin ZM} \cdot \sin ZPM = \frac{\sin 113°\,16'\,12''.51}{\sin 48°\,4'\,32''.74} \times \sin 42°\,36'\,37''.2$$

$\therefore$ $PZM = A = 123°\,42'\,36''$

$\therefore$ $\frac{A+H}{2} = \frac{1}{2}(123°\,42'\,36'' + 42°\,36'\,37''.20) = 83°\,9'\,36''.6$

$\frac{A-H}{2} = \frac{1}{2}(123°\,42'\,36'' - 42°\,36'\,37''.20) = 40°\,32'\,59''.4$

$\frac{PM - ZM}{2} = \frac{1}{2}(113°\,16'\,12''.51 - 48°\,4'\,32''.74) = 32°\,35'\,49''.9$

Substituting these values in Equ. 1 above, we get

$$\tan\frac{ZP}{2} = \frac{\sin 83°\,9'\,36''.6}{\sin 40°\,32'\,59''.4} \cdot \tan 32°\,35'\,49''.9$$

$\therefore$ $\frac{ZP}{2} = 44°\,20'\,29''.4$

or $ZP = 88°\,40'\,58''.8$ = co-latitude

$\therefore$ Latitude of the place $= 90° - 88°\,40'\,58''.8 = \mathbf{1°\,19'\,1''.2}$.

Example 13.63. *Observations on a star α-aldebaran were made at a place in N-latitude for determining the latitude of the place by prime vertical transit. The following is the record obtained :*

Interval between the passage of α-aldebaran across prime vertical = $9^h\,22^m\,6^s$ *mean time.*

Mean readings of the bubble on striding level = 11^S *and* 16^N

Value of each division = 16″

Declination of the star = 15° 20′ 48″ N

Determine the latitude of the place of observation.

Solution. When the observations are made on a star at its prime vertical transit, the latitude (Fig. 13.60) is given by

$$\tan\theta = \tan\delta \,.\, \sec H \qquad \ldots(1)$$

Let us first calculate the hour angle (H) of the star at its prime vertical transit.

Interval between the passage across prime vertical $= 9^h\ 22^m\ 6^s$ meantime.

To convert it into sidereal time interval add acceleration at the rate of 9.8565 seconds per hour of meantime.

$9^h \times 9.8565 = 88.71$ seconds

$22^m \times 0.1642 = 3.61$ seconds

$6^s \times 0.0027 = 0.02$ second

Total acceleration $= 92.34$ seconds $= 1^m\ 32.34^s$

$\therefore$ Sidereal time interval $= 9^h\ 22^m\ 6^s + 1^m\ 32.34^s$

$= 9^h\ 23^m\ 38.34^s = 140°\ 54'\ 35''.1$

$\therefore$ H = half the time interval $= 70°\ 27'\ 17''.55$

Hence $\tan\theta = \tan 15°\ 20'\ 48'' \sec 70°\ 27'\ 17''.55$

$\therefore$ $\theta = 39°\ 20'\ 25''.6$

Since the trunnion axis is inclined, let us correct the value,

Error due to striding level $= \dfrac{N-S}{2} \times d = \dfrac{16-11}{6} \times 16 = 40''$

As the north end of the axis is higher, the correction is additive.

Hence correct $\theta = 39°\ 20'\ 25''.6 + 40'' =$ **$39°\ 21'\ 5''.6$**.

Example 13.64. *In longitude 7° 20′ W, an observation for latitude was made on Polaris on a certain day. The mean of the observed latitude was 48° 36′ 40″ and the average of the local mean times, $20^h\ 24^m\ 50^s$. The readings of the barometer and thermometer were 30.42 inches and 58° F respectively. Find the latitude, given the following:*

R.A. of Polaris = $1^h\ 41^m\ 48.64^s$

Declination of Polaris = 88° 58′ 28″.26

G.S.T. of G.M.M. = $16^h\ 48^m\ 20.86^s$.

Solution

(a) *Calculation of polar distance.*

From Chamber's Mathematical Tables (page 431)

Mean refraction for 48° 36′ 40″	= 51″
Correction for 58°F temp.	= − 1″
Correction for barometer	= + 1″
∴ Refraction correction	= 51″ (subtractive)

$\therefore$ True altitude = observed altitude − refraction = 48° 36′ 40″ − 51″ = 48° 35′ 49″.

(b) *Calculation of hour angle (H).*

The hour angle can be calculated by subtracting the R.A. from L.S.T.

Longitude $= 7°\ 20'\ W = 0^h\ 29^m\ 20^s\ W$

Acceleration at the rate of 9.8565 seconds per hour of longitude :

$29^m \times 0.1642 = 4.76$ seconds

$20^s \times 0.0027 = 0.05$ seconds

$\therefore$ Acceleration $= 4.81$ seconds

$\therefore$ L.S.T. of L.M.M. = G.S.T. of G.M.M. + acceleration

$= 16^h\ 48^m\ 20.86^s + 4.81^s = 16^h\ 48^m\ 25.67^s$

L.M.T. of observation $= 20^h\ 24^m\ 50^s$

To convert it into sidereal interval, add acceleration at the rate of 9.8565 seconds per mean hour.

$20^h \times 9.8565 = 197.13$ seconds

$24^m \times 0.1642 = 3.94$ seconds

$50^s \times 0.0027 = 0.14$ second

Total acceleration $= 201.21$ seconds $= 3^m\ 21.21^s$

Sidereal interval since L.M.M. = Meantime interval + acceleration.

$= 20^h\ 24^m\ 50^s + 3^m\ 21.21^s$.

$= 20^h\ 28^m\ 11.21^s$

Add L.S.T. of L.M.M. $= 16^h\ 48^m\ 25.67^s$

$\therefore$ L.S.T. $= 37^h\ 16^m\ 36.88^s - 24^h$

$= 13^h\ 16^m\ 36.88^s$

Deduct R.A. of Polaris $= 1^h\ 41^m\ 48.64^s$

$\therefore$ Hour angle $(H) = 11^h\ 34^m\ 46.24^s = 173°\ 42'\ 3''.6$.

Now, latitude $\theta = \alpha - p\cos H + \frac{1}{2}\sin 1''\, p^2 \sin^2 H \,.\, \tan\alpha$.

p = polar distance $= 90° - 88°\ 58'\ 28''.26 = 1°\ 1'\ 31''.74 = 3691''.74$

First correction $= p\cos H = 3691''.74 \cos 173°\ 42'\ 3''.6 = -3669''.5 = -1°\ 1'\ 9''.5$

$\therefore$ Second correction $= \frac{1}{2}\sin 1''\, p^2 \sin^2 H \,.\, \tan\alpha$

$= \frac{1}{2} \times \dfrac{1}{206265}(3691.74)^2 \sin^2(173°\ 42'\ 3''.6) \tan 48°\ 35'\ 49'' = +0''.5$.

(*Note.* The above calculations for first and second corrections may be done with a five figure log-table if the answer is required to the nearest 1″.)

Hence $\theta = 48°\ 35'\ 49'' - (-1°\ 1'\ 9''.5) + 0''.5 =$ **$49°\ 36'\ 59''$ N.**

Example 13.65. *The latitude of a station 4° 20′ E of the 120° W meridian was determined by reducing an observation of β Aquilae to meridian, the true altitude of the star being 39° 20′ 30″ and the approximate latitude of the station 56° 54′ 30″ N.*

The time of the observation, $10^h 55^m 30^s$ was taken with a mean time chronometer, which was $1^m 25^s$ fast on the standard time of the 120° meridian. The R.A. and declination of the star were respectively $19^h 52^m 16^s$ and 6° 15′ 02″ N, G.S.T. at G.M.N. being $8^h 30^m 20^s$.

Determine the exact latitude by applying the circum-meridian correction to the observed latitude.

Solution. The meridian altitude h is given by $h = \alpha + Bm$

where $$B = \frac{\cos\theta \,.\, \cos\delta}{\cos h} \approx \frac{\cos\theta \,.\, \cos\delta}{\cos\alpha}$$

and $$m = \frac{2\sin^2\frac{H}{2}}{\sin 1''},$$ where H is in arc measure.

Let us first calculate the hour angle.

G.S.T. of G.M.N. = $8^h 30^m 20^s$

Longitude = 4° 20′ E of 120° W meridian = 115° 40′ W = $7^h 42^m 40^s$

∴ Acceleration for $7^h 42^m 40^s$ at 9.8565 sec. per hour = 1′ 16″

∴ L.S.T. of L.M.N. = G.S.T. of G.M.N. + acceleration

$= 8^h 30^m 20^s + 1^m 16^s = 8^h 31^m 36^s$

L.S.T. = R.A. = $19^h 52^m 16^s$

∴ S.I. after L.M.N. = L.S.T. – L.S.T. of L.M.N.

$= 19^h 52^m 16^s - 8^h 31^m 36^s = 11^h 20^m 40^s$

To convert it to mean time interval, subtract the retardation at the rate of 9.8296 per sidereal hour.

∴ Retardation = $(9.8296^s)(11^h 20^m 40^s) = 1^m 51.95^s$

∴ M.T. interval after L.M.N. = S.I. after L.M.N. – retardation.

$= 11^h 20^m 40^s - 1^m 51.95^s = 11^h 18^m 48.05^s$

Observed standard mean time = $10^h 55^m 30^s$

Chronometer correction = – $1^m 25^s$

∴ Corrected standard mean time = $10^h 45^m 05^s$

Correction for 4° 20′ Longitude (E) = + $17^m 20^s$

∴ L.M.T. of observation = $11^h 11^m 25^s$

∴ Mean time interval before transit = M.T. interval after L.M.N. – L.M.T.

$= 11^h 18^m 48.05^s - 11^h 11^m 25^s = 0^h 7^m 23.05^s$

Acceleration for $7^m 23.05^s$ of meantime = 1.21^s

∴ S.I. before transit = M.T. interval before transit + acceleration

$= 7^m 23.05^s + 1.21^s = 7^m 24.26^s = 444.3^s$ ...(*i*)

Now $$m = \frac{2\sin^2\frac{1}{2}H}{\sin 1''} \quad \text{(where 1 sec. time } H = 15'' \text{ arc.)}$$

$$= \frac{2\sin^2\frac{1}{2}(15\,H)}{\sin 1''} = \frac{2\left(\frac{15}{2}\right)^2 H^2 (\sin 1'')^2}{\sin 1''}$$

$$= \frac{225\,H^2}{2 \times 206265} = \frac{H^2}{1834}, \text{ with } H \text{ in seconds} = \frac{(444.3)^2}{1834} = 107''.6$$

$$B = \frac{\cos\delta\cos\theta}{\cos\alpha} = \frac{\cos 6°\,15'\,02''\,.\,\cos 56°\,54'\,30''}{\cos 39°\,20'\,30''}$$

Hence $$mB = \frac{\cos 6°\,15'\,02''\,.\,\cos 56°\,54'\,30''}{\cos 39°\,20'\,30''} = 444.3^s = 1'\,15''.51$$

Hence correct meridian altitude = $h = \alpha + mB$ = 39° 20′ 30″ + 1′ 15″.51 = 39° 21′ 45″.51

and θ = 90° – h + δ = 90° – 39° 21′ 45″.51 + 6° 15′ 02″ = **56° 53′ 16″.49 N.**

13.18. DETERMINATION OF LONGITUDE

Since the difference in longitudes between two places is equal to the difference in their local times, the longitude of a place can be determined by determining the local time (mean or sidereal) at the place and subtracting it from the Greenwich time (mean or sidereal) at the same instant. The local time can be determined by any of the methods discussed earlier. However, the finding of the Greenwich time at the instant of observations is the main important part of the longitude determination. If the local time is *greater* than the Greenwich time (or the standard time), the place is to the *east* of Greenwich meridian (or the standard meridian). Similarly, if the local time is *lesser* than the Greenwich time (or the standard time), the place is to the *west* of Greenwich meridian. The various methods of determining the longitude are :

(1) By transportation of chronometers.
(2) By electric telegraph.
(3) By wireless time signals.
(4) By observing the moon and the stars which culminate at the same time.
(5) By celestial signals.
(6) By lunar distances.

Methods (4) to (6) are only of historical interest and will not be discussed here.

(1) LONGITUDE BY TRANSPORTATION OF CHRONOMETERS

In this method, the chronometer time is noted at the instant of making the observations for the local time. The chronometer reading is then corrected for its time and rate. For this, the chronometer should be previously compared with Greenwich time and its error and rate should be known. Thus, at the instant of the celestial observations we know the correct Greenwich time. Comparing the calculated local time with that of the chronometer time, we can find the longitude of the place of observation.

Chronometer is a very delicate instrument. The main difficulty arises from the fact that its rate while being transported, and while it is stationary is not the same. Hence the *travelling rate* of the chronometer should also be ascertained for precise determination. Suppose it is required to determine the difference in longitude between two stations A and B, the chronometer being regulated to give the time of station A. The *'rate'* of the chronometer *i.e.*, the amount by which it gains or loses in 24 hours is found at A. The chronometer is then transported to the station B of unknown longitude and its error is determined with reference to this meridian. *If the chronometer runs perfectly, the two watch corrections will differ by just the difference in longitude.*

The method is now not used by surveyors except where wireless or telegraphic communication are not available. However, it is still used for the determination of longitude at sea.

(2) LONGITUDE BY ELECTRIC TELEGRAPH

If the two places are connected by an electric telegraph, the longitude can be determined very accurately by sending telegraphic signals in opposite directions for the chronometer times (local). Let A and B be the stations, A being to the east of B.

Let t_1 = local time of A at which the signal is sent from A to B.

and t_2 = local time of B at which the signal is received at B.

If the transmission time is neglected, the difference in longitude (ϕ) is given by $\phi = t_1 - t_2$, t_1 being greater than t_2.

If, however, s is the time of transmission, $(t_1 + s)$ is the actual local time of A corresponding to the local time t_2 at B. Hence the difference in longitude is

$$\phi = (t_1 + s) - t_2 = (t_1 - t_2) + s \quad ...(1)$$

Similarly, let a signal be sent in the reverse direction from B to A.

Let t_2' = local time of B at which the signal is sent from B to A,

t_1' = local time of A at which the signal is received.

If the transmission time is neglected, we get

$$\phi = t_1' - t_2'.$$

If, however, s is the time taken in transmitting the signal $(t_2' + s)$ is the actual local of B corresponding to the local time t_1' of A. Hence the difference in longitude is

$$\phi' = t_1' - (t_2' + s) = t_1' - t_2' - s$$

By averaging the two results, we get

$$\text{Difference in longitude} = \frac{1}{2}\left\{ (t_1 - t_2 + s) + (t_1' - t_2' - s) \right\} = \frac{1}{2}\left\{ (t_1 - t_2) + (t_1' - t_2') \right\}.$$

(3) LONGITUDE BY WIRELESS SIGNALS

The advent of wireless signals has rendered the carrying of the time of the reference meridian comparatively easy and most accurate. Time signals are now sent out from various wireless stations at stated intervals, and the surveyor, by their aid, may check his chronometer in almost any part of the world. A list of wireless signals, their times and durations of emission together with their wave lengths and type of signals, is given in the *Admiralty list of wireless signals,* which is published annually ; and changes or any corrections are notified in the weekly *Notices to Mariners.* Greenwich meantime signals are sent and usually

continue, for a period of five minutes. The signals are rhythmic and consist of a series of 61 Morse dots to the minute, the beginning and end of each minute being denoted by a dash, which is counted as zero of the series which follows.

PROBLEMS

1 At a point A in latitude 50° N, a straight line is ranged out, which runs due east of A. This straight line is prolonged for 60 Nautical miles to B. Find the latitude of B, and if it be desired to travel due North from B so as to meet the 50° parallel again at C, find the angle ABC at which we must set out, and the distance BC. (*U.L.*)

2. The R.A. of a star being $20^h 24^m 13.72^s$, compute the L.M.T. of its culmination at Madras (Long. 80° 14′ 19″.5 E) on Sept. 6, the G.S.T. at 0^h G.M.T. on that date being $22^h 57^m 06.95^s$

3. Find the L.S.T. at a station in longitude 76° 20′ E at 9.30 A.M. (Indian Zone Time) on August 10 on that date at G.M.M. The R.A. of mean sun is $9^h 13^m 30.9^s$. (*G.U.*)

4. From the N.A., it is found that on the date of observation, G.S.T. of G.M.N. is $3^h 14^m 26^s$. Taking retardation as 9.85 sec. per hour of longitude, find the L.M.T. in a place 75° W, when the local sidereal time is $5^h 20^m 0^s$. (*B.U.*)

5. Find the local mean time at which β Leonis made its upper transit on 1st May 1940 at a place 60° E. Given R.A. of β Leonis on 1st May was $11^h 46^m 02^s$ and G.S.T. of G.M.N was $9^h 23^m 23^s$. (*B.U.*)

6. Find the R.A. of the meridian of Bombay at 4.30 P.M. Given : Longitude of Bombay 72° 48′ 46″.8 East ; G.S.T. at G.M.M.= $10^h 10^m 40.73^s$ on that day.
(*Note*: R.A. of a place = L.S.T.)

7. What are the systems of co-ordinates employed to locate position of a heavenly body ? Why it is necessary, to have several systems instead of one ?

8. Explain the systems of time reckoning known as sidereal apparent solar and mean solar time, and show how they differ from each other. (*I.R.S.E.*)

9. What is equation of time ? Show, by means of sketches, that it vanishes four times a year.

10. Explain with aid of sketches how the quantities of the following groups are related to each other:
(*i*) The R.A. of a star, the hour angle of the star at any instant and the sidereal time at that instant.
(*ii*) Equation of time, apparent time and mean time.
Show that the equation of time vanishes four times in a year. (*A.M.I.E.*)

11. (*a*) Explain the following terms :
(*i*) Equation of time , (*ii*) Celestial sphere, (*iii*) Parallax, and (*iv*) Sidereal time.
(*b*) An observation was made on Dec. 30, 1919 in longitude 82° 17′ 30″ E; the meridian altitude of the sun's lower limb was 40° 15′ 13″. The sun was on the south of the observer's zenith. Calculate the approximate latitude of the place. Correction for refraction 1′ 10″ ; for parallax = 6″.9 ; correction for semi-diameter 16′ 17″.5. Declination of star at G.A.N.= 23° 13′ 15″. decreasing at the rate of 9″.17 per hour (*B.U.*)

12. What are 'parallax' and 'refraction' and how do they affect the measurement of vertical angles in astronomical work ?
Give rough values of the corrections necessary when measuring a vertical angle of 45°. (*A.M.I.C.E.*)

13. In longitude 60° W, an observation was made on β Tauri, whose R.A. was $5^h 21^m 59.48^s$. If the hour angle of the star was $9^h 15^m 8^s$, find the local mean time of observation.
Given G.S.T. at G.M.N. = $14^h 46^m 39.53^s$.

14. On a certain date, the right ascension of α-Draconis was $14^h 2^m 5^s$. From the N.A. and the longitude of the place, the local sidereal time of local mean noon was found to be $6^h 35^m 44^s$. The declination of the star was 64° 47′ 33″ *N*. Find the local mean time of east elongation. Assume the latitude of 60° *N*. *(A.M.I.C.E.)*

15. If the time be found by a single altitude, show that a small error in the latitude will have no effect on the time when the body is in the prime vertical.

16. Determine the G.M.T. at which the star α-Aurigae crossed the meridian of a station in longitude 28° 31′ *E* in the northern hemisphere at upper culmination on May 31st 1926, the declination of the star being 45° 55′ 25″ *N*, and its right ascension $5^h 11^m 6^s$ with G.S.T. of G.M.N. $4^h 32^m 55^s$. If the true altitude of the star was 76° 30′ 50″, find also the latitude of the station. *(B.U.)*

17. Draw a diagram to show the celestial sphere for a point 15° *N*, 75° *E*, showing the horizon, meridian, zenith, pole and celestial equator.
Mark also the path of the sun at mid-summer, and the position of α-Bootes (decl. of 20° *N*; R.A. $14^s 10^m$) at 22^h G.S.T. *(U.R.)*

18. An observation of time was made on Aldebaran (α-Tauri) on Oct. 1, 1940 in altitude 52° 12′ 50″ *N*, the mean of two observed altitude being 28° 36′ 20″. The average sidereal time of observing these altitudes was $0^h 15^m 28.4^s$ by the sidereal chronometer.
Find the error of the chronometer given that the star's R.A. and declination were $4^h 32^m 31.1^s$ and 16° 23′ 30″.5 respectively and that the star was east of the meridian.

19. On 7th Feb., a star (R.A. $5^h 9^m 44^s$) is in transit at Sidney (Longitude 155° 12′ 23″ *E*) when the time by the observer's watch which should keep local times is $8^h 0^m 33^s$. Given that the mean sun's R.A. at a mean noon at Greenwich on 7th Feb. is $21^h 8^m 36.1^s$ and that 1 hour of S.T. is equivalent to $59^m 50.2^s$ of meantime, find to the nearest second how much the watch is slow or fast. *(Math. Trip.)*

20. Reduce the following meridian observations for latitude :

Star	Declination	Right Ascension	Observed Altitude	Altitude Level object end	Altitude Level eye end
M_1	60° 02′ 50″ *S*	$13^h 59^m 00^s$	49° 28′ 15″ *S*	5.4	4.6
M_2	19° 32′ 10″ *N*	$14^h 12^m 33^s$	50° 58′ 10″ *N*	5.2	4.8

The value of level division is 14″. Take the refraction correction as −58″ cot altitude. If the longitude is 142° 36′ *E* and the sidereal time of mean noon at Greenwich is $4^h 6^m 17^s$, at what local mean times will the two transits occur ?

21. Your longitude is 75° *E* of Greenwich.
You are required to find the error to the nearest second of a meantime chronometer at mid-night 1st–2nd March.
In order to find this, you have timed the transit of two stars near mid-night as follows :
Transit of α Mali $23^h 32^m 14^s$ } by the chronometer
,, ,, β Gemini $1^h 43^m 52^s$ }
Relevant extracts from the Nautical Almanacs are
R.A. of α Mali $6^h 19^m 01^s$
R.A. of β Gemini $8^h 30^m 56^s$
Sidereal time of Greenwich mean moon 1st March : $18° 45^m 12^s$.

22. Criticise the method of determining azimuths from elongation observations, stating its limitations in high altitudes.

A star α of declination 84° 42′ *N* is observed at eastern elongation when its clockwise angle from a survey line is 118° 20′. Immediately afterwards another star β of declination 72° 24′ *N* is observed at western elongation, its clockwise angle from *OP* being 94° 6′.
Determine the azimuth of the line *OP*. *(U.P.)*

23. At a point in latitude *N* 55° 46′ 12″ the altitude of the sun's centre was found to be 23° 17′ 32″ at $5^h 17^m$ P.M. (Greenwich meantime). The theodolite was first pointed to a reference mark, the vernier reading being 0° 00′ 00″ ; the horizontal angle between the sun's centre and the reference mark at the time of observation was found to be 68° 24′ 30″. Find graphical azimuth of the reference mark from the centre of the instrument.
Data : Sun's declination at Greenwich apparent noon on day of observation ... 17° 46′ 52″ *N*
Variation of declination per hour ... −38″
Refraction for altitude of 30° 20′ ... 2′ 12″
Parallax in altitude ... 0′ 8″
Equation of time (apparent − mean) ... $6^m 6^s$. *(U.L.)*

24. To determine the azimuth of reference object from station *B*. (Lat. 51° 30′ 30″ *N*) of a triangulation survey, the sun was observed at $4^h 30^m 13^s$ P.M. (G.M.T.) after crossing the meridian. The observed altitude of the sun's centre was 38° 28′ 25″ and the horizontal angle measured anticlockwise from R.O. to the sun was 161° 35′ 20″. The apparent declination of the sun at G.M.N. was 20° 5′ 38″.1 *N* increasing 30″.42 per hour. The sun's horizontal parallax may be taken as 8″.7 and the refraction correction −58″ cot α. Calculate the azimuth of R.O. *(I.R S.E.)*

25. A star was observed at Western elongation at a place in lat. 28° 20′ *S* and longitude 124° 24′ *W*, when its clockwise bearing from a survey line was 164°.
Determine the local mean time of elongation, also the azimuth of the line, given that the star's declination was 76° 36′ 55″ *S* and its right ascension $6^h 41^m 52^s$, the G.S.T. of G.M.N. being $5^h 12^m 20^s$. *(U.L.)*

26. An observation of azimuth was made during the early hours of the morning of 1 Jan. 1940, on α Ursae Minoris (Polaris) at elongation at a place of latitudde 45° *N*, and longitude 5° *E*.
The declination of the star on that date was + 88° 59′ 03″ and its R.A. was $1^h 43^m 32^s$.
The mean observed horizontal angle between the star and the R.O. was 42° 37′ 22″, R.O. being to the west of the star.
Find (*a*) which elongation was used ?
(*b*) the exact local mean time of elongation.
(*c*) the azimuth of the R.O.
Given G.S.T. of G.M.T. 0^h on 1 Jan. 1940 was $6^h 38^m 01.9^s$ (U.B.)

27. At a place in longitude 31° 41′ 40″ S, 121° 32′ 30″ E, a star whose R.A. = $0^h 22^m 15.6^s$, declination 77° 37′ 54″ S is observed at eastern alongation when its clockwise horizontal angle from a survey line *ZO* is 110° 14′ 30″. Find the azimuth of the survey line and the local mean time of the elongation, if the mean time of the transit of ϒ at Greenwich is $1^h 20^m 57^s$ from mid-night.

28. To determine the latitude of a place (longitude 37° W) observations were made on Polaris and its corrected altitude was found to be 46° 17′ 28″ when the mean time of observation was $7^h 43^m 35^s$ P.M. Find the latitude of the place, given the following
G.S.T. at G.M.M on the day of observation = $10^h 51^m 31.5^s$
R.A. of Polaris = $1^h 27^m 37.7^s$
Declination of Polaris = + 88° 51′ 08″

29. A meridian altitude of the lower limb of the sun is taken on 5th Nov. 1934 in latitude N, longitude 78° 25′ W. Given the observed altitude = 47° 18′ 44″, parallax = 6″, refraction = 53″.6; declination of the sun at mid-night 4/5 Nov. 1934 = S 15° 24′ 27″.4 with an hourly variation of 46″.23 increasing; semi-dia. 16′ 9″.5.

Calculate the latitude of the observer's station, the equation of time at mid-night 4/5 Nov. 1934 is $+ 16^m 21.5^s$ with an hourly variation $- 0.044^s$. *(A.I.M.E.)*

Answers

1. Latitude of $B = 49° 59' 22''.6$; $\angle ABC = 88° 48' 40''$; $BD = 0.624$ Nautical miles.
2. $21^h 24^m 28.48^s$.
3. $6^h 19^m 30.3^s$.
4. $2^h 4^m 24.31^s$.
5. $2^h 22^m 54.49^s$ P.M..
6. $2^h 42^m 35.51^s$
11. (*b*) N 26° 17′ 7″.91.
12. + 6″ ; – 57″
13. $23^h 45^m 54.29^s$.
14. $5^h 4^m$ P.M. nearly.
16. $10^h 45^m 23.27^s$; 32° 26′ 16″ N.
18. Chronometer slow 2.5^s
19. 56.67^s slow.
20. $\theta = 19° 30' 22''.6$; L.M.T.'s : $9^h 52^m 40^s$ P.M. for M_1 ; $10^h 06^m 10^s$ P.M. for M_2
21 Choronometer slow 28^s
22. 112° 43′ 56″
23. 24° 2′ 8″ from south.
24. 4° 24′ 12″.
25. $7^h 58^s 19.13^s$: 180° 45′ 7″.75 from S point.
26. (*a*) West (*b*) $0^h 57^m 27.6^s$ Jan. 2; 315° 56′ 38″
27. 55° 10′ 40″ ; $8^h 12^m 34.2^s$ P.M.
28 46° 03′ 36″ N.
29. 26° 47′ 56″.7 N.

14

Photogrammetric Surveying

14.1. INTRODUCTION

Photogrammetric surveying or photogrammetry is the science and art of obtaining accurate measurements by use of photographs, for various purposes such as the construction of planimetric and topographic maps, classification of soils, interpretation of geology, acquisition of military intelligence and the preparation of composite pictures of the ground. The photographs are taken either from the air or from station on the ground. *Terrestrial photogrammetry* is that branch of photogrammetry wherein photographs are taken from a fixed position on or near the ground. *Aerial photogrammetry* is that branch of photogrammetry wherein the photographs are taken by a camera mounted in an aircraft flying over the area. Mapping from aerial photographs is the best mapping procedure yet developed for large projects, and are invaluable for military intelligence. The major users of aerial mapping methods are the civilian and military mapping agencies of the Government.

The conception of using photographs for purposes of measurement appears to have originated with the experiments of Aime Laussedat of the Corps of Engineers of the French Army, who in 1851 produced the first measuring camera. He developed the mathematical analysis of photographs as perspective projections, thereby increasing their application to topography. Aerial photography from balloons probably began about 1858. Almost concurrently (1858), but independently of Laussedat, Meydenbauer in Germany carried out the first experiments in making critical measurements of architectural details by the intersection method on the basis of two photographs of the building. The ground photography was perfected in Canada by Capt. Deville, then Surveyor General of Canada in 1888. In Germany, most of the progress on the theoretical side was due to Hauck.

In 1901, Pulfrich in Jena introduced the stereoscopic principle of measurement and designed the *sterecomparator*. The *stereoautograph* was designed (1909) at the Zeiss workshops in Jena, and this opened a wide field of practical application. Scheimpflug, an Australian captain, developed the idea of double projector in 1898. He originated the theory of perspective transformation and incorporated its principles in the *photoperspectograph*. He also gave the idea of radial triangulation. His work paved the way for the development of aerial surveying and aerial photogrammetry.

In 1875, Oscar Messter built the first aerial camera in Germany and J.W. Bagloy and A. Brock produced the first aerial cameras in U.S.A. In 1923, Bauersfeld designed the *Zeiss stereoplanigraph*. The optical industries of Germany, Switzerland, Italy and France, and later also those of the U.S.A. and U.S.S.R. took up the manufacture and constant further development of the cameras and plotting instruments. In World War II, both the sides made extensive use of aerial photographs for their military operations. World War II gave rise to new developments of aerial photography techniques, such as the application of radio control to photoflight navigation, the new wide-angle lenses and devices to achieve true vertical photographs.

TERRESTRIAL PHOTOGRAMMETRY

14.2. BASIC PRINCIPLES

The principle of terrestrial photogrammetry was improved upon and perfected by Capt. Deville, then Surveyor General of Canada in 1888. In terrestrial photogrammetry, photographs are taken with the camera supported on the ground. The photographs are taken by means of a phototheodolite which is a combination of a camera and a theodolite. Maps are then compiled from the photographs.

The principle underlying the method of terrestrial photogrammetry is exactly similar to that of plane table surveying, *i.e.* if the directions of same objects photographed from two extremities of measured base are known, their positions can be located by the intersection of two rays to the same object. However, the difference between this and plane tabling is that more details are at once obtained from the photographs and their subsequent plotting etc. is done by the office while in plane tabling all the detailing is done in the field itself.

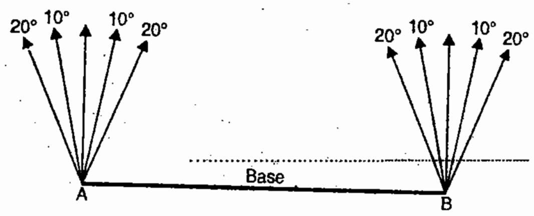


FIG. 14.1. DIRECTION OF POINTINGS IN TERRESTRIAL PHOTOGRAMMETRY.

Thus in Fig 14.1, *A* and *B* are the two stations at the ends of base *AB*. The arrows indicate the directions of horizontal pointings (in plan) of the camera. For each pair of pictures taken from the two ends, the camera axis is kept parallel to each other. From economy and speed point of view, minimum number of photographs should be used to cover the whole area and to achieve this, it is essential to select the best positions of the camera stations. A thorough study of the area should be done from the existing maps, and a ground reconnaissance should be made. The selection of actual stations depends upon the size and ruggedness of the area to be surveyed. The camera should be directed downward rather than upward, and the stations should be at the higher points on the area.

The terrestrial photogrammetry can be divided into two branches :

(*i*) Plane-table photogrammetry.

(*ii*) Terrestrial stereophotogrammetry.

The *plane table photogrammetry* consists essentially in taking a photograph of the area to be mapped from each of the two or three stations. The photograph perpendiculars may be oriented at any angle to the base, but usually from an acute angle with the latter. The main difficulty arises in the identifications of image points in a pair of photographs.

In the case of homogeneous areas of sand or grass, identification becomes impossible. The principles of stereophotogrammetry, however, produced the remedy.

In *terrestrial stereophotogrammetry*, due to considerable improvement of accuracy obtained by the stereoscopic measurement of pairs of photographs, the camera base and the angles of intersection of the datum rays to the points to be measured can be considerably reduced since the camera axes at the two stations exhibit great similarity to each other. The image points which are parallactically displaced relative to each other in the two photographs are fused to a single spatial image by the stereoscopic measurement.

14.3. THE PHOTO-THEODOLITE

The photo-theodolite is a combination of a 1 second theodolite and a terrestrial camera. Fig. 14.2 illustrates a back view of Bridges-Lee photo-theodolite made by Messers L.Cassella, London. Fig. 14.3 shows the photograph of a modern photo-theodolite manufactured by M/s Wild Heerbrugg Ltd.

A photo-theodolite essentially consists of the following parts. (Fig. 14.2) :

(1) *A camera box A of fixed focus type*. The focal length of the lens is generally 15 cm or more. The camera box is mounted on the axis exactly in the same manner as the vernier plate of a theodolite. Thus, the box can be rotated in azimuth about its vertical axis.

(2) *A hollow rectangular frame I placed vertically to the rear side*. The frame carries two cross-hairs *k* and *k'*, the intersection of which is exactly opposite to the optical centre of the lens. The line of collimation is defined as the line joining the intersection of the cross-hairs to the optical centre of the lens. The cross wires are pressed tightly against sensitive plate and are thus photographed on the photographic plate along with the field object. Two small celluloid strips can be fitted into the grooves in the lower corners of the frame I, and can be easily removed to write any description upon them in ink which is also photographed.

(3) *Across the rear of the vertical frame is also carried a straight transparent celluloid tangent scale*. Upon the base of the frame is pivoted a magnetic needle carrying a vertical cylindrical transparent scale (*M*) graduated to 30 minutes.

(4) The *sensified photographic* plate is *placed* between the vertical frame (*I*) and the back which is held by the spring. Before uncapping the lens, the front of the side is withdrawn to expose the plate and the vertical frame (*I*) is moved backward and forward by the screw (*J*) until the hair lines and the tangent scale are in contact with the plate. The magnetic needle is also set free to swing on its pivot. When the lens is uncapped (after the needle comes to rest), the photographs of hair lines, tangent scale, and the circular scale of the needle are imprinted on the negative. The reading of the scale at its intersection with the vertical hair on the photograph gives the magnetic bearing of the principal vertical plane (*i.e.* the vertical plane containing the optical axis).

(5) The box is supported on the tripod and is furnished with an inner and an outer axis, each of which is fitted with a clamp and fine adjusting screw. The graduated horizontal circle carries verniers reading to single minutes. These are supported on a levelling head carrying three foot screws.

(6) On the top of the box, a *telescope* is fitted. The telescope can be rotated in a vertical plane, about a horizontal axis, and is fitted with vertical arc with verniers, clamp, and slow motion screw. The line of sight of the telescope is set in the same vertical plane as the optical axis of camera.

14.4. DEFINITIONS (Fig. 14.4)

Camera Axis. Camera axis is the line passing through the centre of the camera lens perpendicular both to the camera plate (negative) and the picture plane (photograph). The optical axis coincides with the camera axis in a camera free from manufacturing imperfections.

Picture Plane. Picture plane is the plane perpendicular to the camera axis at the focal distance in front of the lens. It is represented by the positive contact print or *photograph* taken from a plate or film.

Principal Point. Principal point (k or k') is defined by the intersection of the camera axis with either the picture plane (positive) or the camera plate (negative).

Focal Length. Focal length (f) is the perpendicular distance from the centre of the camera lens to either the picture plane or the camera plate. It satisfies the following relation

$$f = \frac{uv}{u + v}$$

where u and v are conjugate object and image distances.

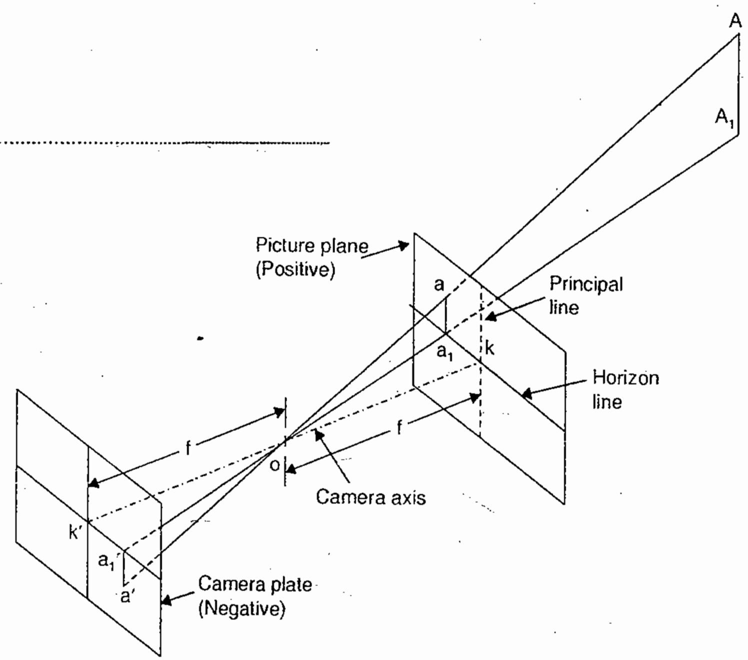


FIG. 14.4

Focal Plane (Image Plane). Focal plane is the plane (perpendicular to the axis of the lens) in which images of points in the object space of the lens are focused.

Nodal Point. Nodal point is either of two points on the optical axis of a lens (or a system of lenses) so located that when all object distances are measured from one point and all image distances are measured from the other, they satisfy the simple lens relation

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}.$$

Also a ray emergent from the second point is parallel to the ray incident at the first.

Perspective Centre. Perspective centre is the point of origin or termination of bundles of perspective rays. The two such points usually associated with a survey photograph are the interior perspective centre and the exterior perspective centre. In a distortionless lens camera system, one perspective centre encloses the same angles as the other, and in a perfectly adjusted lens camera system, the interior and exterior centres correspond to the rear and front nodal points, respectively.

Principal Distance. When the contact prints from original negatives are enlarged (or reduced) before their use in the compilation of subsequent maps, the value of the focal length (f) of the camera is not applicable to the revised prints. The changed value of f, holding the same geometrical relations, is known as the *principal distance*. In other words, it is the perpendicular distance from the internal perspective centre to the plane of a particular finished negative or print. This distance is equal to the calibrated focal length corrected for both the enlargement or reduction ratio and the film (or paper) shrinkage (or expansion) and maintains the same perspective angles as the internal perspective centre to points on the finished negative or print as existed in the camera at the moment of exposure. This is a geometrical property of each particular finished negative or print.

Principal Plane. Principal plane is plane which contains principal line and the optical axis. It is, therefore, perpendicular to the picture plane and the camera plate.

Print. A print is a photographic copy made by projection or contact printing from a photographic negative or from a transparent drawing as in blue-printing.

Fiducial Mark. A fiducial mark is one of two, three or four marks, located in contact with the photographic emulsion in a camera image plane to provide a reference line or lines for the plate measurement of images.

Fiducial Axis. Opposite fiducial marks define a reference line. Two pairs of opposite fiducial marks define two reference lines that intersect at 90°. These two lines are referred to as the x and y axes or the fiducial axes.

Film Base. Film base is a thin, flexible, transparent sheet of cellulose nitrate, cellulose acetate or similar material, which is coated with a light sensitive emulsion and used for taking photographs.

14.5. HORIZONTAL AND VERTICAL ANGLES FROM TERRESTRIAL PHOTOGRAPH

The horizontal and vertical angles to various points in a photograph can easily be found analytically, graphically or instrumentally. Fig. 14.5 (*a*) shows two points A and B photographed with camera axis horizontal so that the picture plane is vertical and the horizon line is horizontal. The image of the ground points A and B appear at a and b respectively

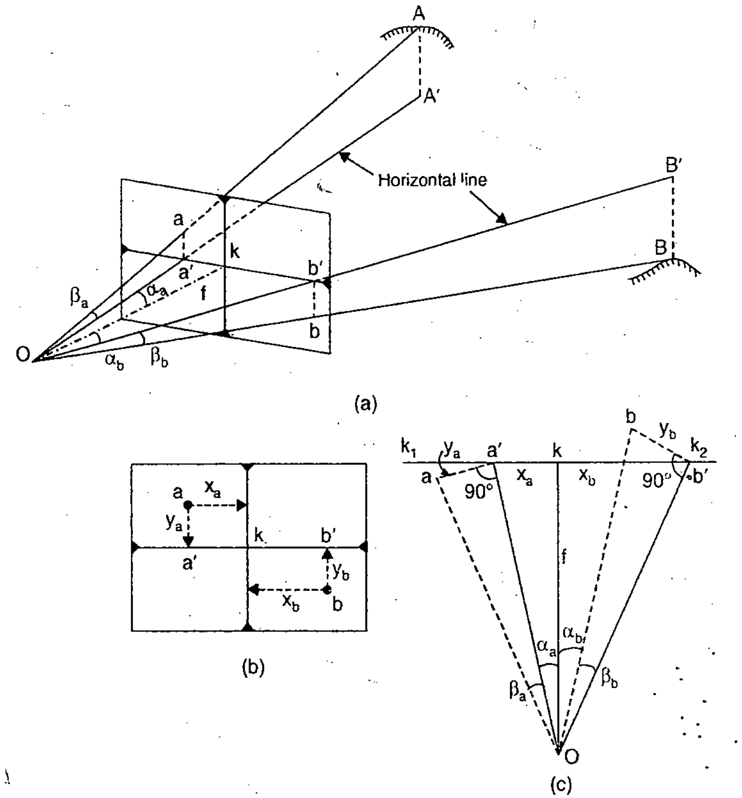


FIG. 14.5 HORIZONTAL AND VERTICAL ANGLES FROM TERRESTRIAL PHOTOGRAPHS WITH CAMERA AXIS HORIZONTAL.

and their projections on horizon line are at a' and b' respectively. If $f = ok$ = focal length of the lens, the horizontal angles α_a and α_b are given by

$$\tan \alpha_a = \tan \angle a'ok = \frac{x_a}{f} \quad \ldots[14.1(a)]$$

and

$$\tan \alpha_a = \tan \angle b'ok = \frac{x_b}{f} \quad \ldots[14.1(b)]$$

where x_a and x_b are the photographic x-co-ordinates of a and b with respect to the principal line as the y-axis. The horizontal angle between A and B is then equal to $(\alpha_a + \alpha_b)$ or in general $\alpha_a \pm \alpha_b$.

Similarly, let β_a and β_b be the vertical angles to A and B, as marked in Fig. 14.5 (*a*) then we have

$$\tan \beta_a = \tan \angle aoa' = \frac{aa'}{oa'} \quad \text{and} \quad \tan \beta_b = \tan \angle bob' = \frac{bb'}{ob'}$$

But $\quad aa' = y_a,\ bb' = y_b \ ; \quad oa' = f \sec \alpha_a$ and $ob' = f \sec \alpha_b$

Hence $\quad \tan \beta_a = \dfrac{y_a}{f \sec \alpha_a} \quad \ldots[14.2(a)] \ ; \quad \tan \beta_b = \dfrac{y_b}{f \sec \alpha_b} \quad \ldots[14.2(b)]$

The algebraic sign of vertical angle depends on the sign of y co-ordinates. Evidently, β_b will be a depression angle.

The horizontal and vertical angles can also be determined graphically, as shown in Fig. 14.5 (*c*) where the line $k_1 k_2$ represents the true horizon of the photograph. The line ko is constructed perpendicular to $k_1 k_2$ and represents the optical axis, the distance ko being made equal to f. With a pair of dividers, make $ka' = x_a$ and $kb' = x_b$ by making the measurements from the photographs. Join $a'o$ and $b'o$. The angles α_a and α_b can then be measured.

To find the vertical angle [Fig. 14.5 (*c*)], erect perpendiculars $a'a$ and $b'b$ to oa' and ob' respectively. Make $a'a = y_a$ and $b'b = y_b$, thus getting points a and b respectively. Join ao and bo. The angles aoa' and bob' are the desired vertical angles.

14.6. HORIZONTAL POSITION OF A POINT FROM PHOTOGRAPHIC MEASUREMENT: CAMERA AXIS HORIZONTAL

In plane table terrestrial photogrammetry, two photographs are taken from the ends of a base line. The position of the points can be plotted by graphical intersection as illustrated in Fig. 14.6.

Let P and Q be the known positions of the camera stations. Knowing the camera azimuths (*i.e.*, bearings of camera axis) ϕ_1 and ϕ_2 at both the stations, the horizon lines

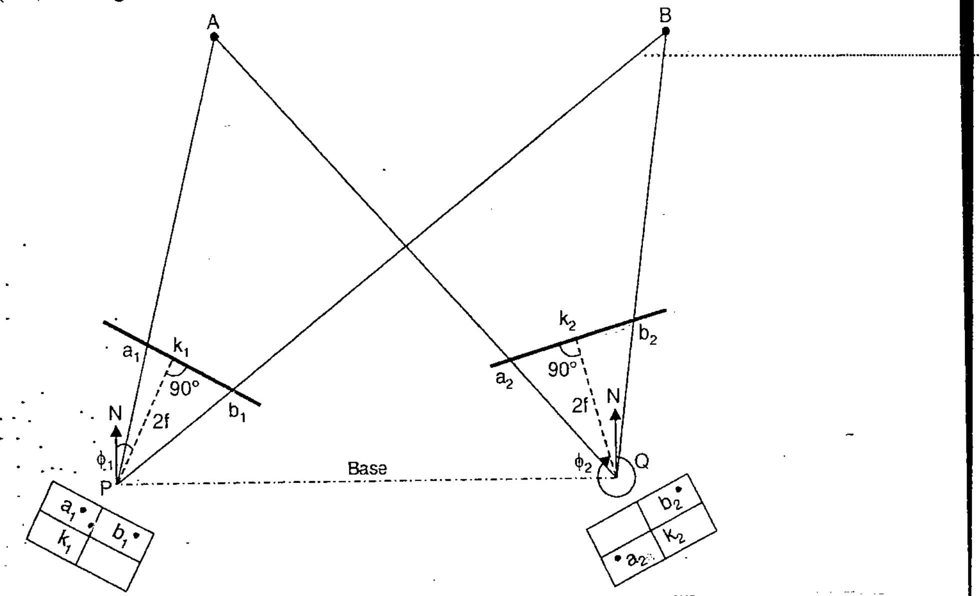


FIG. 14.6. GRAPHICAL INTERSECTION.

$a_1 k_1 b_1$ and $a_2 k_2 b_2$ can be drawn at perpendicular distances of $2f$ from P and Q respectively. On each photograph, the x-co-ordinates of points a and b are scaled by a pair of proportionate dividers set for a 2 to 1 ratio, and transferred to the photograph traces, as shown by the positions a_1, b_1 and a_2 and b_2 respectively in both the photographs taken with the camera axes horizontal at the time of exposure. Join Pa_1 and Pb_1 and prolong them. Similarly, join Qa_2 and Qb_2 and prolong them to intersect the corresponding lines in A and B respectively, thus giving horizontal positions of A and B.

Camera Position by Resection. To fix the positions of the camera stations, a separate ground control is necessary. However, the camera station can also be located by three point resection if the positions of three prominent points (which may be photogrammetric triangulation stations) are known and they are also photographed.

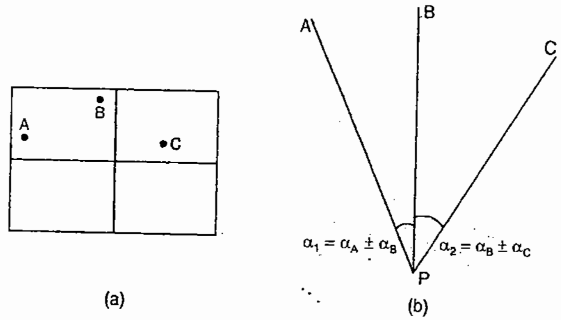


FIG. 14.7. CAMERA POSITION BY THREE POINT RESSECTION.

Thus, in Fig. 14.7. (*a*), let A, B and C be the three stations photographed. From § 14.5, the angles to A, B and C can be determined either graphically or analytically and hence angles $\alpha_1 (= \alpha_A \pm \alpha_B)$ and $\alpha_2 (= \alpha_B \pm \alpha_C)$ are known. If these angles are known graphically, a tracing paper resection on the plotted positions of A, B and C (on the map) will fix the map position of the camera station (P). If, however, the angles α_1 and α_2 are known analytically, the values may be set off by a three armed protractor for a graphical resection, or the values may be used to solve the three-point problem analytically for determining the position of the camera station.

Azimuth of a line from Photographic Measurement. The magnetic bearing or azimuth of the principal vertical plane is given by the reading of the cylindrical scale at its intersection with the vertical hair on the photograph. The horizontal angles of the lines with the principal plane can be calculated as discussed in § 14.5.

Thus, in Fig. 14.8(*a*), a, b and c are the positions of the three points A, B and C. The horizontal angles α_A, α_B and α_C (Fig. 14.8 *b*) can be determined. If ϕ is the azimuth of the principal plane (or the camera azimuth), we have

$$\phi_B = \text{azimuth of } B = \phi + \alpha_B$$

$$\phi_C = \text{azimuth of } C = \phi + \alpha_C$$

$$\phi_A = \text{azimuth of } A = \phi - \alpha_A + 360°$$

In general, therefore, we have

$$\textit{Azimuth of line} = \textit{camera azimuth} + \alpha$$

Due regard must be given to the algebraic sign of α. It may be considered positive when measured to the right of ok and negative when measured to the left. If the azimuth calculated from the above relation comes out to be negative, 360° must be added to the result.

Orientation of Picture Traces

The accuracy in the plotted positions of various points depends upon the correct orientation and placing of picture traces on the plan. The two conditions that are to be fulfilled are: (1) the picture trace should be perpendicular to the line joining the plotted position (O) of the station and the principal point (k), and (2) the principal point (k) should be at the focal distance from O. When enlargements are used, the enlarged focal length should be laid down.

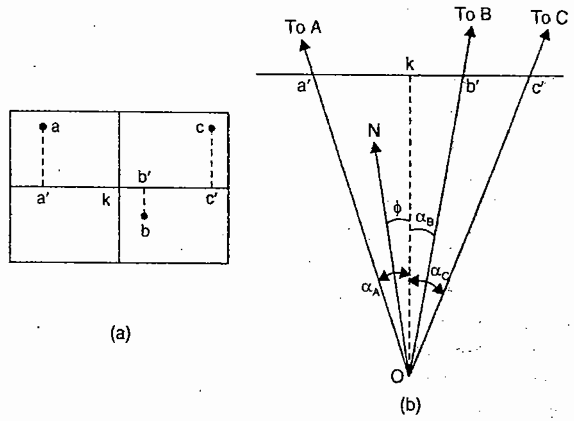


FIG. 14.8. AZIMUTH OF LINES FROM PHOTOGRAPHIC MEASUREMENT

In the case of photo-theodolite used for the photographic surveying, the bearing of the principal plane is known. In that case, the principal plane is laid at the known bearing, the principal point (k) is marked at a distance (f) from the camera station (O) and the picture trace is drawn perpendicular to that of the principal plane.

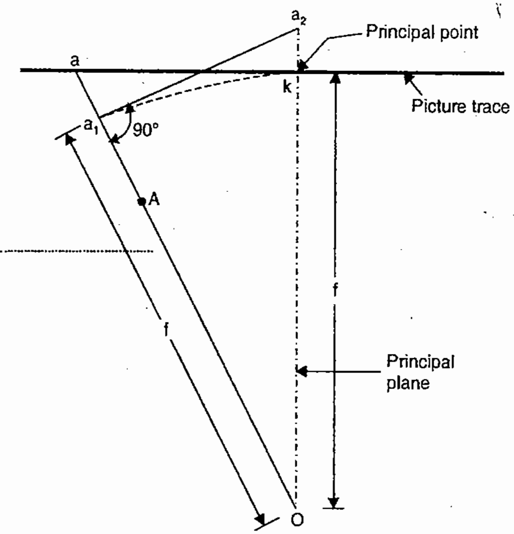


FIG. 14.9. ORIENTATION OF PICTURE TRACE FROM KNOWN POSITION OF POINT

If, however, the photograph includes any point whose position is known on the plane, the orientation may be performed with respect to it as follows : (Fig. 14.9).

Let A be the known position (on the plane) of the point and O be the known position of the camera station. Let ka be the distance (on the photographs) of the point A from the principal plane. Join OA and produce it. With O as the centre and radius equal to $f (= oa_1)$, draw an arc. At a_1, draw a line a_1a_2 perpendicular to oa_1, making a_1a_2 equal to the photographic distance ak. Join a_2o, cutting the arc in k. Thus, the position of the principal point and that of the principal plane is known. Through k, draw ka perpendicular to ok, thus giving us position of the picture trace.

14.7. ELEVATION OF A POINT BY PHOTOGRAPHIC MEASUREMENT

The elevation of a point photographed from two camera stations can be easily calculated from the measured co-ordinates of the images.

Thus, in Fig. 14.10 (*a*), (*b*), let A be the point whose elevation is to be determined with respect to the camera axis. A_1 is the projection of A on a horizontal plane passing through O. Let x and y be the co-ordinates of the photographic image (*a*) of the point A. As determined earlier, the horizontal angle (α) and vertical angle (β) are given by Fig. 14.10 (*a*):

$$\tan \alpha = \frac{x}{f} \quad ...(1)$$

$$\tan \beta = \frac{y}{oa_1} = \frac{y}{f \sec \alpha} = \frac{y}{f} \cos \alpha \quad ...(2)$$

or $$= \frac{y}{\sqrt{f^2 + x^2}} \quad ...(3)$$

In Fig. 14.10 (*b*),

$$oa_1 = \sqrt{f^2 + x^2} = f \sec \alpha$$

$$\angle a_1 oa = \beta = \angle AOA_1$$

Hence, from the similar triangles,

$$\frac{y}{oa_1} = \frac{AA_1}{OA_1}$$

$\therefore$ $$V = AA_1 = OA_1 . \frac{y}{oa_1}$$

$$= D . \frac{y}{\sqrt{f^2 + x^2}}$$

or $$V = \frac{Dy}{f \sec \alpha} = \frac{Dy}{f} \cos \alpha \quad ...(14.4)$$

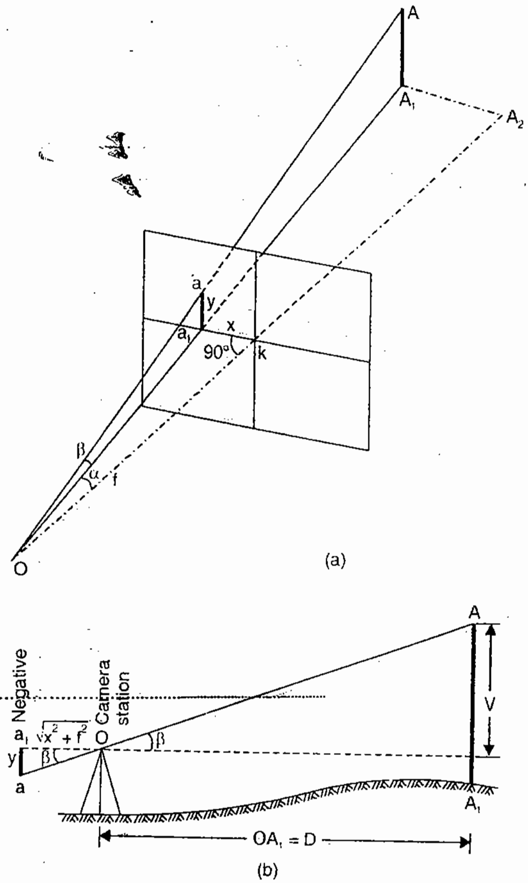


FIG. 14.10. ELEVATION BY PHOTOGRAPHIC MEASUREMENT.

Due regard must be paid to the sign of y.

If the elevation of the camera axis is known, the elevation of the point can be calculated from the relation : $h = H_c + V + c$

where h = elevation of the point

H_c = elevation of the camera lens

c = correction for curvature and refraction.

Elevation by Graphical Construction

In Fig. 14.11, let $a'kb'$ be the picture trace correctly oriented on the plane, and A and B be the plotted positions of two points, obtained by the intersection of the corresponding rays from the two photographic traces.

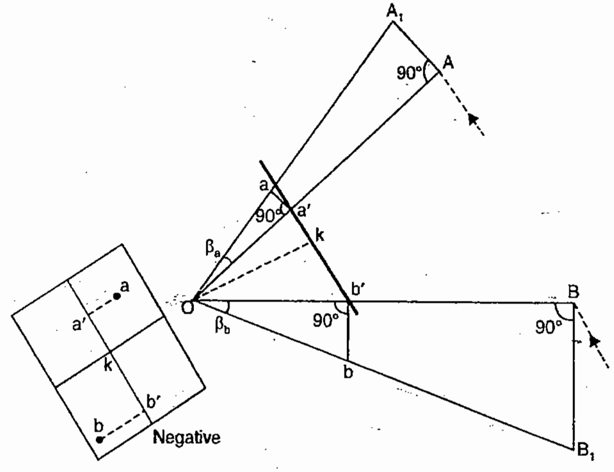


FIG. 14.11. ELEVATION BY GRAPHICAL CONSTRUCTION.

At a', erect perpendicular $a'a$, making $a'a = y$ co-ordinate of a. Join oa and extend it. Evidently, $\angle aoa' = \beta_a$. At A, draw AA_1 perpendicular to OA_1 to meet the line oa in A_1. Scale off AA_1, thus getting the elevation of A *above* the camera axis.

Similarly to get the elevation of B, erect $b'b$ perpendicular to ob', making $b'b = y$ co-ordinates of b. Join ob and prolong it. Draw BB_1 perpendicular to OB. Thus BB_1 is the elevation of B above the camera axis.

14.8. DETERMINATION OF FOCAL LENGTH OF THE LENS

Generally, the focal length of the camera lens is given by the manufacturer. Since the accurate knowledge of the focal length is very essential, it can be determined experimentally as below (Fig. 14.12).

Select two suitable points A and B. Measure the horizontal angle AOB ($= \theta$) accurately with the theodolite. Expose off the plate to show A and B. Let X_a and X_b be the co-ordinates of the two points. Then, we have

$$bk = x_b, \; ak = x_a$$

$$\tan \alpha_a = \frac{x_a}{f} \; ; \; \tan \alpha_b = \frac{x_b}{f}$$

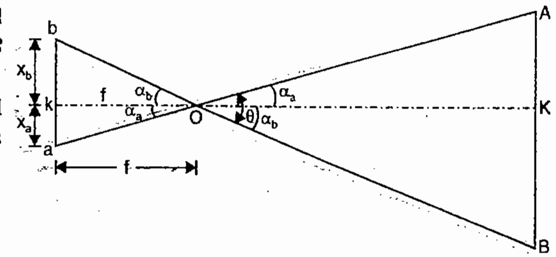


FIG. 14.12. FIELD DETERMINATION OF FOCAL LENGTH.

$\therefore \quad \tan \alpha_a \cdot \tan \alpha_b = \dfrac{x_a x_b}{f^2}$

Now $\quad \tan \theta = \tan (\alpha_a + \alpha_b) = \dfrac{\tan \alpha_a + \tan a_b}{1 - \tan \alpha_a \tan \alpha_b} = \dfrac{\dfrac{x_a}{f} + \dfrac{x_b}{f}}{1 - \dfrac{x_a \cdot x_b}{f^2}}$

$\therefore \quad \tan \theta (f^2 - x_a \cdot x_b) = f(x_a + x_b)$ or $f^2 - \dfrac{(x_a + x_b) f}{\tan \theta} - x_a x_b = 0$

which gives,

$$f = \frac{\dfrac{x_a + x_b}{\tan \theta} + \sqrt{\dfrac{(x_a + x_b)^2}{\tan^2 \theta} + 4 x_a x_b}}{2} = \frac{x_a + x_b}{2 \tan \theta} + \sqrt{\frac{(x_a + x_b)^2}{4 \tan^2 \theta} + x_a \cdot x_b} \quad ...(14.5)$$

Thus, the value of f can be calculated.

Example 14.1. *Three points A, B and C were photographed and their co-ordinates with respect to the lines joining the collimation marks on the photograph are :*

Point	*x*	*y*
a	*– 35.52 mm*	*+ 21.43 mm*
b	*+ 8.48 mm*	*– 16.38 mm*
c	*+ 48.26 mm*	*+ 36.72 mm*

The focal length of the lens is 120.80 mm. Determine the azimuths of the lines OB and OC, if that of OA is 354° 30′. The axis of the camera was level at the time of the exposure at the station O.

Solution

Fig. 14.8 shows the position of the points.

$\tan \alpha_a = \dfrac{x_a}{f} = \dfrac{-35.52}{120.80} \quad \therefore \quad \alpha_a = -16° 23'$

$\tan \alpha_b = \dfrac{x_b}{f} = \dfrac{+8.48}{120.80} \quad \therefore \quad \alpha_b = +4° 0'$

$\tan \alpha_c = \dfrac{x_c}{f} = +\dfrac{48.26}{120.80} \quad \therefore \quad \alpha_c = +21° 47'$

Azimuth of camera axis $= \phi = \phi_a - \alpha_a = 354° 30' - (-16° 23') = \mathbf{10° 53'}$

Azimuth of $\quad B = \phi + \alpha_b = 10° 53' + 4° = \mathbf{14° 53'}$

Azimuth of $\quad C = \phi + \alpha_c = 10° 53' + 21° 47' = \mathbf{32° 40'}$.

Example 14.2. *Photographs of a certain area were taken from P and Q, two camera stations, 100 m apart. The focal length of the camera is 150 mm. The axis of the camera makes an angle of 60° and 40° with the base line at stations P and Q respectively. The image of a point A appears 20.2 mm to the right and 16.4 mm above the hair lines on the photograph taken at P and 35.2 mm to the left on the photograph taken at Q.*

Calculate the distance PA and QA and elevation of point A, if the elevation of the instrument axis at P is 126.845 m.

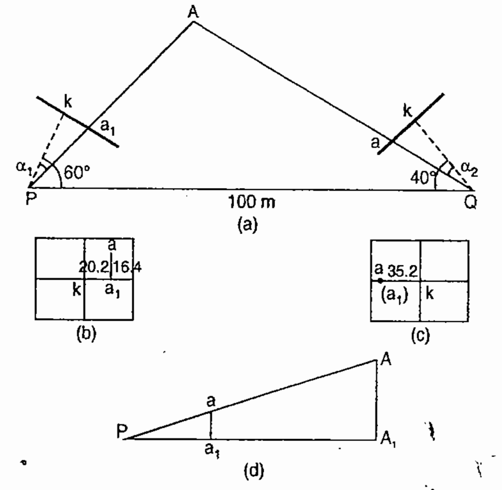


FIG. 14.13

Solution

Fig. 14.13 (a) shows the position of the ground point A with respect to the stations P and Q and the picture traces. Fig. 14.13 (*b*) shows the photograph taken at P and Fig. 14.13 (*c*) shows the photograph taken at Q, with the positions of a properly marked.

From the photograph at P,

$$\alpha_1 = \tan^{-1} \frac{ka_1}{f} = \tan^{-1} \frac{20.2}{150} = 7° 40'$$

$\therefore \quad \angle APQ = 60° - \alpha_1 = 60° - 7° 40' = 52° 20'$

From the photograph at Q,

$$\alpha_2 = \tan^{-1} \frac{ka}{f} = \tan^{-1} \frac{35.2}{150} = 13° 12'$$

$\therefore \quad \angle AQP = 40° - \alpha_2 = 40° - 13° 12' = 26° 48'$

$\therefore \quad \angle PAQ = 180° - 52° 20' - 26° 48' = 100° 52'$

From the triangle APQ,

$$AP = PQ \cdot \frac{\sin AQP}{\sin PAQ} = 100 \cdot \frac{\sin 26° 48'}{\sin 100° 52'} = \mathbf{45.9\ m}$$

and $$AQ = PQ \cdot \frac{\sin APQ}{\sin PAQ} = 100 \cdot \frac{\sin 52° 20'}{\sin 100° 52'} = \mathbf{80.6\ m}$$

Calculation of R.L. of A

From the photograph at P,

$$Pa_1 = \sqrt{x_a^2 + f^2} = \sqrt{(20.2)^2 + (150)^2} = 151.33 \text{ mm.}$$

Let A_1 be the projection of A on the horizontal line Pa_1 drawn through P (Fig. 14.15 *d*). Then from the similar triangles,

$$\frac{AA_1}{aa_1} = \frac{PA}{Pa_1}$$

$\therefore \quad AA_1 = aa_1 \cdot \dfrac{PA_1}{Pa_1} = \dfrac{16.4 \times 45.9}{151.53} = 4.975$ m

$\therefore \quad$ R.L. of A = R.L. of instrument axis $+ AA_1 = 126.845 + 4.975$ m $= \mathbf{131.820\ m}$.

Example 14.3. *The distance from two points on a photographic point to the principal line are 68.24 mm to the left, and 58.48 mm to the right. The angle between the points measured with a transit is 44° 30′. Determine the focal length of the lens.*

Solution

Distance of first point from principal line = x_1 = 68.24 mm

Distance of second point from principal line = x_2 = 58.48 mm

Angle between the two points = θ = 44° 30′

The focal length is given by the expression (Eq. 14.5),

$$f = \frac{x_1 + x_2}{2 \tan \theta} + \sqrt{\frac{(x_1 + x_2)^2}{4 \tan^2 \theta} + x_1 x_2}$$

where $\frac{x_1 + x_2}{2 \tan \theta} = \frac{68.24 + 58.48}{2 \tan 44° 30'} = 64.47$; $\left(\frac{x_1 + x_2}{2 \tan \theta}\right)^2 = (64.67)^2 = 4156.9$

$x_1 x_2 = 68.24 \times 58.48 = 3990.4$

Substituting the values, we get

$$f = 64.47 + \sqrt{4156.9 + 3990.4} = 64.47 + 90.26 = \mathbf{154.73 \ mm}$$

AERIAL PHOTOGRAMMETRY

14.9. AERIAL CAMERA

The primary function of the terrestrial camera as well as the aerial camera is the same, *i.e.*, that of taking pictures. However, since the aerial camera is mounted on a fast moving aeroplane, its requirements are quite different. The aerial camera requires : (*i*) fast lens, (*ii*) high speed and efficient shutter, (*iii*) high speed emulsion for the film, and (*iv*) a magazine to hold large rolls of film. As such, an aerial camera may be considered to be a surveying instrument of great precision.

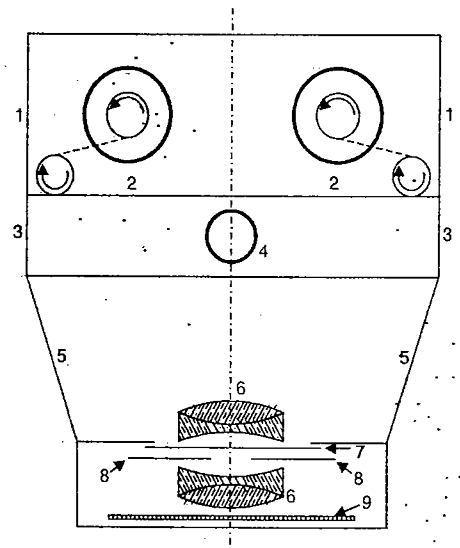
Fig. 14.14 shows the photograph of the wild RC-9 automatic supper wide angle camera. Fig. 14.16 shows the schematic diagram of an aerial camera.

An aerial camera consists of the following essential parts :

(*i*) the lens assembly (including lens, diaphragm, shutter and filter)

(*ii*) the camera cone

(*iii*) the focal plane



1. MAGAZINE 2. FOCAL PLANE 3. BODY
4. TRUNNION 5. CONE 6. LENS
7. SHUTTER 8. DIAPHRAGM 9. FILTER

FIG. 14.16 SCHEMATIC DIAGRAM OF AERIAL CAMERA.

(*iv*) the camera body

(*v*) the drive mechanism

(*vi*) the magazine

(*i*) The Lens Assembly :

The lens assembly consists of the lenses, the diaphragm, the shutter and the filter.

Fig. 14.15 shows the cross-section of the high performance lenses manufactured by Wild Heerbrugg Ltd. Wild Aviotar f : 4 [Fig. 14.15 (*a*)] is normal angle lens while Wild Universal-Aviogon f : 5.6 [Fig. 14.15 (*b*)] and Wild Super-Aviogon f : 5.6 [Fig. 14.15 (*c*)] are wide angle lenses and super wide angle lenses respectively. The following are the details of the lenses manufactured by M/s Wild Heerbrugg Ltd. (Table below)

The other lenses commonly used are : (*i*) Bausch and Lomb *Metrogen f* : 6.3 wide angle lens with 93° coverage, most commonly used in the United States, (*ii*) Zeiss Topogon f : 6.3 with 93° coverage, and (*iii*) Goertz Aerotar f : 6.8 with 75° coverage.

Since the air-craft is at a considerable distance from the terrain to be photographed, all the points can be considered to be at an infinite distance from the lens and *hence the focal plane of the aerial camera can be fixed at one location.* Thus, an aerial camera is always of a fixed focus type, the focus being set for infinity.

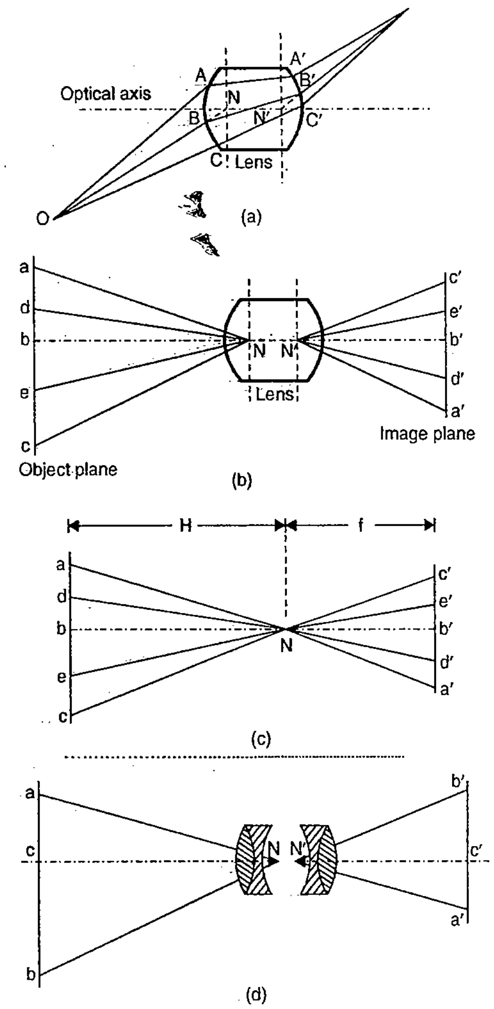


FIG. 14.17. THE LENS NODES.

Camera	*Lens*	*Focal Length (f) cm*	*Picture size (cm)*	*Field Angle*
Wild RC 8 for 19 cm and 24 cm film width	Aviotar f : 4 Aviogon f : 5.6 Universal Aviogon f : 5.6	21 11.5 15.2	18 × 18 18 × 18 23 × 23	60° 90° 90°
Wild RC 9 for 24 cm film width	Super-Aviogon f : 5.6 Super-Infrangon f : 5.6	8.8 8.8	23 × 23 23 × 23	120° 120°
Wild RC 7a for plates 15 × 15 cm	Aviotar f : 4 Aviogon f : 5.6	17 10	14 × 14 14 × 14	60° 90°

Fig. 14.17 (*a*) shows a lens forming the image *I* of an object *O*. The ray *OA*, which meets the lens near its top, emerges with its direction changed slightly downward, such as *O AA' I*. The ray *OC*, which meets the lens near its bottom, emerges with its direction changed slightly upward, such as *OCC'I*. In between these, there must be a ray that emerges parallel to its original direction, such as *OBB'I*. Let *OB* extended cut the optical axis at point *N*, and *IB'* extended cut the optical axis at *N'*. In an ideal lens, the cardinal points *N* and *N'* will be common for all the rays of an object, *i.e.*, they will meet at the common point *N* and *N'* as shown in Fig. 14.17 (*b*). These two points (*N* and *N'*) are known as the *front nodal point* and the *rear nodal point* respectively. *If a ray of light is directed at the front nodal point, it is refracted by the lens system that it appears to emerge from the rear nodal point without having undergone any change in direction.*

Fig. 14.17 (*d*) shows the optical diagram of a simple lens system composed of four elements with an air space between the two doublets. *CNN'C'* is the optical axis which pierces the two principal planes of the doublet at the two nodal points *N* and *N'*. The distance *N'C* between the rear nodal point and the plane of infinite focus is called the focal length of the lens system.

In photogrammetric computations it is often convenient to eliminate the distance *NN'* and to superimpose *N'* and *N* as shown in Fig. 14.17 (*c*). Under this condition, each ray is a straight line and the image is an identical representation of the object to the scale *f/H*. An actual lens system is designed to approach this ideal as closely as possible.

The Shutter **:** The **camera** shutter controls the interval of time during which light is allowed to pass through the lens. Since the aircraft moves at a high speed, a fast speed shutter is required to prevent blurring of the image caused by camera vibrations and the forward motion of the aircraft. The shutter speed generally varies from 1/100 second to 1/1000 second. There are three types of shutters used in aerial cameras :

(*a*) Between-the-lens type (*b*) Focal plane type (*c*) Louvre type.

In the *between-the-lens type*, the shutter is fixed in the space between the elements of the lens system, the space being equal to the fraction of an inch. With this type of shutter, the film is exposed only during the interval the shutter is open. The *focal plane type* shutters operate near the focal plane of the camera. These types of shutters permit higher shutter speeds and are provided in the cameras used for military operations. The film is progressively exposed throughout the time of passage of slit across the focal plane. This type of shutter is not useful for mapping purposes since it includes a distortion in the scale of the photograph in the direction of the movement of the shutter and position errors in the relationship of object points on photographs. The *louvre type* shutters are usually employed for large lens aperture with high speed. It consists of a number of metal strips about 5 mm wide supported on a metal frame and is placed either in front of the lens or at its back.

The Diaphragm **:**

A diaphragm is placed between the lens elements and acts as a physical opening of the lens system. It consists of a series of leaves which can be rotated to increase or decrease the size of the opening to restrict the size of the bundle of rays to pass through the lens. If the diaphragm opening is larger, the shutter speed has to be greater.

The Filter **:**

A filter consists of a piece of coloured glass placed in front of the lens. It filters the stray light (blue and violet) in the atmosphere caused by haze and moisture. It also protects the lens from the flying particles in the atmosphere.

(*ii*) Camera Cone :

The camera cone supports the entire lens assembly including the filter. At the top of it are provided the collimation marks which define the focal plane of the camera. The cone is made up of the material having low co-efficient of thermal expansion so that the collimation marks and the lens system are held in the same relative positions at operational temperatures. The elements of *interior orientation* are fixed by the relative positions of the lens, the lens axis, the focal plane and the collimation marks.

(*iii*) **The Focal Plane** :

The collimation marks are provided at the upper surface of the cone. The focal plane is provided exactly above the collimation marks. It is kept at such a distance from the near nodal point that best possible image is obtained.

(*iv*) **The Camera Body** :

The camera body is the part of the camera provided at the top of the cone. Sometimes, it forms the integral part of the cone in which case they act as an integral part to preserve the interior orientation once the camera is calibrated.

(*v*) **The Drive Mechanism** :

The camera drive mechanism is housed in the camera body and is used for (*i*) winding and tripping the shutter (*ii*) operating the vacuum system for flattening the film, and (*iii*) winding the film. It may be either operated manually or automatically.

(*vi*) **Magazine** :

A magazine holds the exposed and unexposed films and houses the film flattening device at the focal plane. The power operation of the movable parts of the magazine is supplied from the drive mechanism. The film is flattened at the focal plane either by inserting a piece of optical glass in the focal plane opening or by applying a vacuum to ribbed plate criss-crossed with tiny grooves and provided to the back of the film.

14.10. DEFINITIONS AND NOMENCLATURE

1. Vertical Photograph. A vertical photograph is an aerial photograph made with the camera axis (or optical axis) coinciding with the direction of gravity.

2. Tilted photograph. A tilted photograph is an aerial photograph made with the camera axis (or optical axis) *unintentionally* tilted from the vertical by a small amount, usually less than 3° (Fig. 14.18).

3. Oblique Photograph. An oblique photograph is an aerial photograph taken with the camera axis directed *intentionally* between the horizontal and the vertical. If the apparent horizon is shown in the photograph, it is said to be *high oblique*. If the apparent horizon is not shown, it is said to be *low oblique*.

4. Perspective Projection. A perspective projection is the one produced by straight lines radiating from a common (or selected) point and passing through point on the sphere to the plane of projection. A *Photograph* is a perspective projection.

5. Exposure station. Exposure station is a point in space, in the air, occupied by the camera lens at the instant of exposure. Precisely, it is the space position of the front nodal point at the instant of exposure.

6. Flying height. Flying height is the elevation of the exposure station above sea level or any other selected datum.

7. Flight line. It is a line drawn on a map to represent the track of the aircraft.

8. Focal length. It is the distance from the front nodal point of the lens to the plane of the photograph (*i.e.*, *OK* in the Fig. 14.18). It is also the distance of the image plane from the rear nodal point. *Equivalent focal length* is the distance of the image plane from the rear nodal point (or the distance of the plane of the photograph from the front nodal plane) yielding the best average definition.

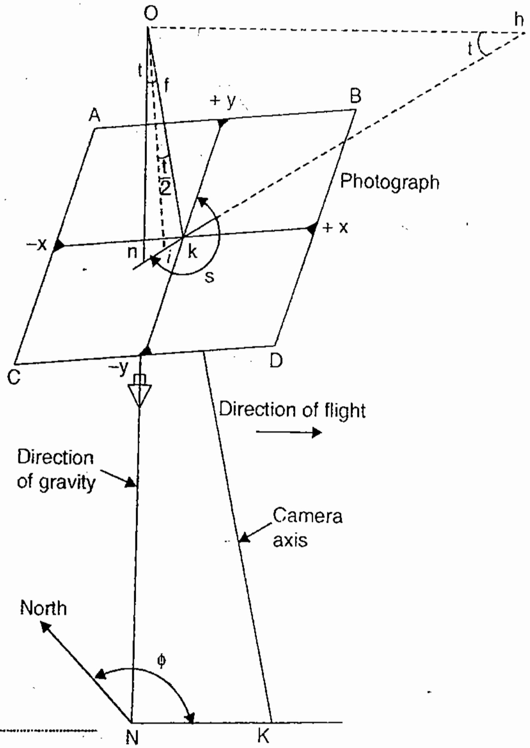


FIG. 14.18. TILTED PHOTOGRAPH

9. Principal point. Principal point is a point where a perpendicular dropped from the front nodal point strikes the photograph. (Also, it is the foot of a perpendicular to the image plane from the rear nodal point in a camera lens system free from manufacturing errors). This principal point is considered to coincide with the intersection of the *x*-axis and the *y*-axis. In Fig. 14.18, *k* is the principal point. The point *K* is known as the *ground principal point* where the line *OK* produced meets the ground.

10. Nadir point. Nadir point is a point where a plumb line dropped from the front nodal point pierces the photograph. Thus, in Fig. 14.18, *n* is the nadir point, which is a point on the photograph vertically beneath the exposure station. This point is also known as the *photo-nadir* or *photo plumb point*.

11. Ground nadir point. Ground nadir point or ground plumb point is the datum intersection with the plumb line through the front nodal point. It is the point on the ground vertically beneath the exposure station such as point *N* in Fig. 14.18.

12. Tilt. Tilt is the vertical angle defined by the intersection, at the exposure station, of the optical axis with the plumb line. In Fig. 14.18, $\angle kon = t$ = tilt.

13. Principal Plane. A principal plane is the plane defined by the lens (*O*), the ground nadir point (*N*) and the principal point produced to the ground (*K*). It is the thus a vertical plane containing the optical axis, such as the plane *nok* or *NOK* in Fig. 14.18.

14. Principal line. A principal line is the line of intersection of the principal plane with the plane of the photograph. It is thus the line on a photograph obtained by joining the principal point and the photo nadir point, such as the line *nk* in Fig. 14.18.

15. Isocentre. Isocentre is the point in which the bisector of the angle of tilt meets the photographs. Thus, in Fig. 14.18, *oi* is the bisector and *i* is the isocentre. The angle of tilt lies in the principal plane, and hence the isocentre (*i*) lies on the principal line at a distance of $f \tan \frac{t}{2}$ from the principal point.

On a vertical photograph, the isocentre and the photo-nadir point coincide with the principal point.

16. Swing. Swing is the angle measured in the plane of the photograph from the positive *y*-axis clockwise to the nadir point. Thus, in Fig. 14.18, *s* is the swing.

17. Azimuth of the principal plane. The azimuth of the principal plane (sometimes also known as the azimuth of the photograph) is the clockwise horizontal angle measured about the ground nadir point from the ground survey north meridian to the principal plane of the photograph, such as the angle ϕ in Fig. 14.18. It is thus the ground-survey direction of the tilt.

18. Horizon point. Horizon point is the intersection of the principal line with the horizontal line through the perspective centre, such as point *h* in Fig. 14.19. In a near vertical or tilted photograph, this point is generally outside the photograph. In a high oblique photograph, however, it is in the photograph.

19. Axis of tilt. Axis of tilt is a line in the plane of the photograph and is perpendicular to the principal line at the isocentre such as $i_1\, i\, i_2$ in Fig. 14.19. The plane of the photograph is tilted to the horizontal about this axis. The axis of tilt is a horizontal line, as are all line perpendicular to the principal line. It is also known as *isometric parallel.*

Fig. 14.19 shows a *high oblique* photograph illustrating the perspective principles.

In Fig. 14.19,

ABCD = Oblique plane of the photograph negative

CDEF = Ground horizontal plane

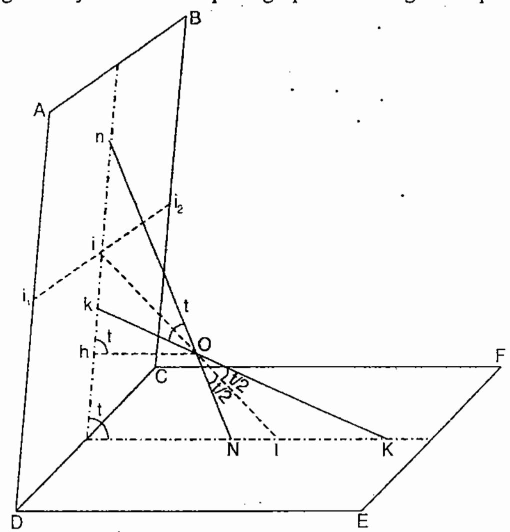


FIG. 14.19. OBLIQUE PHOTOGRAPH

O = perspective centre or the rear nodal point of the camera lens (or the exposure station)
k = principal point
K = ground principal point
Ok = principal distance
t = angle of tilt = $\angle kon$ = angular deviation of the photograph perpendicular from the plumb line
n = photo-nadir or photo plumb point
N = ground nadir or ground plumb point
noN = plumb line or vertical line through the perspective centre
i = isocentre
I = ground isocentre
nik = principal line
h = horizon point
$i_1 i i_2$ = axis of tilt = isometric parallel

Relation Between Principal Point, Plumb Point and Isocentre :

From Figs. 14.18 and 14.19,

(1) nk = distance of the nadir point from the principal point

$$\frac{nk}{kO} = \tan t \quad \text{or} \quad nk = kO \cdot \tan t = f \tan t \qquad \ldots(14.6)$$

since $kO = f$ = principal distance

(2) ki = distance of the isocentre from the principal point

$$\frac{ki}{kO} = \tan\frac{t}{2} \quad \text{or} \quad ki = kO \cdot \tan\frac{t}{2} = f \tan\frac{t}{2} \qquad \ldots(14.7)$$

(3) kh = distance along the principal line, from the principal point to the horizon point

$$\frac{kh}{kO} = \cot t \quad \text{or} \quad kh = kO \cdot \cot t = f \cot t. \qquad \ldots(14.8)$$

14.11. SCALE OF A VERTICAL PHOTOGRAPH

Since a photograph is the perspective projection, the images of ground points are displaced where there are variations in the ground elevation. Thus, in Fig. 14.20 (a) the images of two points A and A_0, vertically above each other, are displaced on a *vertical* photograph and are represented by a and a_0 respectively. Due to this displacement, there is no uniform scale between the points on such a photograph, except when the ground points have the same elevation. If the elevation of points vary, the scale of the vertical photograph will vary from point to point on the photograph.

Let us first take the case when the ground is horizontal, *i.e.*, all the points are having the same elevation, such as shown in Fig. 14.20 (*a*).

Let $$S = \text{scale} = \frac{\text{map distance}}{\text{ground distance}}$$

From Fig. 14.20 (*a*), $$S = \frac{ka}{KA} = \frac{Ok}{KA} = \frac{f}{H-h} = \frac{f}{H-h} \qquad \ldots(14.9)$$

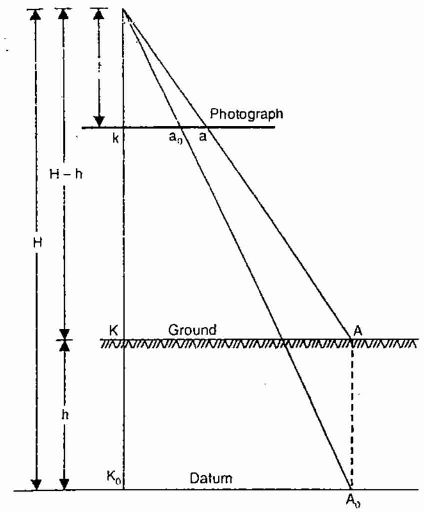


FIG. 14.20. (*a*) SCALE OF A VERTICAL PHOTOGRAPH

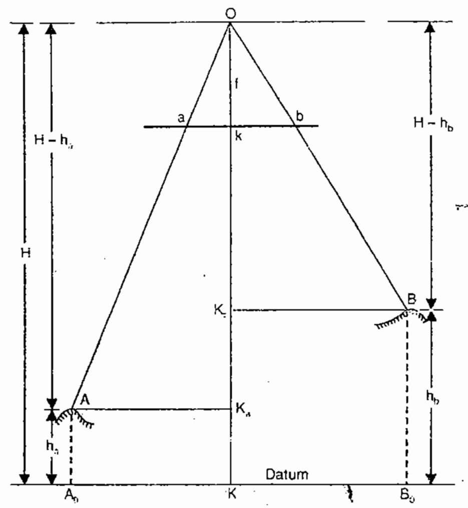


FIG. 14.20. (*b*) SCALE OF A VERTICAL PHOTOGRAPH.

where H = height of the exposure station (or the air plane) above the mean sea level
f = focal length of the camera
h = height of the ground above mean sea level

Let us now take the case when the points are not having the same elevation, as represented in Fig. 14.20 (*b*).

Let A and B be two points having elevations h_a and h_b respectively above mean sea level. They are represented by a and b respectively on the map. k is the principal point of the vertical photograph taken at height H above mean sea level.

The scale of the photograph at the elevation h_a is evidently equal to the ratio $\frac{ak}{AK_a}$.

From similar triangles, $$\frac{aK}{AK_a} = \frac{Ok}{OK_a} = \frac{f}{H-h_a}$$

Hence the scale of the photograph at the elevation h_a is equal to $\frac{f}{H-h_a}$.

Similarly, the scale of the photograph at the elevation h_b is equal to the ratio $\frac{bk}{BK_b}$.

From similar triangles, $$\frac{bk}{BK_b} = \frac{Ok}{OK_b} = \frac{f}{H-h_b}.$$

Hence the scale of the photograph at the height h_b is equal to $\frac{f}{H-h_b}$.

In general, therefore, the scale of the photograph is given by

$$S_h = \frac{f}{H-h}$$

where S_h = scale at the elevation h.

The scale of the photograph can also be designated by the representative fraction (R_h), *i.e.*

$$R_h = \frac{1}{\left(\frac{H-h}{f}\right)}$$

where $(H-h)$ and f are expressed in the same unit (*i.e.* metres).

Datum Scales (S_d)

The *datum scale* of a photograph is that scale which would be effective over the entire photograph if all the ground points were projected vertically downward on the mean sea level before being photographed. Thus, from Fig. 14.20 (*a*),

$$\text{Datum scale} = S_d = \frac{ka}{KA_0} = \frac{Ok}{OK} = \frac{f}{H} \quad ...(14.10)$$

where K and A_0 are the projections of k and A on the datum plane.

Average Scale (S_{av})

The average scale of a vertical photograph is that which would be effective over the entire photograph if all the ground points were projected vertically downward or upward on a plane representing the average elevation of the terrain before being photographed.

Thus, $$S_{av} = \frac{f}{H-h_{av}} \quad ...(14.11)$$

where h_{av} = average elevation of the terrain

To Find the Scale of a Photograph

If the images to ground points of *equal elevation* and known horizontal distance appear on the photograph, the scale of the photograph can be determined by comparing the ground length and the corresponding length on the photograph. Thus, if l is the distance on the photograph, between the two points A and B having the same elevation h and the horizontal distance (ground) between them to be L, the scale at the height h is given by

$$S_h = \frac{l}{L} \quad ...(14.12)$$

The distance L measured on the ground either directly or by the triangulation, or it can be taken from the existing maps, if available. To find the average or fairly representative scale of photograph several known lines on the photograph should be measured and compared and the average scale should be adopted. In case a reliable map of the area is available, the photographic scale can be found by comparing the photo distance and the map distance between two well-defined points at the same elevation.

Thus, $$\frac{\text{Photo scale}}{\text{Map scale}} = \frac{\text{photo distance}}{\text{map distance}}$$

If the focal length of the lens and the flying height (H) above *m.s.l.* is known, the scale can be found from the relation

$$S_h = \frac{f}{H-h} \quad ...(14.13)$$

14.12. COMPUTATION OF LENGTH OF LINE BETWEEN POINTS OF DIFFERENT ELEVATIONS FROM MEASUREMENTS ON A VERTICAL PHOTOGRAPH

In Fig. 14.21, let A and B be two ground points having elevations h_a and h_b above datum, and the co-ordinates (X_a, Y_a), (X_b, Y_b) respectively with respect to the ground co-ordinate axes which coincide in direction with the photographic co-ordinates x and y-axis. The origin of the ground co-ordinates lie vertically beneath the exposure station.

Let a and b be the corresponding points of the photograph, and (x_a, y_a), (x_b, y_b) be the corresponding co-ordinates. From similar triangles,

$$\frac{Ok}{OK_a} = \frac{x_a}{X_a} = \frac{y_a}{Y_a} = \frac{f}{H-h_a} \quad ..(1)$$

Also, $$\frac{Ok}{OK_b} = \frac{x_b}{X_b} = \frac{y_b}{Y_b} = \frac{f}{H-h_b} \quad ..(2)$$

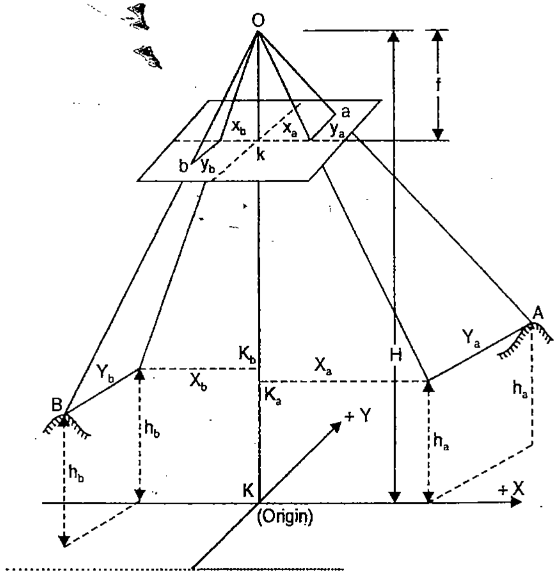


FIG. 14.21. COMPUTATION OF LENGTH OF A LINE.

Hence, we have $$X_a = \frac{H-h_a}{f} \cdot x_a \quad ...[14.14\ (a)]$$

$$Y_a = \frac{H-h_a}{f} \cdot y_a \quad ...[14.14\ (b)]$$

$$X_b = \frac{H-h_b}{f} \cdot x_b \quad ...[14.14\ (c)] \; ; \; Y_b = \frac{H-h_b}{f} \cdot y_b \quad ...[14.14\ (d)]$$

And, in general, the co-ordinates X and Y of any point at an elevation are :

$$X = \frac{H-h}{f} x \; ; \quad Y = \frac{H-h}{f} y.$$

The length L between the two points A and B is then given by

$$L = \sqrt{(X_a - X_b)^2 + (Y_a - Y_b)^2} \quad ...(14.15)$$

The value of X_a, X_b and Y_a and Y_b must be substituted with their proper algebraic signs.

14.13. DETERMINATION OF HEIGHT (H) OF LENS FOR A VERTICAL PHOTOGRAPH

If the images of two points A and B having different known elevations and known length between them appear on the photograph, the elevation or height H of the exposure station can be calculated by a reversed procedure from that of the preceding article.

As proved in the previous article, the ground length L is given by

$$L^2 = (X_a - X_b)^2 + (Y_a - Y_b)^2$$

Substituting the values of X_a, X_b, Y_a, Y_b as obtained in the previous article, we get

$$L^2 = \left[\frac{H - h_a}{f} x_a - \frac{H - h_b}{f} x_b\right]^2 + \left[\frac{H - h_a}{f} y_a - \frac{H - h_b}{f} y_b\right]^2 \quad \ldots(14.16)$$

In the above expression, the ground distance L, and elevations h_a and h_b are known quantities. The photographic co-ordinates (x_a, y_a), (x_b, y_b) can be measured. The only unknown is H. Collecting the terms for H, the equation takes the quadratic form

$$pH^2 + qH + r = 0$$

where p, q and r are the numbers obtained after substituting the values of the known quantities. The value of H is then obtained by

$$H = \frac{-q + \sqrt{q^2 - 4pr}}{2p}$$

The computation of H by the solution of quadratic equation is rather very tedious and time consuming. Alternately, the value of H can be determined by successive approximations as follows :

Step 1 :

The first approximate value of H is obtained from the scale relationship

$$\frac{f}{H_{approx.} - h_{ab}} = \frac{ab}{AB} = \frac{l}{L} \quad \ldots(14.17)$$

where
h_{ab} = average elevation of line AB
$AB = L$ = known ground distance
$ab = l$ = measured photographic distance

Step 2 :

The approximate value of H so obtained is used for calculating the co-ordinates (X_a, Y_a) and (X_b, Y_b). Using these co-ordinates, the approximate value of H and the elevations h_a and h_b, the length of the line is computed. Length is then compared with the actual distance to get a more correct value of H. Thus,

$$\frac{H - h_{ab}}{H_{approx.} - h_{ab}} = \frac{\text{correct } AB}{\text{computed } AB} \quad \ldots(14.18)$$

Step 3 :

With this value of H, step 2 is repeated till the computed length of AB and the correct length agree within necessary precision, usually 1 in 5000.

14.14. RELIEF DISPLACEMENT ON A VERTICAL PHOTOGRAPH

We have seen that if the photograph is truly vertical and the ground is horizontal, and if other sources of errors are neglected, the scale of the photograph will be uniform. Such a photograph represents a true *orthographic projection* and hence the true map of the terrain. In actual practice, however, such conditions are never fulfilled. When the ground is not horizontal, the scale of the photograph varies from point to point and is not constant.

Since the photograph is the perspective view, the ground relief is shown in perspective on the photograph. Every point on the photograph is therefore, displaced from their true orthographic position. This displacement is called *relief displacement*.

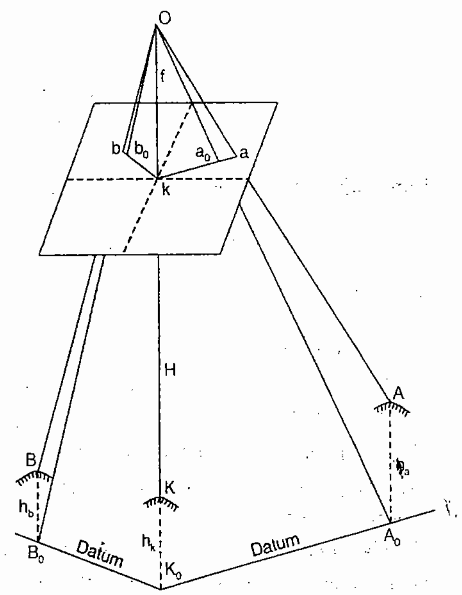


FIG. 14.22. RELIEF DISPLACEMENT ON VERTICAL PHOTOGRAPH.

Thus, in Fig. 14.22, A, B and K are three ground points having elevations h_a, h_b and h_k above datum. A_0, B_0 and K_0 are their *datum positions* respectively, when projected vertically downwards on the datum plane. On the photograph, their positions are a, b and k respectively, the points k being chosen vertically below the principal point. If the datum points A_0, B_0 and K_0 are imagined to be photographed along with the ground points, their positions will be a_0, b_0 and k respectively. As is clear from the figure, the points a and b are *displaced outward* from their datum photograph positions, the displacement being along the corresponding radial lines from the principal point. The radial distance aa_0 is the *relief displacement* of A while bb_0 is the relief displacement of B. The point k has not been displaced since it coincides with the principal point of the photograph.

To calculate the amount of relief displacement, consider Fig. 14.23 which shows a vertical section through the photograph of Fig. 14.22 along the line ka.

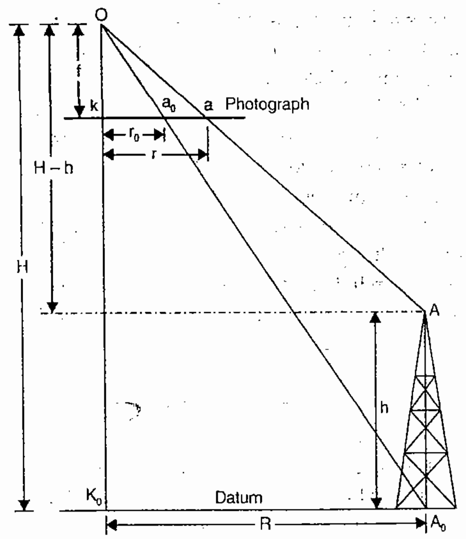


FIG. 14.23. CALCULATION OF RELIEF DISPLACEMENT.

In Fig. 14.23,

Let r = radial distance a from k
r_0 = radial distance of a_0 from k
$R = K_0 A_0$

Then, from similar triangles,

$$\frac{f}{H - h} = \frac{r}{R}, \text{ from which } r = \frac{Rf}{H - h} \quad \ldots(1)$$

Also $\frac{f}{H} = \frac{r_0}{R}$, from which $r_0 = \frac{Rf}{H}$ ...(2)

Hence the relief displacement (d) is given by

$$d = r - r_0 = \frac{Rf}{H-h} - \frac{Rf}{H}$$

or
$$d = \frac{Rfh}{H(H-h)} \quad ...(3)$$

But
$$R = \frac{r(H-h)}{f} = \frac{r_0 H}{f} \quad \text{from (1) and (2).}$$

Substituting the values of R in (3), we get

$$d = \frac{r(H-h)}{f} \cdot \frac{fh}{H(H-h)} = \frac{rh}{H} \quad ...(4a) \; [14.19\ (a)]$$

Also
$$d = \frac{r_0 H}{f} \cdot \frac{fh}{H(H-h)} = \frac{r_0 h}{H-h} \quad ...(4b) \; [14.19\ (b)]$$

From equations (3) and (4) above, we conclude the following :

(1) The relief displacement *increases* as the distance from the principal point *increases*.

(2) The relief displacement *decreases* with the *increase* in the flying height.

(3) For point *above datum*, the relief displacement is *positive* being radially *outward*.

(4) For point below datum (h having negative value), relief displacement is *negative*, being radially *inward*.

(5) The relief displacement of the point vertically below the exposure station is zero.

In the above expressions, H and h must be measured above the *same datum*.

Height of Object from Relief Displacement

If the scale of the photograph is known (or computed by the method discussed earlier), ~~equation (4-a) can be used~~ to determine the height of any object, such as a tower TB shown in Fig. 14.24. Let h be the height of the tower above its base, and H be the height (unknown) of the exposure station above the selected datum passing through the base of the tower. Let t and b be the top and bottom positions of the tower on the photograph. The radial distance r and the relief displacement can very easily be measured. If the scale S of the photograph is *known*, the height H can be calculated from the relation

$$S = \frac{f}{H} \quad ...(i)$$

Knowing H, and measuring d and r, the height h is calculated from equation (4a). Thus,

$$h = \frac{dH}{r} \quad ...(14.20)$$

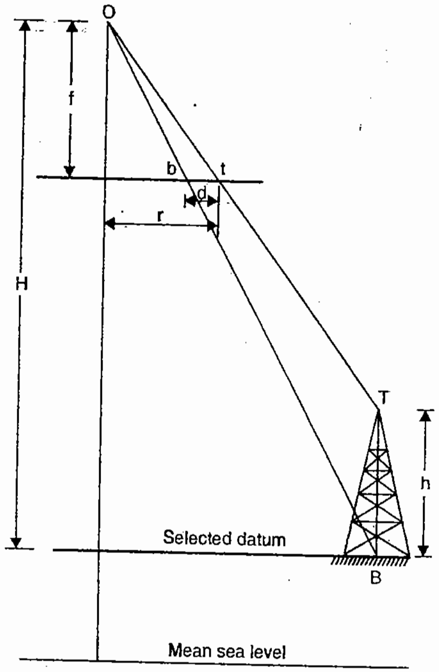


FIG. 14.24. HEIGHT OF A TOWER FROM RELIEF DISPLACEMENT.

where h is the height of the tower above the selected datum with reference to which H has been computed. Incidently, if the elevation of the bottom of the tower is known, the height of flight above mean sea level can be known.

Example 14.4. *A vertical photograph was taken at an altitude of 1200 metres above mean sea level. Determine the scale of the photograph for terrain lying at elevations of 80 metres and 300 metres if the focal length of the camera is 15 cm.*

Solution

The scale at any height h is given by

$$S_h = \frac{f}{H-h}$$

When $h = 80$ m, we have ; $S_{80} = \frac{15 \text{ cm}}{(1200-80)\text{ m}} = \frac{1 \text{ cm}}{74.67 \text{ m}}$

or **1 cm = 74.67 m.**

As a representative fraction, the scale is

$$R_{80} = \frac{\frac{15}{100}\text{ m}}{(1200-80)\text{ m}} = \frac{1}{\frac{1120 \times 100}{15}} = \frac{\mathbf{1}}{\mathbf{7467}}$$

Similarly, at $h = 300$ m, $S_{300} = \frac{15 \text{ cm}}{(1200-300)\text{ m}} = \frac{15 \text{ cm}}{900 \text{ m}} = \frac{1 \text{ cm}}{60 \text{ m}}$ or **1 cm = 60 m**

As a representative fraction, the scale is

$$R_{300} = \frac{\frac{15}{100}}{(1200-300)} = \frac{\mathbf{1}}{\mathbf{6000}}$$

Example 14.5. *A camera having focal length of 20 cm is used to take a vertical photograph to a terrain having an average elevation of 1500 metres. What is the height above sea level at which an air-craft must fly in order to get the scale of 1 : 8000 ?*

Solution

The scale expressed as R.F. is given by

$$R = \frac{f}{H-f}$$

Substituting the values, $\frac{1}{8000} = \frac{\left(\frac{20}{100}\right)\text{ m}}{(H-1500)\text{ m}}$ or $H - 1500 = \frac{20 \times 8000}{100} = 1600$

$\therefore$ $H = 1600 + 1500 = \mathbf{3100 \text{ m}}$ above *m.s.l.*

Example 14.6. *A line AB, 2000 m long, lying at an elevation of 500 m measures 8.65 cm on a vertical photograph for which focal length is 20 cm. Determine the scale of the photograph in an area the average elevation of which is about 800 m.*

Solution.

$$\text{Scale} = \frac{\text{Map distance}}{\text{Ground distance}} = \frac{f}{H-h}$$

$$\therefore \quad \frac{8.65 \text{ cm}}{2000 \text{ m}} = \frac{20 \text{ cm}}{(H - 500) \text{ m}}$$

or $$(H - 500) = \frac{20 \times 2000}{8.65} = 4624 \text{ m}$$

$$\therefore \quad H = 4624 + 500 = 5124 \text{ m}$$

$$\therefore \quad S_{800} = \frac{20 \text{ cm}}{(5124 - 800) \text{ m}} = \frac{1 \text{ cm}}{216.2 \text{ m}}$$

Hence S_{800} is **1 cm = 216.2 cm.**

Example 14.7. *A section line AB appears to be 10.16 cm on a photograph for which the focal length is 16 cm. The corresponding line measures 2.54 cm on a map which is to a scale 1/50,000. The terrain has an average elevation of 200 m above mean sea level. Calculate the flying altitude of the aircraft, above mean sea level, when the photograph was taken.*

Solution.

The relation between the photo scale and map scale is given by

$$\frac{\text{Photo scale}}{\text{Map scale}} = \frac{\text{Photo distance}}{\text{Map distance}}$$

Here, map scale $= \frac{1}{50,000}$; Let the photo scale be $\frac{1}{n}$

$$\therefore \quad \frac{1/n}{1/50,000} = \frac{10.16}{2.54}$$

$$\therefore \quad \frac{1}{n} = \frac{10.16}{2.54} \times \frac{1}{50,000} = \frac{1}{12,500} \quad \text{or} \quad n = 12,500$$

Again, $$S_{200} = \frac{1}{n} = \frac{f}{(H - h)} \quad \text{or} \quad \frac{1}{12,500} = \frac{(16/100) \text{ m}}{(H - 200) \text{ m}}$$

or $$(H - 200) = \frac{16}{100} \times 12500 = 2000 \text{ m}$$

Hence $$H = 2000 + 200 = \mathbf{2200 \text{ m}}.$$

Example 14.8. *Two points A and B having elevations of 500 m and 300 m respectively above datum appear on the vertical photograph having focal length of 20 cm and flying altitude of 2500 m above datum. Their corrected photographic co-ordinates are as follows:*

Point	*Photographic Co-ordinates* x (cm)	y (cm)
a	+ 2.65	+ 1.36
b	− 1.92	+ 3.65

Determine the length of the ground line AB.

Solution

The ground co-ordinates are given by

$$X_a = \frac{H - h_a}{f} \cdot x_a = \frac{2500 - 500}{20} \times (+2.65) = +265 \text{ m}$$

$$Y_a = \frac{H - h_a}{f} \, y_a = \frac{2500 - 500}{20} \times (+1.36) = +136 \text{ m}$$

$$X_b = \frac{H - h_b}{f} \cdot x_b = \frac{2500 - 300}{20} \times (-1.92) = -211.2 \text{ m}$$

$$Y_b = \frac{H - h_b}{f} \cdot y_b = \frac{2500 - 300}{20} \times (+3.65) = +401.5 \text{ m}$$

$$(X_a - X_b)^2 = (265 + 211.2)^2 = 22.677 \times 10^4 \text{ m}^2$$

$$(Y_a - Y_b)^2 = (136 - 401.5)^2 = 7.049 \times 10^4 \text{ m}^2$$

Hence $$AB = \sqrt{(X_a - X_b)^2 + (Y_a - Y_b)^2} = \sqrt{(22.677 + 7.049)\, 10^4} = \mathbf{545 \text{ m}}.$$

Example 14.9. *The ground length of a line AB is known to be 545 m and the elevations of A and B are respectively 500 m and 300 m above m.s.l. On a vertical photograph taken with a camera having focal length of 20 cm include the images a and b of these points, and their photographic co-ordinates are :*

($x_a = +2.65$ cm, $y_a = +1.36$ cm); ($x_b = -1.92$ cm, $y_b = +3.65$ cm).

The distance ab scaled directly from the photograph is 5.112 cm. Compute the flying height above the mean sea level.

Solution

From the scale relationship, the approximate height can be calculated from

$$\frac{f}{H_{approx.} - h_{ab}} = \frac{ab}{AB}$$

Here, $$h_{ab} = \tfrac{1}{2}(500 + 300) = 400 \text{ m}$$

$$\therefore \quad \frac{20 \text{ (cm)}}{(h_{approx.} - 400) \text{ m}} = \frac{5.112 \text{ (cm)}}{545 \text{ (m)}}$$

$$\therefore \quad H_{approx.} - 400 = \frac{20 \times 545}{5.112} \quad \text{or} \quad H_{approx.} = 400 + 2132.2 = 2532.2 \text{ m}$$

Using this approximate height, the ground co-ordinates of A and B are calculated from Eq. 14.14.

$$X_a = \frac{H - h_a}{f} \cdot x_a = \frac{2532.2 - 500}{20} \times 2.65 = +269.3 \text{ m}$$

$$Y_a = \frac{H - h_a}{f} \cdot y_a = \frac{2532.2 - 500}{20} \times 1.36 = +138.2 \text{ m}$$

$$X_b = \frac{H - h_b}{f} \cdot x_b = \frac{2532.2 - 300}{30} \times (-1.92) = -214.3 \text{ m}$$

$$Y_b = \frac{H - h_b}{f} \cdot y_b = \frac{2532.2 - 300}{20} \times (3.65) = +407.3 \text{ m}$$

The ground length based on the approximate height is given by

$$L = \sqrt{(269.3 + 214.3)^2 + (138.2 - 407.3)^2} = 553.4 \text{ m}$$

The actual ground length is 545 m. The second approximate height is calculated as follows :

$$\frac{H - h_{ab}}{H_{approx.} - h_{ab}} = \frac{\text{Correct } AB}{\text{Computed } AB}$$

$$\therefore \quad \frac{H - 400}{2532.2 - 400} = \frac{545}{553.4} \quad ; \text{ From which } \quad H = 400 + 2100 = 2500$$

Using this value of H to calculate the co-ordinates, we get

$$X_a = \frac{2500 - 500}{20} \times 2.65 = +265 \quad ; \quad Y_a = \frac{2500 - 500}{20} \times 1.36 = +136$$

$$X_b = \frac{2500 - 300}{20} \times (-1.92) = -211.2 \quad ; \quad Y_b = \frac{2500 - 300}{20} \times 3.65 = +401.5$$

$$\therefore \quad L = \sqrt{(265 + 211.2)^2 + (136 - 401.5)^2} = 545$$

This agrees with the measured length. Hence height of lens = **2500** m.

Example 14.10. *The distance from the principal point to an image on a photograph is 6.44 cm, and the elevation of the object above the datum (sea level) is 250 m. What is the relief displacement of the point if the datum scale is 1/10,000 and the focal length of the camera is 20 cm ?*

Solution.

The datum scale is given by

$$S_d = \frac{1}{10,000} = \frac{(20/100) \text{ m}}{H \text{ m}}$$

From which $\quad H = \frac{20}{100} \times 10,000 = 2000$ m above *m.s.l.*

Again, the relief displacement (d) is given by

$$\therefore \quad d = \frac{r\,h}{H} = \frac{6.44 \times 250}{2000} = \mathbf{0.805 \text{ cm.}}$$

Example 14.11. *A tower TB (Fig. 14.24), 50 m high, appears in a vertical photograph taken at a flight altitude of 2500 m above mean sea level. The distance of the image of the top of the tower is 6.35 cm. Compute the displacement of the image of the top of the tower with respect to the image of its bottom. The elevation of the bottom of the tower is 1250 m.*

Solution

Let H = height of the lens above the bottom of the tower.

The displacement d of the image of the top with respect to the image of the bottom is given by

$$d = \frac{h\,r}{H}$$

where $\quad h$ = height of the tower above its base = 50 m ; $H = 2500 - 1250 = 1250$ m

$$\therefore \quad d = \frac{50 \times 6.35}{1250} = \mathbf{0.25 \text{ cm.}}$$

Example 14.12. *A vertical photograph of a flat area having an average elevation of 250 metres above mean sea level was taken with a camera having a focal length of*

20 cm. A section line AB, 250 m long in the area, measures 8.50 cm on the photograph. A tower TB in the area also appears on the photograph. The distance between the images of top and bottom of the tower measures 0.46 cm on the photograph. The distance of the image of the top of the tower is 6.46 cm. Determine height of the tower.

Solution. (Fig. 14.24)

$$\text{Scale} = \frac{\text{Map distance}}{\text{Ground distance}} = \frac{f}{H}$$

where $\quad H$ = height of camera above the selected datum.

Let the average ground be the selected datum.

$$\therefore \quad \frac{8.50 \text{ cm}}{250.0 \text{ m}} = \frac{20 \text{ cm}}{H \text{ m}} \quad \text{or} \quad H = \frac{20 \times 250.0}{8.50} = 588.2 \text{ m}$$

Again, the height of the tower above its base is given by

$$h = \frac{dH}{r} = \frac{0.46 \times 588.2}{6.46} \text{ m} = \mathbf{41.89 \text{ m.}}$$

14.15. SCALE OF A TILTED PHOTOGRAPH

We have seen that in the case of a perfectly vertical photograph the scale is uniform, from point to point, only if the ground is flat and has uniform elevation throughout. If the elevations of the points vary, the scale also varies. If a tilted photograph (or near vertical photograph) is taken over an area having no relief, the scale will not be uniform. The downward half of the photograph will have a larger scale than the upward half. The problem becomes still more complicated if a tilted photograph is taken over an area with relief. To determine the scale of the photograph from point to point in such a case, the position of the points must be known with respect to the principal line since the tilt takes place along the principal line. In addition, the tilt, swing, flying height, focal length and the elevation of the point must also be known.

Fig. 14.25 shows a tilted photograph which includes the image a of a point A at an altitude of h above datum. k is the principal point and n the photo nadir. nk is the principal line. From a, draw am perpendicular to the principal line. am is, therefore, a horizontal line. From m draw mm' perpendicular to the plumb line. mm' is, therefore, a horizontal line. Hence the triangle amm' lies in a horizontal plane.

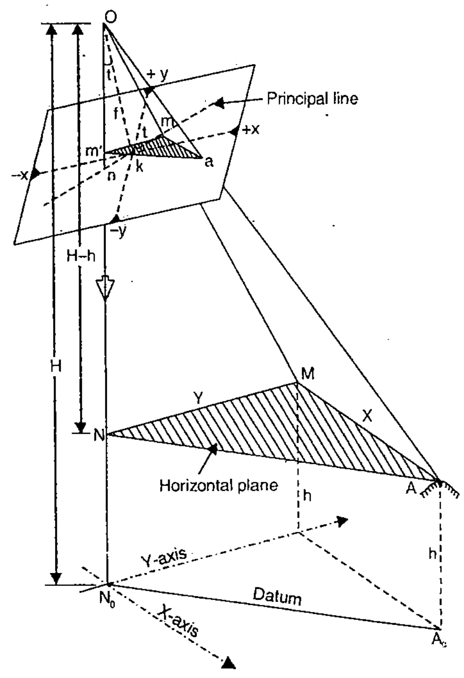


FIG. 14.25. SCALE OF A TILTED PHOTOGRAPH.

Let N and M be the points on on and om extended, at heights of h above datum. Thus N, M and A have the same elevation. The triangle NMA is in a horizontal plane.

From the similar triangles $om'a$ and ONA, we get

$$\frac{m'a}{NA} = \frac{Om'}{ON}$$

But $Om' = On - m'n = f \sec t - mn \sin t$; $ON = ON_0 - NN_0 = H - h$

$$\frac{m'a}{NA} = \frac{\text{Map distance}}{\text{Ground distance}} = \text{scale at a point whose elevation is } h = S_h$$

Substituting the values in (1), we get

$$S_h = \frac{f \sec t - mn \sin t}{H - h} \quad \ldots(2)$$

In the above expression mn is the distance along the principal line, between the photo nadir and the foot of the perpendicular from the point under consideration. To find its value, let us consider the system of co-ordinates axes illustrated in Fig. 14.26.

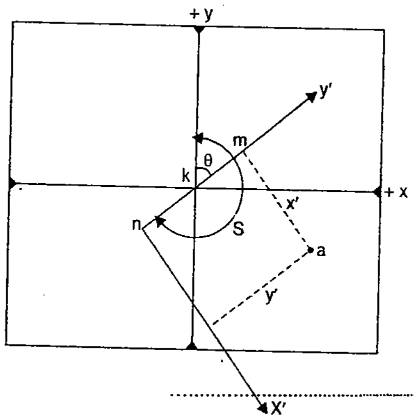


FIG. 14.26. CO-ORDINATE AXES THROUGH PLUMB POINT.

Let the photographic co-ordinates of the image a be x and y. Let s be the angle of swing and θ be the angle between the y-axis and the principal line. If the y-axis be rotated to the position of the principal line, the new axis (or y'-axis) will be inclined to the original axis by an angle θ given by

$$\theta = 180° - s \quad \ldots(14.21)$$

As in analytic geometry, the angle θ is considered to be *positive* when the rotation is in the *counter-clockwise* direction and *negative* when it is in the clockwise direction. Thus, the angle θ in Fig. 14.26. is negative. Let the new x-axis (or x'-axis) be selected through the nadir point n. The distance kn is equal to $f \tan t$ (see Eq. 14.6). The new co-ordinates (x', y') of the point a with reference to the x' and y' axis are given by

$$x' = x \cos\theta + y \sin\theta \quad \ldots[14.22\ (a)]$$

$$y' = -x \sin\theta + y \cos\theta + f \tan t \quad \ldots[14.22\ (b)]$$

The distance nm is therefore equal to y'. Substituting this in (2), we get

$$S_h = \frac{f \sec t - y' \sin t}{H - h} \quad \ldots(14.23)$$

It is clear that the co-ordinates y' is the same for the points on the line ma. *Hence the scale, which is the linear function of y, is constant for all the points on a line perpendicular to the principal line.*

For finding the scale at a given point on the photograph by Eq. 14.23, the following data is essential :

(1) focal length (2) tilt (3) height of flight
(4) height of the point (5) swing
and (6) the position of the point which respect to principal line.

14.16. COMPUTATION OF LENGTH OF LINE BETWEEN POINTS OF DIFFERENT ELEVATIONS FROM MEASUREMENTS ON A TILTED PHOTOGRAPH

In Fig. 14.25, triangles $m'ma$ and NMA are in horizontal planes. From scale relationship, we have

$$S_h = \frac{am}{AM}$$

But $am = x'$ (from Fig. 14.26) ; $AM = X$; $S_h = \dfrac{f \sec t - y' \sin t}{H - h}$

Substituting the values, we get

$$AM = X = \frac{H - h}{f \sec t - y' \sin t} \cdot x' \quad \ldots[14.24\ (a)]$$

Similarly, from scale relationships, we have $S_h = \dfrac{m'm}{NM}$

But $m'm = mm \cos t = y' \cos t$; $NM = Y$; and $S_h = \dfrac{f \sec t - y' \sin t}{H - h}$

Substituting the values, we get

$$NM = Y = \frac{H - h}{f \sec t - y' \sin t} \cdot y' \cos t \quad \ldots14.24\ [(b)]$$

Thus, the ground co-ordinates (x, y) of any point are known.

If there are two points A and B, their ground co-ordinates (X_a, Y_a) and (X_b, Y_b) can be calculated, and the length L of the line AB computed from the expression

$$L = \sqrt{(X_a - X_b)^2 + (Y_a - Y_b)^2} \quad \ldots[14.24\ (c)]$$

14.17. DETERMINATION OF FLYING HEIGHT FOR A TILTED PHOTOGRAPH

If the images of two points A and B having different known elevations and known length between them appear on a tilted photograph, the elevation or height H of the exposure station can be determined exactly in the same way as discussed in § 14.13. The method is outlined in the following steps :

Step 1 :

From the photographic co-ordinates (x_a, y_a), (x_b, y_b), calculate the photographic length ab (or scale it directly from the photograph). From the existing maps, or by another source, the ground length of AB is known. Calculate the approximate flying height from the scale relationship :

$$\frac{f}{H_{approx.} - h_{ab}} = \frac{ab}{AB} \; ; \text{ where } h_{ab} = \text{average elevation of } AB.$$

Step 2 :

Using the approximate H so obtained, and the photographic co-ordinates, compute the ground co-ordinates (X_a, Y_a), (X_b, Y_b) by solution of equations [14.24 (a)] and [14.24 (b)].

The new co-ordinates x' and y' may be computed from Eq. 14.22. The length L of the line AB can then be calculated from Eq. 14.24 (*c*).

Compare the computed length of AB with that of the correct length from the relationship

$$\frac{H - h_{ab}}{H_{approx.} - h_{ab}} = \frac{\text{Correct } AB}{\text{Computed } AB}$$

where H is the new value of the flying height.

***Step 3*:**

Repeat step 2 till the computed length of AB agrees with its correct length within the required degree of accuracy.

14.18. TILT DISTORTION OR TILT DISPLACEMENT

If a terrain is photographed, once with a tilted photograph and then with a vertical photograph, both taken at the same flight altitude and with the same focal length, the two photographs will match at the axis of tilt only. The image of any other point, not on the axis of tilt, will be *displaced* either outward or inward with respect to its corresponding position on a vertical photograph.

Tilt distortion or tilt displacement is defined as the difference between the distance of the image of a point on the tilted photograph from the isocentre and the distance of the image of the same point on the photograph from the isocentre if there had been no tilt.

Fig. 14.27 shows a vertical photograph and tilted photograph of the same terrain, intersecting each other in a line which is the axis of tilt. n is the nadir point of the tilted photograph, and serves as the principal point of the vertical photograph. k is the principal point of the tilted photograph. The portion of the tilted photo above the axis of tilt is known as the *upper part* while the portion below it is known as the *lower part* of the photograph.

Let us consider two ground points A and B photographed both on the vertical photograph as well as on the tilted photograph. a and b are their images on the tilted photo while a' and b' are the corresponding images on the vertical photograph. If the vertical photograph is now rotated about the axis of tilt until it is in the plane of

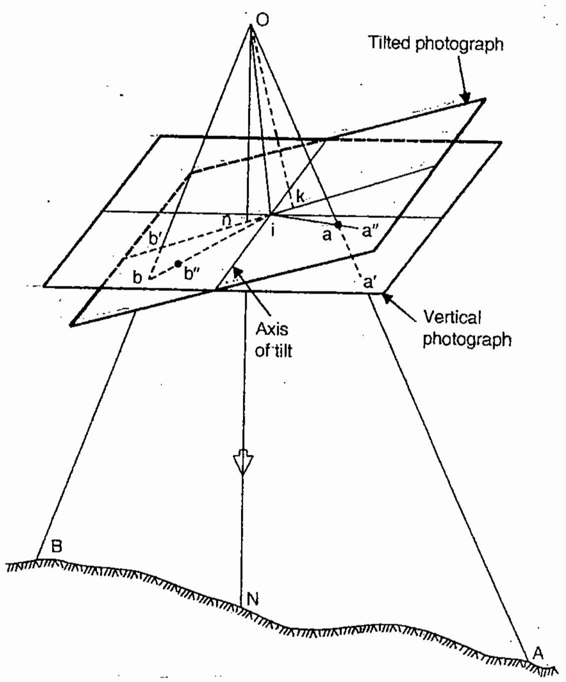


FIG. 14.27. TILT DISTORTION.

the tilted photograph, point a' would fall at a'' while point b' would fall at b''. The tilt displacement of points a and b are therefore aa'' and bb''. *It is to be noted that these displacement occur along lines which radiate from the isocentre.*

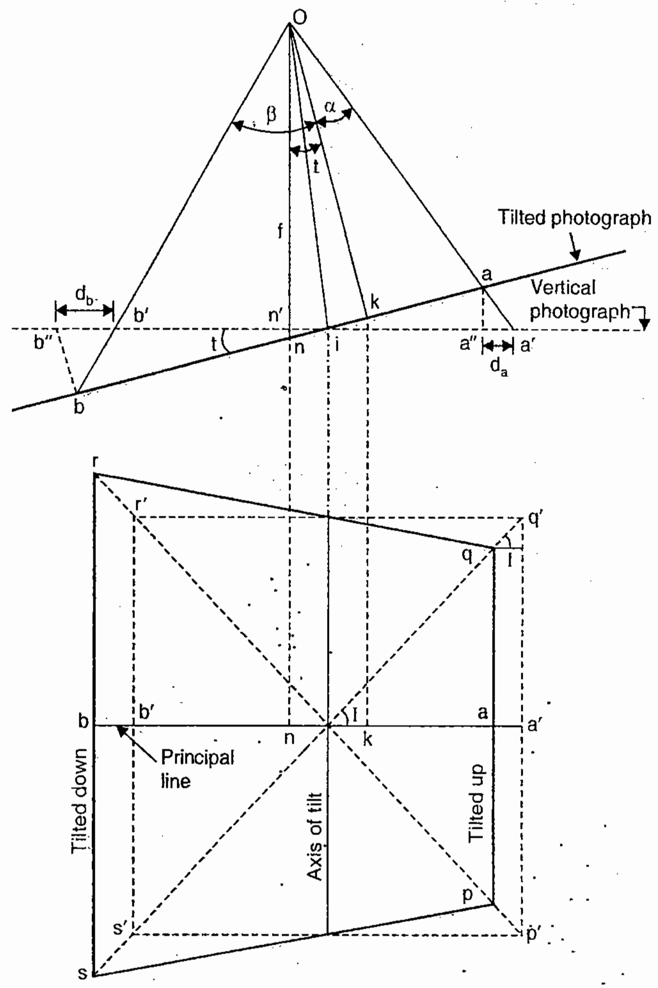


FIG. 14.28. DISTORTION OF A SQUARE.

To calculate the amount of tilt distortion or displacement, consider Fig. 14.28 which shows the effect of tilt along a line perpendicular to the axis of tilt. The line ab represents the principal line of the tilted photograph and $a'b'$ represents the principal line of the vertical photograph. The principal point for the tilted photograph is at k while that of the vertical photograph is at n'. The axis of tilt (perpendicular to the plane of the paper in the sectional view) is at i. O is the exposure station which is common for both the photographs. a

and b are the images of two points on the tilted photograph, along its principal line, while a' and b' are the corresponding positions on the vertical photograph. Since i is the point of rotation, d_a and d_b represent the displacements of the points a and b with respect to a' and b' respectively. Let α be the inclination of the ray Oa with Ok. Similarly, β is the inclination of the ray Ob to Ok.

Thus d_a = tilt displacement of a with respect to a'

or $d_a = ia' - ia$

But $ia' = n'a' - n'i = f\tan(t+\alpha) - f\tan t/2$ and $ia = ka + ki = f\tan\alpha + f\tan t/2$

Hence $d_a = f\tan(t+\alpha) - f\tan t/2 - f\tan\alpha - f\tan t/2$

or $$d_a = f[\tan(t+\alpha) - \tan\alpha - 2\tan t/2] \quad ...[14.25](a)$$

Similarly, $d_b = ib - ib'$

$ib = kb - ki = f\tan\beta - f\tan t/2$; $ib' = n'b' + n'i = f\tan(\beta - t) + f\tan t/2$

$\therefore$ $d_b = f\tan\beta - f\tan t/2 - f\tan(\beta - t) - f\tan t/2$

or $$d_b = f[\tan\beta - \tan(\beta - t) - 2\tan t/2] \quad ...[14.25(b)]$$

In the above expressions, the angles α and β can be found by the relations :

$$\tan\alpha = \frac{ka}{f}, \text{ and } \tan\beta = \frac{kb}{f}.$$

It can be shown that equations [14.25 (a,b)] can be represented by the approximate formula

$$d = \frac{(ia)^2 \sin t}{f} \quad ...(14.26)$$

It is quite clear from the figure that the tilt displacement of a point on the upward half of a tilted photograph is *inward* (such as for point a) while the tilt displacement of a point on the downward or nadir point half is *outward* (such as for b)

Equations 14.25 give the tilt displacements for the points on the principal line. The tilt displacement of a point not lying on the principal line is greater than that of a corresponding point on the principal line.

Let I = angle measured at the isocentre from the principal line to the point.

d_u = displacement of the point on the upward half of the tilted photograph.

d_d = displacement of the point on the downward half of the tilted photograph.

In Fig. 14.28 (plan), the point q is not on the principal line while point a is on the principal line. qq' is the displacement of q while aa' is the displacement of point a. Since both q and a are equidistant from the axis of tilt, we have

$$qq' = aa' \sec I$$

where I is the angle at the isocentre from the principal line to the point q.

Hence the ratio of the tilt displacement of a point not on the principal line to that of a point on the principal line is equal to the secant of the angle at the isocentre from the principal line to the point.

Thus, the expressions for d_u and d_d can be written as :

$$d_u = f\sec I[\tan(t+\alpha) - \tan\alpha - 2\tan t/2] \quad ...[14.27(a)]$$

$$d_d = f\sec I[\tan\beta - \tan(\beta - t) - 2\tan t/2] \quad ...[14.27(b)]$$

In Fig. 14.28 (plan), $p'q'r's'$ represents a square on the vertical photograph. The corresponding displaced points on the tilted photographs are p, q, r and s. Since the tilt displacements are always radial from the isocentre, the corresponding figure $p\,q\,r\,s$ becomes a rhombus.

14.19. RELIEF DISPLACEMENT ON A TITLED PHOTOGRAPH

It has been shown in § 14.14 that the relief displacement on a vertical photograph is radial from the principal point. The points are displaced radially outward from their datum photograph positions. Fig. 14.29 shows the relief displacement on a tilted photograph.

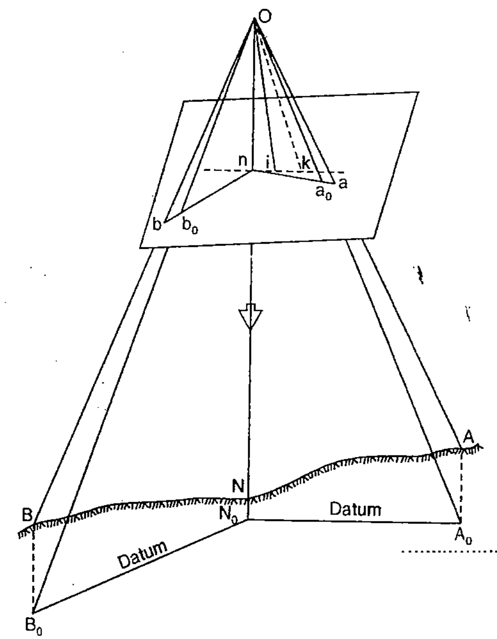


FIG. 14.29. RELIEF DISPLACEMENT ON A TILTED PHOTOGRAPH.

A, B and N are ground points, and A_0, B_0, N_0 are their corresponding datum positions. N and N_0 being vertically below the nadir point n. A and B are imaged at a and b respectively, a_0 and b_0 are the datum photograph positions of A_0 and B_0. i is the isocentre and k is the principal point. The plane $O\,N\,N_0\,A_0\,A$ is a vertical plane since it contains the plumb line ON. The points n, a_0 and a lie on the same vertical plane. Since the points n, a_0 and a also lie on the photograph, they are in the same line, *i.e.*, a_0 and a lie on a radial line from the nadir point. Similarly, the point n, b_0 and b are on the same line, and b_0 and b are radial from the nadir point. *Thus, on a tilted photograph, the relief displacement is radial from the nadir point.* The amount of relief displacement on a tilted photograph depends upon : (*i*) flying height, (*ii*) distance of the image from the nadir point, (*iii*) elevation of the ground point, and (*iv*) position of the point with respect to the principal line and to the axis of tilt. In the case of near vertical photograph, where the tilt is less than 3°, the relief displacement can be calculated from equations 14.19 with the modification that the radial distances r and r' are measured from the nadir point and *not* from the principal point.

$\therefore$ Thus $$d = \frac{rh}{H} \quad ...[14.28(a)] \qquad \text{and} \qquad d = \frac{r_0 h}{H - h} \quad ...[14.28(b)]$$

where d is the relief displacement on a tilted photograph,

r = radial distance of image point from the photographic nadir.

r_0 = radial distance of datum image point from the photographic nadir.

14.20. COMBINED EFFECTS OF TILT AND RELIEF

It has been shown in § 14.18 that on a tilted photograph covering the ground with no relief, the tilt distortion is radial from the isocentre. In the previous article, it has been shown that the relief distortion is radial from the nadir point. To study the combined effect on the tilt and relief, let us refer Fig. 14.30.

Fig. 14.30 shows the displacements of five points A, B, C, D and E in typical positions. a_0, b_0, c_0, d_0, e_0, are their corresponding datum photograph positions. a', b', d', e' are corresponding positions after the image has undergone relief displacement. a, b, c, d and e are the corresponding positions after the image has undergone tilt displacement.

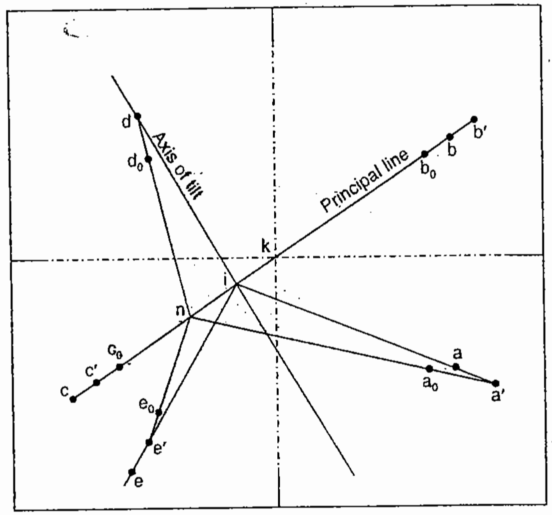


FIG. 14.30. COMBINED EFFECT OF TILT AND RELIEF DISPLACEMENT.

For the point A, the relief displacement is $a_0 a'$ *radially outward* from the nadir point and the tilt displacement is *$a'a$ radially inward* to the isocentre as it lies in the *upper* part of the photograph. Thus, the relief displacement and the tilt displacement tend to compensate each other.

For the point B, the relief displacement $b_0 b'$ is radially outward from the nadir point, and the tilt displacement $b' b$ is radially inward to the isocentre as it lies in the upper part of the photograph. The position of the point has been so chosen that b_0, b' and b lie on the principal line. Here also, both the displacements tend to compensate each other.

For the point D, the relief displacement $d_0 d'$ is radially outward from the nadir point, and the tilt displacement $d'd$ is zero since the image d' happens to fall on the axis of tilt along which there is no tilt displacement.

For the point C, the relief displacement $c_0 c'$ is radially outward from the nadir point while the tilt displacement $c'c$ is radially outward from the isocentre since it lies in the lower part of the photograph. The position of the point has been so chosen that c_0, c' and c lie on the principal line. The relief displacement and the tilt displacement are cumulative.

Lastly, for the point E, the relief displacement $e_0 e'$ is radially outward from the nadir point and the tilt displacement $e'e$ is radially outward from the isocentre since it lies in the lower part of the photograph. Here also, both the displacements are cumulative.

Thus, it can be concluded that the tilt and relief displacements tend to cancel in the upper part of the photograph while they are cumulative in the lower part.

In actual practice, the effects of tilt can be analysed only where precise equipment and trained personnel are available. These effects are more often removed by re-photographing the prints with the aid of accurately established control points in the photograph. Within certain limits of permissible errors, the effect of tilt can be eliminated by means of various projectors. In spite of scale variation, relief displacement and tilt displacement, an aerial photograph taken with a calibrated precision aerial camera is precise perspective view of the terrain, and precise measurements and highly accurate results may be obtained from it.

14.21. FLIGHT PLANNING FOR AERIAL PHOTOGRAPHY

When vertical photographs are to be used for the preparation of maps, all the methods of compilation require that the plumb points of the preceding and succeeding prints are visible in each photograph. Photographs are taken at the proper interval along each strip to give the desired overlap of photographs in the given strip. Each strip is spaced at pre-determined distances to ensure desired side lap between adjacent strips.

The overlap of photographs in the direction of flight line is called *longitudinal overlap or forward overlap* or simply overlap. Along a given flight line, photographs are taken at such frequency as to cause successive photographs to overlap

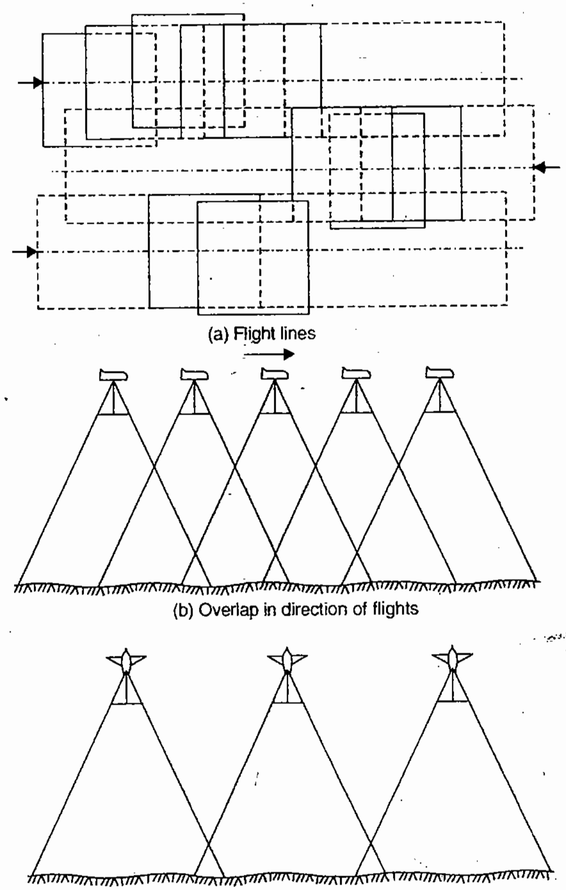


FIG. 14.31. THE OVERLAP AND SIDE LAP OF PHOTOGRAPHS.

each other by 55 to 65 per cent. Fig. 14.31 (*a*) shows three successive flight lines. Fig. 14.31 (*b*) shows the vertical section containing the flight line and showing the overlap. Since the overlap is more than 50 per cent, alternate photographs will overlap one another by 10 to 30 per cent. When photographs are taken with this overlap, the entire area may be examined stereoscopically. The overlap between adjacent flight lines is known as *lateral overlap or sidelap*. The sidelap amounts to about 15 to 35 per cent. Fig. 14.31 (*c*) shows the vertical section taken normal to the three flight lines of Fig. 14.31 (*a*).

The number of individual photographs required to cover a given area increases with the increase in the overlap and sidelap, thus increasing the amount of work both in the field as well as in the office.

Reasons for Overlap

The following are some of the reasons for keeping overlap in the photographs :

To tie the different prints together accurately, it is desirable that the principal point of each print should appear on the edges of as many adjacent strips as possible.

(2) The distortions caused by the lens and by the tilt, and the relief displacements are more pronounced in the outer part of the photograph than near the centre of each photograph. If the overlap is more than 50%, these distortions and displacements can be overcome quite effectively while constructing the maps.

(3) In order to view the pairs of photographs stereoscopically, only the overlapped portion is useful. Hence the overlap should at least be 50%.

(4) Due to the overlap, each portion of the territory is photographed three to four times. Hence any picture distorted by excessive tilt or by cloud shadows etc. can be rejected without the necessity of a new photograph.

(5) If the flight lines are not maintained straight and parallel, the gaps between adjacent strips will be left. These gaps can be avoided by having sidelap.

(6) In the stereoscopic examination, objects can be viewed from more than one angle if sufficient overlap is provided.

Fig. 14.32 shows a photographic flight with an automatic aerial camera, the overlap of successive vertical photographs being 60%.

EFFECTIVE COVERAGE OF THE PHOTOGRAPH

The amount of overlap and sidelap to be used in flight planning depends upon the effective coverage of each photograph. The relation between the separation of flight lines and the separation between photographs must be arranged to give the greatest area to each stereopair.

The effective coverage of each photograph depends upon (*i*) size of format or focal plane opening. (*ii*) focal length and (*iii*) angular coverage of the lens. The effective angular coverage of the lens with the 12 in. (30.4 cm) focal length is represented by a cone the apex of which lies at the front nodal point and the apex angle of which is about 60°. In general, the effective coverage with a 12 in. lens will embrace more than 9″ × 9″ format size, and hence the entire photograph is usable (Fig. 14.33 *a*). The effective angular coverage with a 6″ (15.2 cm) wide angle lens is a cone of rays the apex of which is about 86°. A sizeable portion of the 9″ × 9″ format is not usable, and the useful

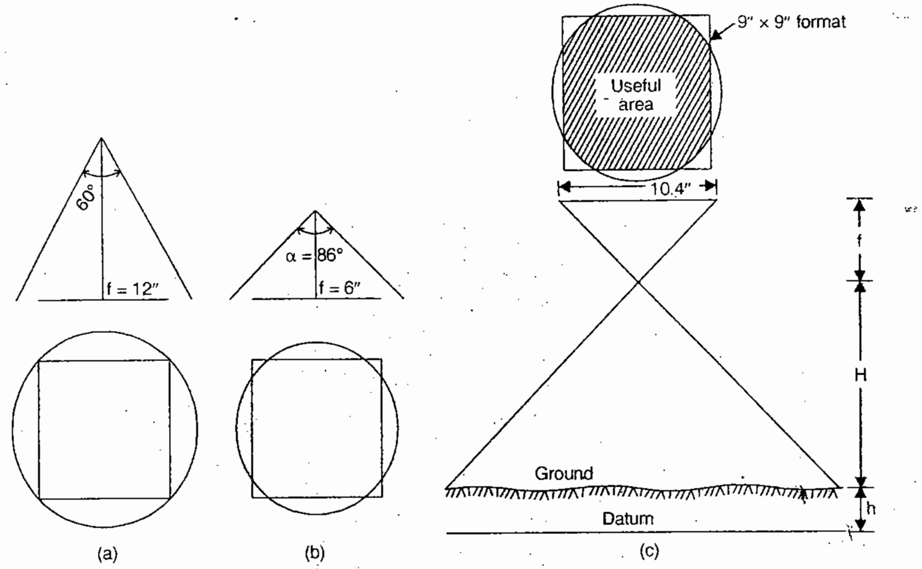


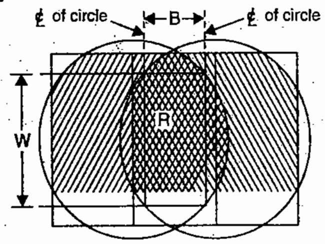
FIG. 14.33. ANGULAR COVERAGE.

circle at the negative plane is equal to $2f \tan \frac{\alpha}{2} = 11.2''$ approximately. Due to errors in directing the camera and in following the flight lines, this should be reduced to at least 10.4 in., as shown in Fig. 14.33 (*c*).

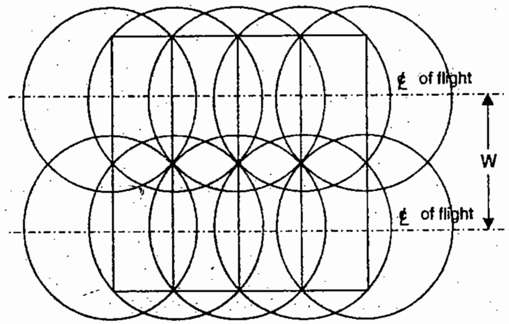
The effective area of overlap between the two photographs is that bound by the overlapping circles representing the effective coverage of the photographs. Since the stereomodels must fit each other, the useful steroareas must be assumed to be rectangles having a width equal to the interval *B* between exposures. the two longer sides of this rectangle pass through the principal points of the photographs. The stereoareas is shown cross hatched, and the largest rectangle possible is drawn within this area.

Let W = distance between the flight strips

A_s = stereo-areas (*i.e.*, area of the rectangle)



(a) Effective overlap



(b) Spacing between flight lines

FIG. 14.34. FLIGHT LINES AND INTERVALS.

Then, $A_s = BW$ ...(1)

But $\frac{1}{2} W = \sqrt{R^2 - B^2}$ ...(2)

$\therefore$ $A_s = 2B\sqrt{R^2 - B^2}$ For A_s to be maximum $\frac{dA_s}{dB} = O = R^2 - 2B^2$

which gives $R = B\sqrt{2}$ ...(3)

Hence when $R = 5.2$ in., the value of $B = 3.67$ in.

Overlap in terms of inches on the photograph = 9.00 – 3.67 = 5.33″

$\therefore$ % overlap = 5.33/9 = 59.2% in the direction of flight.

Again, substituting the value of $R = B\sqrt{2}$ in (2), we get

$$\frac{1}{2} W = \sqrt{2B^2 - B^2} = B$$

$\therefore$ $W = 2B$ ...(4) ...(14.29)

Hence for maximum rectangular area, the rectangle must have the dimension in the direction of flight to be one-half the dimension normal to the direction of flight.

From Fig. 14.33 (*c*), $\frac{2R}{H} = \frac{10.4}{6}$ or $R = 0.867\,H$

Substituting this in (2), we get

$$B = \frac{1}{\sqrt{2}} R = \frac{1}{\sqrt{2}} \times 0.867\,H = 0.61\,H$$

Substituting in (4), $W = 1.22\,H$

where H is the height of lens above ground.

Hence the distance between the successive flights equals to 1.22 times the height of flight above the ground. This is the *maximum* allowable distance when the principal point of the photographs fall directly opposite one another on the two flight lines.

As found earlier, $W = 2B = 2 \times 3.67 = 7.34$ in.

$\therefore$ Side lap between flight lines, in terms of inches on the photograph = 9 – 7.34 = 1.66″

$\therefore$ Side lap = 1.66/9 = 18.4%

If $H = 3000$ metres, $B = 0.61 \times 3000 = 1830$ m and $W = 1.22 \times 3000 = 3660$ m

Hence, an exposure should be taken at every 1830 m and the separation of flight strips should be 3660 m.

The above analysis presumes ideal conditions, *i.e.* (*i*) level terrain, (*ii*) vertical photographs, (*iii*) no crab, (*iv*) no drift of the air craft and (*v*) constant flying height. The flight path centre lines are laid out parallel to the longest dimension of the area, on any existing map. Unless the area to be mapped is exactly covered by a certain number of flight paths spaced at the computed value of W, W should be reduced to introduce one more flight path to utilize the excess of photography for increasing the side overlap.

SELECTION OF FLYING ALTITUDE

The selection of height above ground depends upon the accuracy of the process to be used and the contour interval desired. Several inter-related factors which affect the selection

of flying height, such as desired scale, relief displacement, and tilt, have already been discussed, Since vertical accuracy in a topographic map is the limiting factor in the photogrammetric process, the flying height is often related to the contour interval of the finished map. The process is rated by its *C*-factor which is the number by which the contour interval is multiplied to obtain the maximum height about the ground.

Thus, ***Flying height*** = (*Contour interval*) × (*C factor*)

C-factor for various processes vary from 500 to 1500, and depends upon the conditions surrounding the entire map-compilation operation.

NUMBER OF PHOTOGRAPHS NECESSARY TO COVER A GIVEN AREA

In the preliminary estimate, the number of photographs required is calculated by dividing the total area to be photographed by the net area covered by a single photograph.

Let A = total area to be photographed

l = length of the photograph in the direction of flight

w = width of the photograph normal to the direction of flight

s = scale of photograph = $\frac{H\,(\text{m})}{f\,(\text{cm})}$ (*i.e.* 1 cm = s metres)

L = net ground distance corresponding to l

W = net ground distance to corresponding to w

a = net ground area covered by each photograph = $L \times W$

P_l = percentage overlap between successive photographs in the direction of flight (expressed as a ratio)

P_w = side lap (expressed as a ratio).

Since each photograph has a longitudinal lap of P_l, the actual ground length (L) covered by each photograph is given by

$$L = (1 - P_l)\,sl \quad ...(i)$$

Similarly, the actual ground width (W) covered by each photograph is given by

$$W = (1 - P_w)\,sw \quad ...(ii)$$

Hence the ground area (a) covered by each photograph

$$a = L.W = (1 - P_l)\,sl\,(1 - P_w)\,s\,.\,w = l\,.\,w\,s^2\,(1 - P_l)\,(1 - P_w) \quad ...[14.30\ (a)]$$

The number of the photographs (N) required is given by

$$N = A/a \quad ...(14.30)$$

If, however, instead of the total area A, the rectangular dimensions (*i.e.*, length and width) of the ground are given, the number of the photographs required are computed by calculating the number of strips and the number of photographs required in each strip and multiplying the two.

Let L_1 = dimension of the area parallel to the direction of flight

L_2 = dimension of the area normal to the direction of flight

N_1 = number of photographs in each strip

N_2 = number of strips required

N = total number of photographs to cover the whole area.

Now net length covered by each photograph = $L = (1 - P_l)\,sl$

$\therefore$ Number of photographs in each strip is given by

$$N_1 = \frac{L_1}{L} + 1 = \frac{L_1}{(1 - P_l)\,sl} + 1 \quad \ldots[14.31(a)]$$

Similarly, net width covered by each photograph = $W = (1 - P_w)\,sw$

Hence the number of the strips required are given by

$$N_2 = \frac{L_2}{W} + 1 = \frac{L_2}{(1 - P_w)\,sw} + 1 \quad \ldots[14.31(b)]$$

Thus, the number of photographs required is

$$N = N_1 \times N_2 = \left\{ \frac{L_1}{(1 - P_l)\,sl} + 1 \right\} \times \left\{ \frac{L_2}{(1 - P_w)sw} + 1 \right\} \quad \ldots(14.31)$$

INTERVAL BETWEEN EXPOSURES

The time interval between the exposures can be calculated if the ground speed of the airplane and the ground distance (along the direction of flight between exposures are known.

Let V = ground speed of the airplane (km/hour)

L = ground distance covered by each photograph in the direction of flight $= (1 - P_l)\,sl$ in km

T = time interval between the exposures.

Then $$T = \frac{3600\,L}{V} \quad \ldots(14.32)$$

The exposures are regulated by measuring the time required for the image of a ground point to pass between two lines on a ground-glass plate of the view-finder. Usually, however, the interval is not calculated, but the camera is tripped automatically by syncronising the speed of a moving grid in the view-finder with the speed of the passage of images across a screen.

CRAB AND DRIFT

Crab. Crab is the term used to designate the angle formed between the flight line and the edges of the photograph in the direction of flight, as shown in Fig. 14.35 (*a*). At the instant of exposure, if the focal plane of the camera is not square with the direction of flight, the crab is caused in the photograph. The arrangements are always made to rotate the camera about the vertical axis of camera mount. Crabbing should be eliminated since it reduces effective coverage of the photograph.

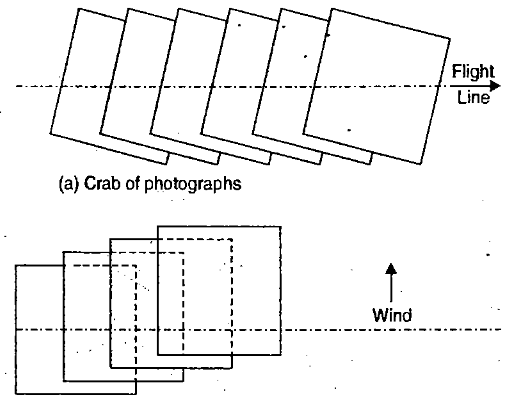


FIG. 14.35. CRAB AND DRIFT

Drift. Drift is caused by the failure of the photograph to stay on the predetermined flight line. If the aircraft is set on its course by compass without allowing for wind velocity, it will drift from its course, and the photographs shall be as shown in Fig. 14.35 (*b*). If the drifting from the predetermined flight line is excessive, reflights will have to be made because of serious gapping between adjacent flight lines.

COMPUTATION OF FLIGHT PLAN

For the computation of the quantities for the flight plan, the following data is required:

1. Focal length of the camera lens
2. Altitude of the flight of the aircraft
3. Size of the area to be photographed
4. Size of the photograph
5. Longitudinal overlap
6. Lateral overlap
7. Position of the outer flight lines with respect to the boundary of the area
8. Scale of the flight map
9. Ground speed of aircraft.

Knowing the above, the amount of film required can be calculated before hand, the flight lines can be delineated on the map and the time interval between exposures can be determined if an intervalometer is to be used.

Example 14.13. *The scale of an aerial photograph is 1 cm=100 m. The photograph size is 20 cm × 20 cm. Determine the number of photographs required to cover an area of 100 sq. km if the longitudinal lap is 60% and the side lap is 30 %.*

Solution.

Here $l = 20$ cm ; $w = 20$ cm ; $P_l = 0.60$; $P_w = 0.30$

$$s = \frac{H\,(\text{m})}{f\,(\text{cm})} = 100 \quad (i.e.\ \ 1 \text{ cm} = 100 \text{ m})$$

$\therefore$ The actual ground length covered by each photograph is

$$L = (1 - P_l)\,sl = (1 - 0.6)100 \times 20 = 800 \text{ m} = 0.8 \text{ km}$$

Actual ground width covered by each photograph is

$$W = (1 - P_w)sw = (1 - 0.3)100 \times 20 = 1400 \text{ m} = 1.4 \text{ km}$$

$\therefore$ Net ground area covered by each photograph is

$$a = L \times W = 0.8 \times 1.4 = 1.12 \text{ sq. km.}$$

Hence number of photographs required is

$$N = \frac{A}{a} = \frac{100}{1.12} = 90$$

Example 14.14. *The scale of an aerial photography is 1 cm = 100 m. The photograph size is 20 cm × 20 cm. Determine the number of photographs required to cover an area 10 km × 10 km, if the longitudinal lap is 60% and the side lap is 30%.*

Solution.

Here $L_1 = 10$ km ; $L_2 = 10$ km

$\therefore$ Number of photographs in each strip is given by

$$N_1 = \frac{L_1}{(1-P_l)\,sl} + 1 = \frac{10{,}000}{(1-0.6)\times 100\times 20} + 1 = 12.5 + 1 \approx 14$$

Number of flight lines required is given by

$$N_2 = \frac{L_2}{(1-P_w)\,sw} + 1 = \frac{10{,}000}{(1-0.3)\,100\times 20} + 1 = 7.6 + 1 \approx 9$$

Hence number of photographs required will be

$$N = N_1 \times N_2 = 14 \times 9 = \mathbf{126}$$

The spacing of the flight lines would be 10/9 = 1.11 km and not 1.4 as calculated theoretically in the previous example.

Example 14.15. *The scale of an aerial photograph is 1 cm = 100 m. The photograph size is 20 cm × 20 cm. Determine the number of photographs required to cover an area of 8 km × 12.5 km, if the longitudinal lap is 60% and the side lap is 30%.*

Solution.

$$N_1 = \frac{12500}{(1-0.6)\times 100\times 20} + 1 = 17$$

$$N_2 = \frac{8000}{(1-0.3)\,100\times 20} + 1 = 7$$

$\therefore$ Number of photographs = 17 × 7 = **119.**

Example 14.16. *An area 30 km long in the north-south direction and 24 km in the east-west direction is to be photographed with a lens having 30 cm focal length for the purpose of constructing a mosaic. The photograph size is 20 cm × 20 cm. The average scale is to be 1 : 12,000 effective at an elevation of 400 m above datum. Overlap is to be atleast 60% and the side lap is to be at least 30%. An intervalometer will be used to control the interval between exposures. The ground speed of the aircraft will be maintained at 200 km per hour. The flight lines are to be laid out in a north-south direction on an existing map having a scale of 1 : 60,000. The two outer flight lines are to coincide with the east and west boundaries of the area. Determine the data for the flight plan.*

Solution.

(*i*) ***Flying height***

We have,

$$\frac{H\,(\text{m})}{f\,(\text{m})} = \frac{H\,(\text{m})}{0.3\,(\text{m})} = \frac{12{,}000}{1}$$

$\therefore$ $H = 12{,}000 \times 0.3 = 3600$ m above ground

$\therefore$ Height above datum = 3600 + 400 = **4000 m.**

(*ii*) ***Theoretical ground spacing of flight lines***

The ground width covered by each photograph, with 30% side lap is given by

$$W = (1 - P_w)\,sw.$$

where w = width of photograph = 20 cm ; s = scale = $\frac{H\,(\text{m})}{f\,(\text{m})} = \frac{3600\,(\text{m})}{30\,(\text{cm})} = \mathbf{120}$

i.e., 1 cm = 120 m ; $P_w = 0.30$

$$W = (1-0.3)\,120 \times 20 = \mathbf{1680\ m.}$$

(*iii*) ***Number of flight lines required***

The number of flight lines is given by Eq. 14.31 (*b*), *i.e.*

$$N_2 = \frac{L_2}{(1-P_w)\,sw} + 1 = \frac{L_2}{W} + 1 = \frac{24{,}000\ \text{m}}{1680\ \text{m}} + 1 = 14.2 + 1 \approx \mathbf{16}.$$

(*iv*) ***Actual spacing of flight lines*** : Since the number of flight lines is to be an integral number, the actual flight lines = 16 and the number of flight strips or spacings = 15. Hence the actual spacing is given by

$$W = \frac{24{,}000}{15} = \mathbf{1600\ m,}$$ against the theoretically calculated value of 1680 m.

(*v*) ***Spacing flight lines on flight map***

Flight map is on a scale of 1 : 60,000 or 1 cm = 600 m. Hence the distance on the flight map corresponding to a ground distance $= \frac{1600}{600} = \mathbf{2.67\ cm.}$

(*vi*) ***Ground distance between exposures***

The ground length covered by each photograph in the direction of flight with an overlap of 60% is given by $L = (1 - P_l)\,sl = (1 - 0.6) \times 120 \times 20 = \mathbf{960\ m.}$

(*vii*) ***Exposure interval***

The time interval between exposures is usually the integral number of seconds.

$$V = 200 \text{ km per hour} = \frac{200 \times 1000}{60 \times 60} \text{ m/sec.} = 55.56 \text{ m/sec.}$$

$\therefore$ The required exposure interval is $\frac{960\ (\text{m})}{55.5\ (\text{m/sec})} = 17.3$ sec. $\simeq$ **17 sec.**

(*viii*) ***Adjusted ground distance between exposures***

Keeping the exposure interval as an integral number of seconds the adjusted ground distance covered by each photograph is given by

$$L = V \times T = 55.56\ (\text{m/sec}) \times 17.0\ (\text{sec}) = \mathbf{945\ m.}$$

(*viii*) ***Number of photographs per flight line***

The number of photographs per flight line is given by

$$N_1 = \frac{L_1}{(1-P_l)\,sl} + 1 = \frac{L_1}{L} + 1 = \frac{30{,}000}{945} + 1 = 31.6 + 1 \approx \mathbf{33}.$$

(*ix*) ***Total number of photographs required is***

$$N = N_1 \times N_2 = 33 \times 16 = \mathbf{528}.$$

14.22. THE GROUND CONTROL FOR PHOTOGRAMMETRY

The ground control survey consists in locating the ground positions of points which can be identified on aerial photographs. The ground control is essential for establishing the position and orientation of each photograph in space relative to the ground. The extent of the ground control required is determined by (*a*) the scale of the map, (*b*) the navigational control and (*c*) the cartographical process by which the maps will be produced. The ground survey for establishing the control can be divided into two parts :

(*a*) Basic control (*b*) Photo control.

The *basic control* consists in establishing the basic net-work of triangulation stations, traverse stations, azimuth marks, bench marks etc.

The *photo control* consists in establishing the horizontal positions or elevations of the images of some of the identified points on the photographs, with respect to the basic control.

Each of these controls introduces *horizontal* control as well as *vertical* control and is known as basic horizontal control, basic vertical control, horizontal photo control, and vertical photo control respectively. The elevation of a vertical photo control point is determined by carrying a line of levels from a basic control bench mark to the point, and then carrying to the original bench mark or to a second bench mark for checking. The horizontal photo control points are located with respect to the basic control by third order or fourth-order triangulation, third order traversing, stadia traversing, trigonometric traversing, substance-bar traversing etc. etc., depending upon the accuracy required. Vertical photo control may be established by third-order leveling, fly levelling, transit-stadia levelling or precision barometric altimetry etc., depending upon the desired accuracy.

The photo control can be established by two methods :

(*i*) Post-marking method

(*ii*) Pre-marking method.

In the *post-marking* method, the photo control points are selected after the aerial photography. The distinct advantage of this method is in positive identification and favourable location of points.

In the pre-marking method, the photo-control points are selected on the ground first, and then included in the photograph. The marked points on the ground can be identified on the subsequent photograph. If the control traverse or triangulation station or bench marks are to be incorporated in the photo-control net work, they are marked with paint, flags etc. in such a way that identification on the photographs becomes easier. The selected control points should be sharp and clear in plan.

14.23. RADIAL LINE METHOD OF PLOTTING (ARUNDEL'S METHOD)

The radial line plot, often called *photo-triangulation* is the most accurate means of plotting a planimetric map from aerial photographs without the use of expensive instruments.

As discussed earlier, the displacement of image due to relief is radial from the principal point of vertical photograph. Hence the angles measured on the photograph at the principal point are true horizontal angles, independent of the height of the object and the flying height. The vertical photograph in space can thus be considered as an angle-measuring device similar to a transit or a plane table. Also, on tilted photographs, angles measured at the isocentre are true horizontal angles independent of tilt, provided that all objects photographed have the same elevation. On a near-vertical photograph, the isocentre is very near to the principal point. Hence the angles measured in the vicinity of these points are very nearly equal to the true horizontal angles, independent of tilt or elevation.

Thus, the radial line method is based on the following perspective properties of a vertical or near vertical photograph :

1. The displacements in a photograph due to ground relief and tilt are, within the limit of graphical measurement, radial from the principal point of the photograph.

2. The images near the principal point are nearly free from errors of tilt, and they are shown in their true orthographic positions, regardless of ground relief.
3. The position of a point included in properly overlapping photographs may be located on the map where three rays from three known points intersect.

Principles of Radial Line Resection and Intersection

Before discussing the procedure for preparing planimetric map from aerial photographs, let us study the principle of radial line resections and intersections. The principle can be best illustrated by the following two problems :

(*a*) To locate the principal point of photographs on a map.

(*b*) To transfer images from a photograph to a map.

(*a*) TO LOCATE THE PRINCIPAL POINT OF PHOTOGRAPHS ON A MAP

A map represents the true horizontal positions of all points at the map scale which is uniform. The map position of principal point of vertical photograph can be located either by (*i*) three point resection or by (*ii*) two point resection.

(*i*) **Map position of principal point by three point resection**

Let *a, b, c* be three photo-control points appearing in both the photographs (Fig. 14.36 *a*), and *A, B* and *C* be their map positions already known by ground survey. It is required to know the map position of the principal points k_1 and k_2 by 3-point resection.

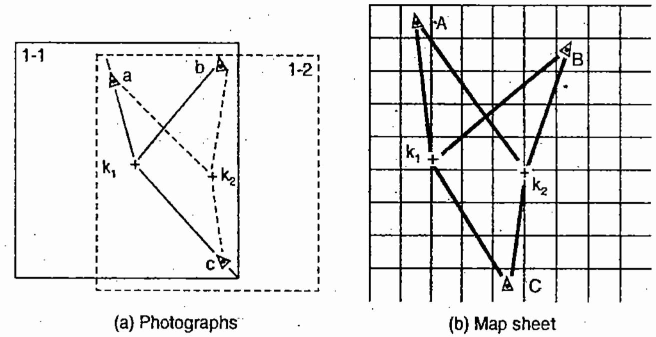


FIG. 14.36. LOCATION OF PRINCIPAL POINTS BY 3-POINT RESECTION.

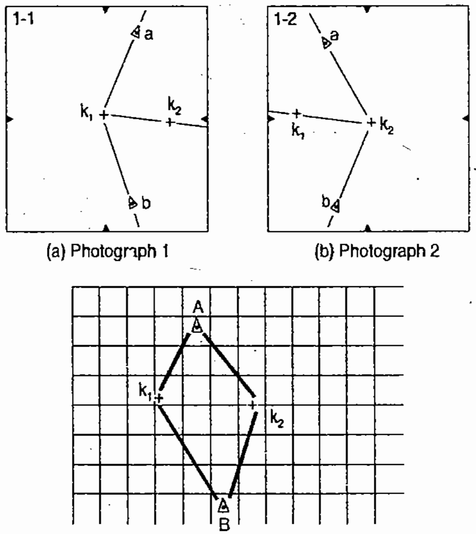
On photograph No. 1 draw rays k_1a, k_1b and k_1c. Evidently, angles ak_1b and bk_1c are the true horizontal angles. Similarly, on photograph No. 2, draw rays k_2a, k_2b and k_2c. A piece of tracing paper is put on photograph No. 1, and the rays are traced. The tracing paper is now put on the map sheet and is oriented in such a way that all the three rays simultaneously pass through the plotted positions *A, B* and *C*. The point of intersection of the three rays is the true map position of the principal point k_1. The principal point k_2 of the second photograph can also be located in a similar manner. A three armed protractor can also be used in the place of a tracing paper.

(*ii*) **Map position of principal points by two point resection**

Let a and b be two photo-control points appearing in both the photographs overlapping each other. A and B are the plotted positions on the map sheet.

On photograph No. 1 (Fig. 14.37 *a*), k_1 is the principal point and k_2 is the principal point of photograph No. 2 transferred on to it. Rays k_1a, k_1b and k_1k_2 are drawn. A tracing paper is put on it and the rays are traced.

On photograph No. 2 (Fig. 14.37 *b*), k_2 is its principal point and k_1 is the principal point of photograph No. 1, transferred on to it. Rays k_2a, k_2b and k_2k_1 are drawn. A tracing paper is put on it and the rays are traced. Both the tracing papers are laid together on the map sheet in such a way that the rays k_1k_2 and k_2k_1 coincide when one is placed over the other. The rays k_1a and k_2a will intersect at a and the rays k_1b and k_2b will intersect at b. The two tracing papers are moved now in such a way that these intersections coincide with A and B respectively, and at the same time the lines k_1k_2 and k_2k_1 coincide each other. The points k_1 and k_2 are then transferred to the map sheet by pricking through. The line k_1k_2 on the map sheet constitutes the base line with k_1 and k_2 as the instrument stations.



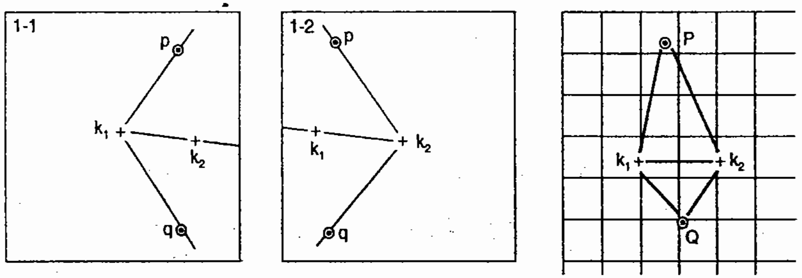
(a) Photograph 1 (b) Photograph 2

(c) Map sheet

FIG. 14.37. LOCATION OF PRINCIPAL POINTS BY 2-POINT RESECTION.

(*b*) **TO TRANSFER IMAGES FROM A PHOTOGRAPH TO A MAP**

Since the angles measured on the photograph at the principal point are true horizontal angles, the position of a point can be located by intersecting the rays to that point from two principal points. In Fig. 14.38, let p and q be the images of two points on two



(a) Photograph 1 (b) Photograph 2 (c) Map sheet

FIG. 14.38. LOCATION OF POINTS BY INTERSECTION.

photographs overlapping each other. k_1 and k_2 are the two principal points. As discussed in the previous paragraph, the principal points k_1 and k_2 can be transferred to the map by the known positions of the photo control points. On each photograph, the rays can be drawn to the points p and q and can be traced on two sheets of tracing paper. Both the sheets are then placed on the map sheet and properly oriented till the map positions of k_1 and k_2 coincide respectively with the traced positions. The intersection of rays k_1p and k_2p gives the position of P, and that of k_1q and k_2q give the position of Q.

The actual plotting of planimetric maps by radial line method is done in the following steps :

(1) Transfer of principal points and plotting the line of flight

(2) Marking the photographs

(3) Plotting the map control

(4) Transferring photographic detail.

(1) Transfer of Principal Points and Plotting the Line of Flight

The principal point of each photograph can be marked on it by means of two intersecting lines drawn between the opposite fiducial marks which are there on the middles of all the four edges of the photograph. The point of intersection of these two collimating lines is the principal point of the photograph. Each photograph is given its serial number and the number of the strip in which it was taken. For example, photograph Nos. 7, 8 and 9 of strip No. 2 will be marked as 2-7, 2-8 and 2-9 respectively. The principal points of these photographs may be marked as k_7, k_8 and k_9 respectively.

Since the longitudinal lap is generally 60% or more, the three photographs will have common overlap of atleast 20% as shown shaded in Fig. 14.39.

If all the photographs of a particular strip are arranged in properly overlapped positions, it will be observed that on the first photograph, its principal point will appear at its middle while the transferred principal point of photograph No. 2 will appear at right hand edge. On photograph No. 2 and all other photographs except the last, three principal points will appear — one of its own at the middle and two at its two edges, as shown in Fig. 14.39 where photograph No. 2 has the principal points k_1, k_2 and k_3. The principal points of each photograph can be transferred on to the adjacent photographs, one to its right and one to the left of it, by fusion under a stereoscope (see § 14.24 for principles of stereoscopic vision and fusion). For this, two adjacent photographs are put under the stereoscope and are oriented correctly with respect to each other till the line of flight of the pictures is parallel to the line joining the centres of two lenses of the stereoscope. The distance between the photographs is adjusted until fusion occurs and the relief of the landscape is clearly visible. In this position, the principal point of one photograph will be seen directly and its image will be projected upon the other photograph. Then with a needle the position of the point can be transferred to the adjacent

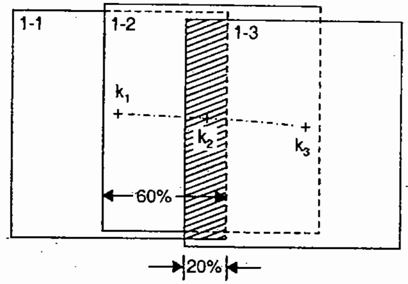


FIG. 14.39

photograph. The line joining the principal points then gives the direction of flight, which can be marked.

(2) Marking the Photographs

Before plotting the map control, each photograph is *marked* by selecting some points on it and drawing radial lines to them from the principal points. For this purpose, let us consider three consecutive photographs 1-1, 1-2 and 1-3 of the first strip, as shown in Fig. 14.40.

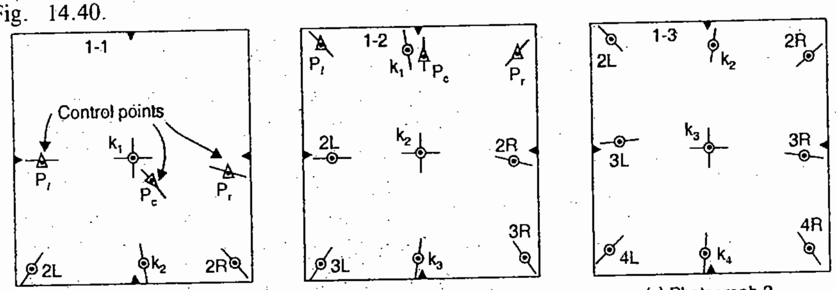


FIG. 14.40. MARKING THE PHOTOGRAPHS

On photograph No. 1, k_1 is its principal point while k_2 is the transferred principal point of photograph No. 2. The images of three ground-control points are identified and marked with needle points at P_l, P_c and P_r. Each of the control-points are enclosed in small triangles drawn with a soft coloured pencil or ink. In addition to three control-points, two additional *pass points* $2L$ and $2R$ are selected at the edge of photograph No. 1 and approximately in line with the transferred principal point k_2 of the second photograph. Thus, on photograph No. 1, in addition to its principal point k_1, there are six more points marked: P_l, P_c, P_r, $2L$, k_2 and $2R$. Short radial lines from the principal point of photograph No. 1 are then drawn through each of these six points. This completes the markings of photograph No. 1.

On photograph No. 2, k_2 is its principal point, while k_1 and k_3 are the transferred principal points of photograph Nos. 1 and 3 respectively. The ground control point P_l, P_c and P_r appear at its edge and nearly in line with k_1. The pass points $2L$ and $2R$ appear approximately in line with k_2. In addition to these, two additional pass points $3L$ and $3R$ are selected at its edge and approximately in line with the transferred principal points k_3 of photograph No. 3. Short radial lines from the principal point of photograph No. 2 are then drawn through each of the points marked on it. This completes the marking of photograph No. 2.

On photograph No. 3, k_3 is its principal point, while k_2 and k_4 are then transferred principal points of photograph Nos. 2 and 4 respectively. The pass points $2L$ and $2R$ appear at its upper edge and in line with k_2. The pass points $3L$ and $3R$ appear in line with k_3. In addition to these, two additional pass points $4L$ and $4R$ are selected at its lower edge and approximately in line with the transferred principal point k_4. Short radial lines from the principal point of photograph No. 3 are then drawn through each of the point marked on it. This completes the marking photograph No. 3.

Each of the succeeding photograph is marked in a similar manner until other ground control points are reached. The end photograph of the strip must include at least one control point.

(3) Plotting the Map Control : The data of the separate photographs are combined into a map showing correct relative locations of the selected points and the control points with the help of a sheet of transparent film base (cellulose acetate) or a good quality tracing paper which exhibits very small changes in its dimensions with changing atmospheric conditions.

The plotted positions of ground control points P_l, P_c and P_r chosen on photographs 1 and 2 are known on the base map. The tracing is stretched on the base map and these control points are transferred by pricking through with a needle. Photograph No. 1 is then slid under the tracing and is oriented in such a way that the radial lines through points P_l, P_c and P_r of the photograph pass through the plotted control points P_l, P_c, P_r on the tracing. In this position, all the rays and points are traced. The principal point k_1 and the transferred principal point k_2 are also traced.

Photograph No. 2 is then slid under the tracing and is oriented in such a way that rays previously drawn on the tracing pass through the corresponding points on the photograph, keeping the traced flight line $k_1 k_2$ coinciding with flight line $k_2 k_1$ on the photograph. Thus, photograph No. 2 is correctly oriented. In this position, all rays and points are traced. In this manner, each of the successive photograph is slid under the tracing, oriented and the rays traced till another ground control point is reached.

Fig. 14.41 shows the plotting of the map control on the tracing. It will be observed that at each of the pass points, there will appear three intersecting rays. The position of each of the points is located on the tracing at the point of intersection of the three rays. This point of intersection may not appear to coincide with the corresponding point on the photograph, because of the displacements due to ground relief. Sometimes, due to errors of plotting, the three rays may not intersect at a point, but may form a small triangle of error. In that case, the centre of the triangle is taken as the position of the point.

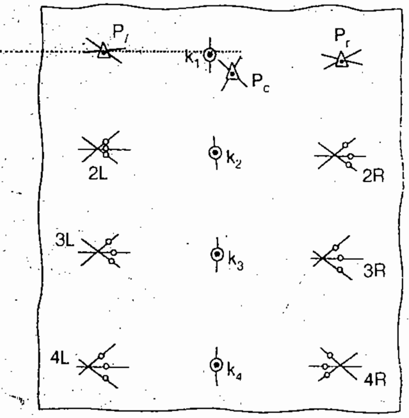


FIG. 14.41. COMPILATION OF MAP CONTROL.

The plotting work is thus continued till the next ground control point is reached. In a perfect map control work, the image of the control point, as located by the intersecting rays, will be the plotted position of the point traced from the base map.

In case, the plotted position of the ground control point does not coincide with its traced position, as usually is the case, the lines of flight, or the positions of the principal points are adjusted as shown in Fig. 14.42. P is the position of the ground control point as located by the intersection of rays, and P' is the corresponding position as traced from the base map. Thus, the total error is $P'P$ in magnitude as well as direction. Each of the principal points $k_7, k_6, k_5 \ldots\ldots k_2$ is shifted to positions $k_7', k_6', k_5', \ldots k_2'$, in a direction parallel to $P\,P'$ by a distance proportional to the distance of that point from the initial fixed point k_1. The positions of other pass points are also adjusted accordingly.

It should be noted that when the tracing is begun, the scale is not at all known. The unknown scale is established by the distance between the two principal points k_1 and k_2. The scale of the data assembled on the film or tracing can then be determined by measuring the distance between the first control point and the last control point on it.

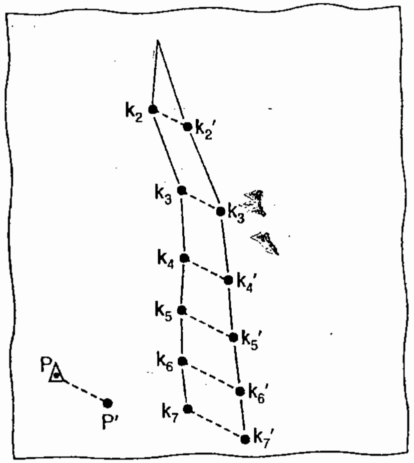


FIG. 14.42. ADJUSTMENT OF THE LINES OF FLIGHT.

(4) Transferring Photographic Details

To transfer the photographic details, each photograph is slid under the tracing and oriented to the map control. The details are then traced on the tracing. Next photograph is then slid and oriented and corresponding rays are drawn to the points. The intersection of the two sets of rays obtained from the two photographs will give the plotted positions of the points, as illustrated in Fig. 14.38. The details can be transferred to the base map either by photograph or by tracing over a carbon sheet.

Plain Templets Method of Control

The templet method is variation of the *radial line* method, and is used with greater convenience when a considerable area is to be plotted by radial-line method.

Templets are of two kinds :

(*i*) Plain templets (*ii*) Slotted templets.

The plain templets are actually the substitutes of the tracing acetate paper of film. The plain templets are transparent sheets, preferably of acetate, and of size slightly bigger than each photograph. The ground control points and pass points or minor control points are selected and marked on each photograph as explained earlier. The principal point of each photograph is also transferred to the adjacent photographs. Separate templets are used for each photograph. The templet is placed on the photograph and the position of principal points are pricked through on to the templet. Radial lines are now drawn on the templet from the principal point over each control point already marked on the photograph. Thus, templets are marked for each and every photograph.

The first templet is then placed on the base map having the plotted positions of the ground control, and is oriented such that the rays of the templet pass through the plotted control points. After this, the second and third templets are also adjusted and oriented properly by the same method as used for the radial line method. All the templets oriented in this way are fastened together by Scotch tape, till another set of ground control points is reached. If the ground control points on the templets do not coincide with the corresponding map positions due to various sources of errors explained already, adjustments are made

by stretching or twisting the whole assembly of templets as whole until the map positions of ground control points coincide with the corresponding positions on the templets. The system of control points established on the combined assembly of templets is then transferred to the base map by pricking through the acetate sheets with needle.

Slotted Templet Method of Control

The slotted templet is an improvement over the plain templet. In this method, all rays from the principal points on minor and ground control points are replaced by slots cut in cardboard or acetate templets (Fig. 14.43).

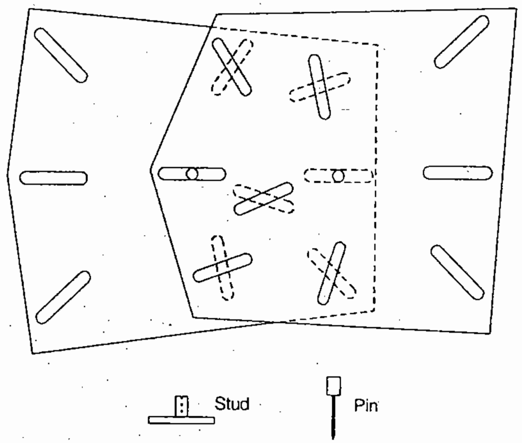


FIG. 14.43. SLOTTED TEMPLET METHOD.

The templets may either be of acetate sheets or sheets of thin, firm cardboard of about the same size of the photographs. As in the radial line method, the ground and minor control points are selected on each photograph and marked with needle points. These points are then transferred to the templet by pricking through the photograph. The templet is thus marked with the principal point of the photograph, the two transferred principal points of the adjacent photographs, and the ground and photo-control points. The templet is then taken to the *slot-cutting machine*. A small hole is then punched at the principal point. Slots representing rays radiating from the principal point to the marked points are cut into the templets by the machine. The slot cutting blade of the machine is designed so that it can be centred accurately over the photograph position of any point marked on the templet. The width of the slots cut are of the same size as the diameter of studs (Fig. 14.43) which are inserted through the slots. One such templet is prepared for each photograph. The metal studs to be inserted in the slots are drilled centrally with a fine hole to accommodate a steel pin.

A specially prepared floor or dais is used for assembly of the slotted templets. The base map containing the accurately plotted ground control points is stretched on the floor. The first templet is put on the base map and is oriented with respect to the ground control points. Those photo-point studs which correspond to known ground control points are then fixed in position on the map by pins (Fig. 14.43) driven through the central holes of the studs. The movable studs (*i.e.* studs having no central pins) are inserted through those slots representing the rays to each selected photo point. The second, third and other templets are then put on the map and oriented one after the other by the method explained earlier, till another set of ground control points are reached. The assembled templets are adjusted until two or more slots belonging to each ground control point will fit over the fixed stud belonging to that point. When this is achieved, all the free studs representing the photo-control points and principal points will be automatically adjusted. The positions thus found for the movable or free studs are the most probable positions for the corresponding photo points, and their positions are then transferred to the map by inserting sharp metal pins through the central holes of the studs.

If the effect of tilt is more in a photograph, its templet will not fit the assembly. Before it can be used, it will need to be rephotographed and the tilt effect removed. The important advantage of the slotted templet method is that any fault due to wrong position of the slot or other sources is mechanically detected when the templets do not fit.

STEREOSCOPY AND PARALLAX

14.24. STEREOSCOPIC VISION

The *depth perception* is the mental process of determining relative distance of objects from the observer from the impressions received through the eyes. Due to binocular vision, the observer is able to perceive the spatial relations, *i.e.*, the three dimensions of his field of view.

The impression of depth is caused mainly due to three reasons : (1) relative apparent size of near and far objects, (2) effects of light and shade, and (3) viewing of an object simultaneously by two eyes which are separated in space. Out of these, the third one is the most important. Each eye views an object from a slightly different position, and by a physiological process the two separate images combine together in the brain enabling us to see in three dimensions.

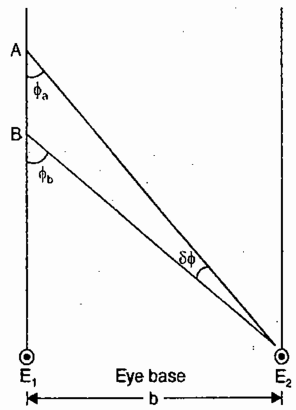


FIG. 14.44. ANGLES OF PARALLAX.

Angle of Parallax (or Parallactic Angle)

In normal binocular vision, the apparent movement of a point viewed first with one eye and then with the other is known as *parallax*. Since an object is viewed simultaneously by two eyes, the two rays of vision converge at an angle upon the object viewed. The angle of parallax or the parallactic angle is the angle of convergence of the two rays of vision. In Fig. 14.44, A and B are two objects in the field of view, and are being viewed by the two eyes represented in space by the positions, E_1 and E_2 . $E_1 E_2 = b$ is known as the *eye base*. The angle E_1AE_2 is the angle of parallax (φ_a) of object A, and the angle E_1BE_2 is the angle of parallax (φ_b) of object B. The object B, for which the parallactic angle φ_b is greater, will be judged to be nearer the observer than the object A for which the parallactic angle φ_a is smaller. The measure of the distance BA is evidently provided by the difference in the parallactic angles of A and B. This difference, *i.e*, $\varphi_b - \varphi_a$ ($= \delta\varphi$) is termed as the *differential parallax*.

Stereoscopic Fusion

The principles of stereoscopic vision can readily be applied to photogrammetry. An aerial camera takes a series of exposures at regular intervals of time. If a pair of photographs is taken of an object from two slightly different positions of the camera and then viewed by an apparatus which ensures that the left eye sees only the left-hand picture and the right eye is directed to the right hand picture, the two separate images of the object will fuse together in the brain to provide the observer with a spatial impression. This is known as a *stereoscopic fusion*. The pair of two such photographs is known as *stereopair*. Two

devices are used for viewing stereopairs : the *stereoscope* and the *anaglyph*. To illustrate the phenomenon of stereoscopic fusion, let us conduct an experiment (see Figs. 14.45 and 14.46) described below.

Fig. 14.45 shows two pairs of dots near the top edge of a sheet of paper. The

FIG. 14.45

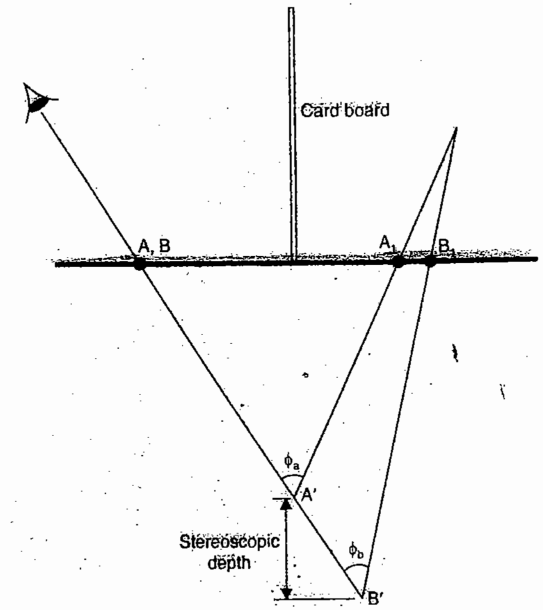


FIG. 14.46. STEREOSCOPIC FUSION.

distance between dots A and A_1 is less than the dots B and B_1. Place a piece of cardboard between AB and $A_1 B_1$, in the plane perpendicular to the sheet so that the left dots A, B are seen with the left eye and the right dots A_1, B_1 are seen with the right eye. By staring hard, it will be observed that A and A_1 fuse together to form a single dot which appears closer than the fused image of B and B_1 (Fig. 14.46).

The apparent difference in level is known as *stereoscopic depth* and depends on the spacing between the dots. The spacing between the dots is called the *parallax difference*.

Clues to Depth Perception

As stated earlier, the depth perception is the mental process of determining relative distance of objects from the observer from the impression received through the eyes. Numerous impressions are received that serve as *clues to depth*, and the following clues are important from photogrammetry point of view :

(1) Head parallax	(2) Accommodation
(3) Convergence	(4) Retinal disparity.

(1) ***Head Parallax*** : Head parallax is the apparent relative movement of object at different distances from the observer when the observer moves.

(2) ***Accommodation*** : Accommodation is the process by which the lens of the eye can be flattened (to focus nearby points on the retina) or made more convex (to focus nearby points on the retina) in accordance with requirements placed on it. Due to the accommodation of lens, the brain gets an approximate clue to distance (or depth). The ability of the eye to accommodate this way become less for weak eyes ; it begins to reduce in the forties and is usually completely lost in the sixties.

(3) *Convergence* : In order to see an object clearly (or sharply) it is necessary that the image of the desired object is placed on the most sensitive part of each retina (the fovea). This causes the two eyes to turn or converge. The convergence of the eyes is therefore a clue to distance since the eyes converge more for nearby points and less for farther points and brain is aware of their relative positions. In Fig. 14.46, the axes of the eyes are directed to points A' and B' *behind* the plane of paper whereas the eyes must be focused for the plane of paper if the dots are to remain sharply defined. Thus, the *convergence* of the eyes (to view A' and B') is not in sympathy with their *accommodation* (to view A and B sharply).

(4) ***Retinal Disparity*** : The picture of an object received by the two eyes are slightly different since the two eyes are at different positions. The difference between the images on the retinas is called *retinal disparity*. Since it is a function of the relative distance of objects viewed, it provides a very strong distance clue. *In photogrammetry, this is the only clue which is actually used.* The range and intensity of stereoscopic perception can be increased by two ways :

(*i*) by apparently increasing the base between view points.

(*ii*) by magnifying the field of view by use of lenses.

Stereoscope

Stereoscope is an instrument used of viewing stereopairs. Stereoscopes are designed for two purposes :

(1) To assist in presenting to the eyes the images of a pair of photographs so that the relationship between convergence and accommodation is the same as would be in natural vision.

(2) To magnify the perception of depth.

There are two basic types of stereoscopes for stereoscopic viewing of photographs:

(1) Mirror stereoscope (2) Lens stereoscope.

(1) The Mirror Stereoscope : The mirror stereoscope, shown diagrammatically in Fig. 14.47 (*b*), consists of a pair of small eye-piece mirrors m and m', and a pair of larger wing mirrors, M and M', each of which is oriented at 45° with the plane of the photographs.

Fig. 14.47 (*a*) shows a nail mounted on a block of timber, and is being photographed by two camera positions. The camera lens is placed first in the position of left eye and then in the position of right eye, and separate photographs are taken in each position. It will be noted that the head of the nail is to the left in the left film and to the right in the right film. ab and $a'b'$ are the images of the nail AB in the two films.

Contact prints from these negatives are placed in the mirror stereoscope as shown in Fig. 14.47 (*b*), where only images of the nail are drawn. The four mirrors transfer

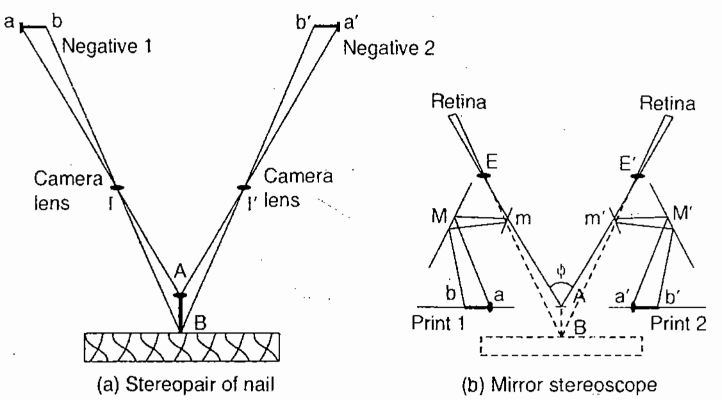


FIG. 14.47. PRINCIPLE OF MIRROR STEREOSCOPE.

the light to the eyes exactly (except for accommodation) as if it had come from nail as shown by dotted line. The convergence and retinal disparity are sufficient for the observer to see the nail in three dimensions.

The total distance $b\ M\ m\ E$ or $b'\ M'\ m'\ E'$, from the eye to the plane of the photographs varies 30 cm to 45 cm, in order that the unaided eye may comfortably view the photographs. The angle φ is determined by the separation of photographs that gives the most eye comfort, and is compatible with the distance *bMmE*. If this distance is to be reduced, a pair of magnifying lenses are placed at E and E'. Each magnifier has a focal length slightly smaller than the distance *bMmE*. Some types of mirror stereoscopes have a set of removable binoculars which are placed at the eye positions E and E'.

Fig. 14.48 shows a Wild ST - 4 mirror stereoscope with a parallax bar manufactured by M/s Wild Heerbrugg Ltd. It is used for spatial observation of stereophotographs upon a maximum model size of approximately 18 cm × 23 cm. The distance between the central point of mirrors is 25 cm for all interpupillary distances. The whole model area can be seen through the two lenses provided for correction of the bundle of rays and for accommodating. A removable set of eyepieces with 3 X magnification can be swung in over these lenses for closer examination of parts of the model and study of details. A pair of eyepieces with 8 X magnification can be inserted in place of those of lower. The 8 X eyepieces are particularly useful when selecting tie points in aerial triangulation. The two inclined binocular eyepiece tubes are adjustable for interpupillarly distance of 56 to 74 mm and have eye-piece adjustments for focusing the separate images.

The greatest single advantage of the mirror stereoscope is the fact that the photographs may be completely separated for viewing, and the entire overlap area may be seen stereoscopically without having to slip the photographs.

(2) The Lens Stereoscope : A lens stereoscope consists of a single magnifying lens for each eye, and no mirrors. The two magnifying lenses are mounted with a separation equal to the average interpupillary distances of the human eyes, but provision is made for changing this separation to suit the individual user.

The distance between the nodal point of the lens and the plane of the photograph depends upon the focal length of the lens. The two photographs can be brought so close to the eyes that proper convergence can be maintained without causing the photographs to interfere with each other as shown in Fig. 14.49. Since the photographs are very close to the eyes, the images occupy larger angular dimensions and therefore appear enlarged. Fig. 14.50 shows a lens stereoscope.

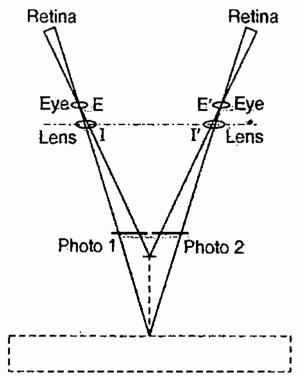


FIG. 14.49. OPTICAL DIAGRAM OF LENS STEREOSCOPE.

The lens stereoscope is apt to cause eye strain as accommodation is not in sympathy with convergence and the axes of the eyes are forced out their normal condition of visions. Most lens stereoscopes are however, quite small and can be slipped in one's pocket, this type being called a *pocket stereoscope*. Because of larger size, mirror stereoscope is not so portable as is the pocket stereoscope.

14.25. PARALLAX IN AERIAL STEREOSCOPIC VIEWS

Parallax of a point is the displacement of the image of the point on two successive exposures.

The difference between the displacements of the images of two points on successive exposures is called the difference in parallax between the two points.

In Fig. 14.51, two points *A* (lower) and *B* (higher) are being photographed by the two positions *O* and *O'* of an aerial camera. The plane is moving at a speed of 200 km/hour and if the exposures are taken at an interval of 20 seconds, the lens centre moves a ground distance of about 1110 metres between the two exposures. Suppose that between the two exposures, the image of the lower object *A* has moved a distance 6.05 cm across the focal plane of the camera, and

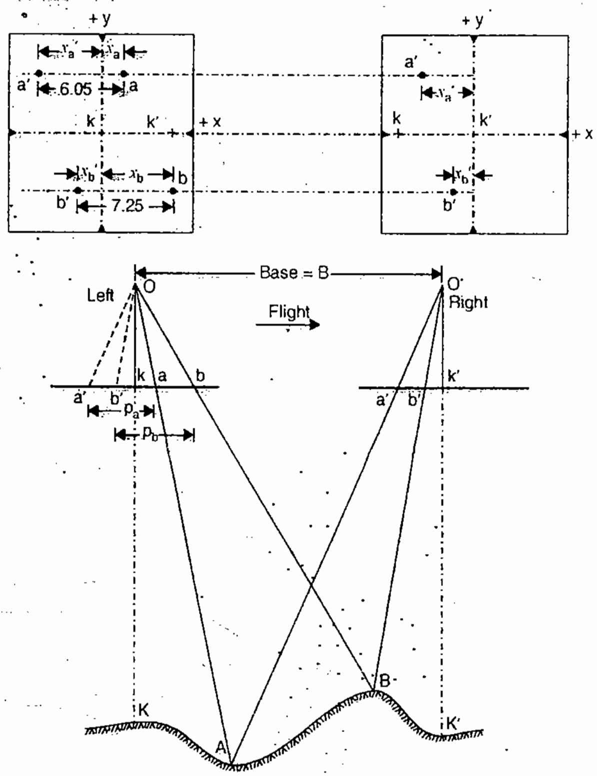


FIG. 14.51. PARALLAX.

the image of the higher object *B* has moved a distance of 7.25 cm. Then the parallax of the lower point is 6.05 cm and that of the higher point is 7.25 cm.

In the left photograph, *a* and *b* are the images of the two points. *k'* is the *transferred* principal point of the right photograph. Both the images *a* and *b* are to the right of the *y*-axis of the left photograph. In the right photograph, *a'* and *b'* are the images of the same points, both the images being to the left of the y-axis. Thus the images (*a*, *b*) of the points have moved to (*a'*, *b'*) between the two exposures. The movement *aa'* (shown on the left photographs) is the parallax of *A*, and *bb'* is the true parallax of *B*. The parallax of the higher point is more than the parallax of the lower point. Thus, each image in a changing terrain elevation has a slightly different parallax from that of a neighboring image. *This point-to-point difference in parallax exhibited between points on a stereopair makes possible the viewing of the photographs stereoscopically to gain an impression of a continuous three dimensional image of a terrain.*

The following are the ideal conditions for obtaining aerial stereoscopic views of the ground surface :

(1) two photographs are taken with sufficient overlap.

(2) the elevation of the camera positions remains the same for the two exposures.

(3) the camera axis is vertical so that the picture planes lie in the same horizontal plane.

Algebraic Definition of Parallax : As defined earlier the displacement of the image of a point on two successive exposures is called the parallax of the point. On a pair of overlapping photographs, the parallax is thus equal to the *x*-coordinate of the point measured on the left-hand photograph (or previous photograph) minus the *x*-coordinate of the point measured on the right-hand photograph (or next photograph). Thus

$$p = x - x' \qquad \text{...(14.23)}$$

Thus, *x*-axis passes through the principal point and is *parallel* to the flight line, while the *y*-axis passes through the principal point and is perpendicular to the line of flight. In general, however, the flight-line *x*-axis is usually very close to the collimation mark *x*-axis, because of the effort made to eliminate drift and crab at the time of photography.

Thus, in Fig. 14.51, the parallax of points *A* and *B* are given by

$$p_a = x_a - x'_a \quad \text{and} \quad p_b = x_b - x'_b$$

In substituting the numerical values of *x* and *x'*, their proper algebraic sign must be taken into consideration. Thus, in Fig. 14.51, if $x_a = 2.55$ cm, $x'_a = -3.50$ cm, $x_b = -4.05$ cm and $x'_b = -3.20$ cm, we have

$$p_a = +2.55 - (-3.50) = 6.05 \text{ cm}$$

$$p_b = +4.05 - (-3.20) = 7.25 \text{ cm.}$$

14.26. PARALLAX EQUATIONS FOR DETERMINING ELEVATION AND GROUND CO-ORDINATES OF A POINT

Let *A* be a point whose ground co-ordinates and elevation are to be found by parallax measurement.

Let K be the ground position of the principal point k of the left photograph, and X and Y be the ground co-ordinates of A with respect to the ground co-ordinate axes, which are parallel to the photographic x and y co-ordinate axes, and with K as the origin. M is an imaginary point which has got the same elevation as that of A, and which lies on the ground X-axis. a and m are the images of A and M on the right photographs, and a' and m' are the corresponding images on the right photograph. *Let* (x, y) be the co-ordinates of a on the left photograph, and (x', y') be the co-ordinates of a' on the right photograph.

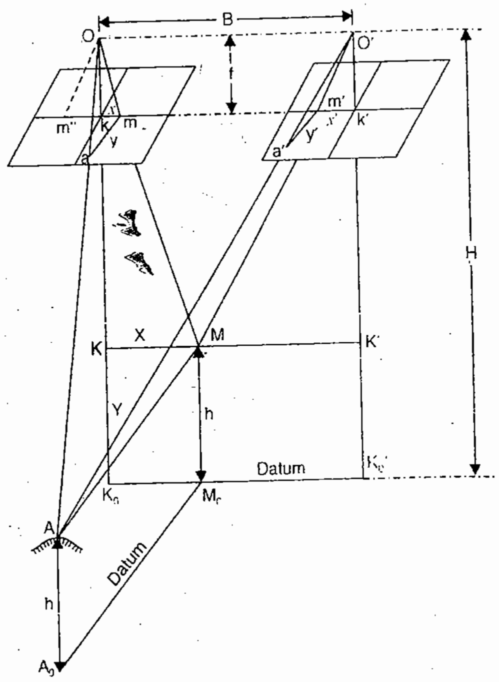


FIG 14.52. PARALLAX EQUATIONS FROM RAPALLAX MEASUREMENTS.

From triangles OKM and Okm, we have

$$\frac{Ok}{OK}=\frac{Om}{OM}=\frac{km}{KM}$$

$$\frac{f}{H-h}=\frac{Om}{OM}=\frac{x}{X} \qquad \ldots(1)$$

From triangles Oam and OAM, we have

$$\frac{Om}{OM}=\frac{am}{AM}$$

But $$\frac{Om}{OM}=\frac{f}{H-h} \text{ from (1) and } \frac{am}{AM}=\frac{y}{Y}$$

Hence $$\frac{am}{AM}=\frac{y}{Y}=\frac{f}{H-h} \qquad \ldots(2)$$

Similarly, from triangles $O'K'M$ and $O'k'm$,

$$\frac{O'k'}{O'K'}=\frac{O'm'}{O'M}=\frac{k'm'}{K'M} \quad \text{or} \quad \frac{f}{H-h}=\frac{x'}{X} \qquad \ldots(3)$$

and from triangles $O'MA$ and $O'm'a'$, we have

$$\frac{O'm'}{O'M}=\frac{a'm'}{AM} \quad \text{or} \quad \frac{f}{H-h}=\frac{y'}{Y} \qquad \ldots(4)$$

From equations (2) and (4), we have $\frac{f}{H-h}=\frac{y}{Y}=\frac{y'}{Y}$

or $$y=y' \qquad \ldots(14.34)$$

Equation 14.34 establishes that there is no y parallax in a stereoscopic pair of photographs.

In the left photograph (Fig. 14.52), draw Om'' parallel to $O'm'$ of right photograph. Then, in the triangles $Om''m$ and OMO' :

OO' is parallel to $m''m$

Om coincides with, and is parallel to OM

Om'' is parallel to $o'm'$

Hence they are similar, and their corresponding altitudes are f and $(H-h)$ respectively.

Thus, $\frac{f}{H-h}=\frac{mm''}{OO'}$ But $mm''=km+km''=x-x'=p$ and $OO'=B=$ air base

$$\therefore \quad \frac{f}{H-h}=\frac{p}{B} \qquad \ldots(5)$$

or $$H-h=\frac{Bf}{p} \qquad \ldots(14.35)$$

This is the parallax equation for the elevation of the point.

Again from equations (1) and (2),

$$X=\frac{H-h}{f}x \quad \text{and} \quad Y=\frac{H-h}{f}y$$

But $$\frac{H-h}{f}=\frac{B}{p}, \text{ from (5)}$$

Hence $$X=\frac{B}{p}x \quad \text{and} \quad Y=\frac{B}{p}y \qquad \ldots(14.36)$$

This is the parallax equation for the ground co-ordinates of the point.

Difference in Elevation by Stereoscopic Parallaxes

In Fig. 14.53, A_1A_2 is a flagpole being photographed from two camera positions O and O'. The top A_2 of the flagpole has an elevation of h_2 above the datum, and the bottom A_1 has an elevation of h_1 above the datum. H is the camera height for both the exposures.

In the left photographs, a_1 and a_2 are the two images of A_1 and A_2, and their x-co-ordinates are x_1 and x_2 respectively.

Similarly, in the right photograph, a'_1 and a'_2 are the images of A_1 and A_2 respectively, and their x-co-ordinates are x'_1 and x'_2 respectively.

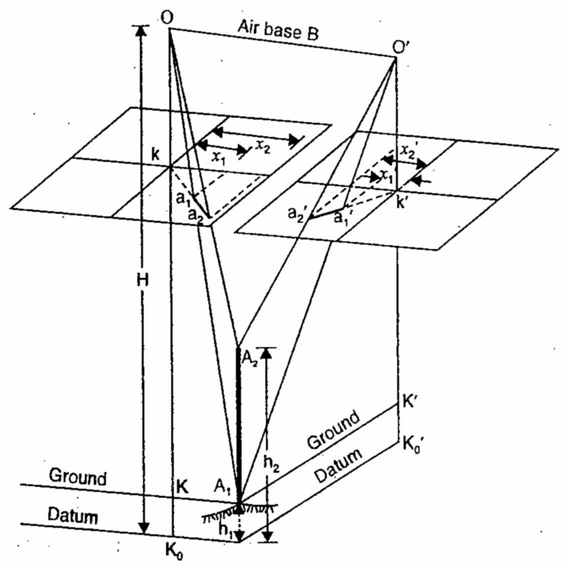


FIG. 14.53. DIFFERENCE IN ELEVATION BY STEREOSCOPIC PARALLAXES.

Evidently, the parallax p_1 for the bottom of the flagstaff is given by

$$p_1 = x_1 - x'_1 \quad \ldots(1)$$

Similarly, the parallax p_2 for the top of the flagstaff is given by

$$p_2 = x_2 - x'_2 \quad \ldots(2)$$

Hence the difference in parallax (Δp) of top and bottom points is given by

$$\Delta p = p_2 - p_1 = (x_2 - x'_2) - (x_1 - x'_1) \quad \ldots(3)$$

From equation 14.35, the elevation of any point is given by

$$h = H - \frac{Bf}{p}$$

Hence, for the top and bottom of flagstaff, we get

$$h_1 = H - \frac{Bf}{p_1} \quad \text{and} \quad h_2 = H - \frac{Bf}{p_2}$$

$\therefore$ Difference in elevation (Δh) is given by,

$$\Delta h = h_2 - h_1 = \left(H - \frac{Bf}{p_2}\right) - \left(H - \frac{Bf}{p_1}\right) = \frac{Bf}{p_1} - \frac{Bf}{p_2}$$

or
$$\Delta h = \left(\frac{p_2 - p_1}{p_1 p_2}\right) Bf \quad \ldots(14.37)$$

or
$$\Delta h = \frac{\Delta p}{p_1 p_2} Bf \quad \ldots[14.37(a)]$$

Now $\quad \Delta p = p_2 - p_1 \quad$ or $\quad p_2 = p_1 + \Delta p$

Hence, we have
$$\Delta h = \frac{\Delta p}{p_1 (p_1 + \Delta p)} \cdot Bf \quad \ldots(14.38)$$

Mean Principal Base **(b_m)** : The distance between the principal point of a photograph and the position of transferred principal point of its next photograph obtained under fusion through stereoscope is called *principal base*. Thus, in Fig. 14.51,

$kk' = b$ = principal base of left photograph

and $\quad k'k = b'$ = principal base of right photograph.

It should be noted that b and b' will not be equal since the elevation of ground positions of the principal-points (K and K') are not the same.

The *mean principal base* is the mean value of the principal bases of the photographs.

Thus,
$$b_m = \frac{b + b'}{2}$$

If the ground principal points (K and K') have the same elevation, then under ideal conditions, $b_m = b$.

Now, in Fig. 14.53, let the datum pass through the bottom A_1 of the flagstaff (*i.e.* $h_1 = 0$). Assuming the ground to be now the datum plane, the ground principal points K and K' will be at the same elevation, and the parallax of the principal points (*i.e.*, the principal base) will be equal to b. If H is the height of camera above the datum (*i.e.* above A_1 now), the general relationship between b and B is given by

$$\frac{B}{b} = \frac{H}{f} \quad \text{or} \quad B = \frac{Hb}{f} = s \times b \quad \ldots(14.39)$$

where s is the scale of the photograph at datum elevation. Substituting this value of air base in equation 14.38, we get

$$\Delta h = \frac{\Delta p}{p_1 (p_1 + \Delta p)} \cdot Hb$$

Since K, K' and A_1 are all at the same elevation, their parallaxes are the same.

Hence $\quad p_1$ = parallax of principal points $= b$

Hence, we get the parallax equation

$$\Delta h = \frac{H \Delta p}{b + \Delta p} = \frac{H \Delta p}{p_1 + \Delta p} \quad \ldots(14.40)$$

While using equations 14.40, the following assumptions must always be kept in mind:

(1) The vertical control point (*i.e.*, point A_1) and the two ground principal points lie at the same elevation.

(2) The flying height (H) is measured above the elevation of the control point and not sea level (unless the control point happens to lie at sea level).

In practical applications, the mean principal base (b_m) is used in place of b, and flying height above the average terrain is taken as the value of H.

Alternative form of Parallax Equation for Δh

We have, $\quad p_1 = \dfrac{fB}{H - h_1}$ and $p_2 = \dfrac{fB}{H - h_2}$

$$\therefore \quad \Delta p = p_2 - p_1 = fB\left(\frac{1}{H - h_2} - \frac{1}{H - h_1}\right) = fB \frac{h_2 - h_1}{(H - h_1)(H - h_2)}$$

But $\quad \Delta h = h_2 - h_1$ and $h_2 = \Delta h + h_1$

$$\therefore \quad \Delta p = fB \frac{\Delta h}{(H - h_1)(H - \Delta h - h_1)}$$

or
$$\Delta p (H - h_1)^2 - \Delta p (H - h_1) \Delta h = fB \Delta h$$

or
$$\Delta h [(H - h_1) \Delta p + fB] = (H - h_1)^2 \Delta p$$

or
$$\Delta h = \frac{(H - h_1)^2 \Delta p}{(H - h_1) \Delta p + fB}$$

Dropping the suffix of h, we get

$$\Delta h = \frac{(H - h)^2 \Delta p}{(H - h) \Delta p + fB} \quad \ldots[14.41(a)]$$

where h is the elevation of lower point above datum.

Putting $\quad fB = Hb$, we get $\quad \Delta h = \dfrac{(H - h)^2 \Delta p}{(H - h) \Delta p + bH} \quad \ldots(14.41)$

It should be noted that the above equation is in its most general form. Eq. 14.40 is the special form of this, and can be obtained by putting $h = 0$ (*i.e.*, lower point at datum) in Eq. 14.41). Thus

$$\Delta h = \frac{(H-0)^2 \Delta p}{(H-0)\Delta p + bH} = \frac{H^2 \Delta p}{H \Delta p + bH} = \frac{H \Delta p}{\Delta p + b} \quad ...(14.42)$$

14.27. EFFECTS OF CHANGES IN ELEVATION h AND PARALLAX p

The difference of elevation between two points is given by equation 14.37, *i.e.*,

$$\Delta h = \frac{p_2 - p_1}{p_1 p_2} Bf.$$

In order to find Δh, therefore, the parallaxes p_1 and p_2 of both the points are to be measured very carefully. If, however many computations are required (as in mapping), the above method of finding Δh is quite inconvenient. In such circumstance, the following two methods are in common use :

(1) the *unit-change* method. (2) the *parallax-table* method.

In both the methods, use of precise instruments (such as *parallax bar, stereocomparator* or contour finder) are used to measure the *difference in parallax* (Δp) directly by means of micrometer scales and the fusion of two dots in the stereoscopic view, into a so-called *floating mark* (see § 14.28).

(1) The Unit-Change Method

From equation 14.35, we have

$$H - h = \frac{Bf}{p} \quad \text{or} \quad h = H - \frac{Bf}{p} \quad ...(1)$$

By differentiation, we get $dh = \frac{Bf}{p^2} dp$

Substituting $p = \frac{Bf}{H-h}$ (from Eq. 14.35), we get $dh = \frac{(H-h)^2}{Bf} \cdot dp$ ...(14.42)

Since $\frac{B}{b} = \frac{H}{f}$, we have also $dh = \frac{(H-h)^2}{bH} \cdot dp$ ...[14.42 (*a*)]

The above equations express the rate of change of p for the infinitesimal change dh in the value of h.

The instruments used for measuring parallaxes are divided in millimeter, and hence the unit of change in parallax is taken as one millimeter. Let the rate of change dh be assumed to be constant for 1 mm change in dp.

Then $dp = 1 \text{ mm} = \Delta p_0$ (say).

If the value of Δh_0 is computed for a corresponding value of $\Delta p_0 = 1$ mm, the total value of Δh (*i.e.*, difference in elevation) is found by multiplying Δh_0 by the number of millimeters in Δp.

Equation 14.42 can then be written as

$$\Delta h_0 = \frac{(h-h)^2 \Delta p_0}{Bf} \quad ...[14.43 \ (a)]$$

or
$$\Delta h_0 = \frac{(H-h)^2}{Bf} = \frac{(H-h)^2}{bH} \quad (\text{since } \Delta p_0 = 1 \text{ mm}) \quad ...(14.43)$$

and
$$\Delta h = \Delta h_0 \times \Delta p \quad ...(14.44)$$

In equations 14.43 and 14.44, H, h, B and Δh_0 are in meters, while Δp and f are millimeters. Also, an average value of $(H-h)$ should be substituted.

(2) The parallax Table Method

$$p = \frac{Bf}{H-h}$$

But $\frac{H-h}{f} = s$ = scale of the photograph at the elevation h.

Hence $p = \frac{B}{s}$

But $\frac{B}{s} = b$ = distance on the photograph corresponding to the air base distance B.

Hence $p = b$ for a given elevation h above datum.

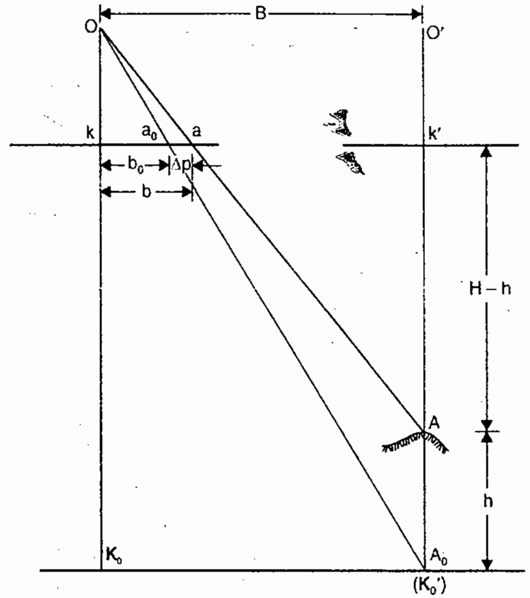


FIG. 14.54.

In Fig. 14.54, A is an object at an elevation h above datum, and A_0 is its datum position. The point A has been so chosen that it is vertically below the second exposure station O', since the image of A and A_0 coincides with the principal point k' of the second photograph, the parallax (p) of A will be a distance ka (= b) and the parallax (p_0) of its datum position A_0 will be a distance ka_0 (= b_0).

Hence $p = b$. ; and $p_0 = b_0$

and b and b_0 are the photograph distances of the air base B, at the elevation h and datum respectively.

Hence $\Delta p = p - p_0 = b - b_0$

But Δp is the radial displacement due to relief of A, and its value is given by equation. 14.19, *i.e.*,

$$\Delta p = \frac{bh}{H} = \frac{b_0 h}{H-h} \quad ...(14.45)$$

where. b = absolute parallax at elevation H

and b_0 = absolute parallax at datum elevation.

Using this equation, a table of total parallaxes designated as $\Sigma\Delta p$ can be computed using a given value of H and the absolute parallax b_o = 100 mm at the datum elevation, for the desired increments in h. The computations can be done as follows :

$$\Sigma\Delta p = \frac{b_0 h}{H-h}$$

Let $H = 8000$ m ; $b_0 = 100$ mm

and h be increased by 10 m for the interval $(H - h) = 8000$ m to $(H - h) = 3000$ m, and then by 5 m for interval $(H - h) = 3000$ to $(H - h) = 1500$ m.

When $h = 0,\ H - h = 8000$ m and $\Sigma\Delta p = 0$

When $h = 10,\ H - h = 7990$ m and $\Sigma\Delta p = \frac{100 \times 10}{7990} = 0.125$ mm

When $h = 1000$ m, $H - h = 7000$ m and $\Sigma\Delta p = \frac{100 \times 1000}{7000} = 14.286$ mm

When $h = 5000$ m, $H - h = 3000$ and $\Sigma\Delta p = \frac{100 \times 5000}{3000} = 166.667$ mm.

Thus, the values of $\Sigma \Delta p$ for the different values of h can be found and a master parallax table between $(H - h)$ and $\Sigma \Delta p$ can be prepared. Such a table will, however, be useful for direct computations only if $H = 8000$ and $b_0 = 100$ mm. The values of $\Sigma \Delta p$ in the table may, however, be adapted to other conditions also if they are multiplied by a constant K *such that*

$$K = \frac{b_0 \text{ (photo)} \times H \text{ (photo)}}{100 \times 8000} \qquad ...(14.46)$$

Example 14.17. *A photogrammetric survey is carried out to a scale of 1 : 20000. A camera with a wide angle lens of $f = 150$ mm was used with 23 cm × 23 cm plate size for a net 60% overlap along the line of flight. Find the error in height given by an error of 0.1 mm in measuring the parallax of the point.*

Solution. $\text{Scale} = \frac{f}{H}$

$$\therefore \quad \frac{1}{20{,}000} = \frac{150/1000 \text{ (m)}}{H\text{(m)}}$$

or $$H = \frac{150}{1000} \times 20{,}000 = 3000 \text{ m}$$

The length of the air base is given by

$$B = \left(1 - \frac{p_l}{100}\right) ls = (1 - 0.6)\,\frac{23}{100} \times 20{,}000 = 1840 \text{ m}$$

From equation 3.41, we have

$$dh = \frac{(H-h)^2}{Bf} \cdot dp$$

Corresponding to the datum elevation, the error dh for $dp = 0.1$ mm is

$$dh = \frac{(3000 - 0)^2}{1840 \times 150} \times 0.1 = \mathbf{3.26\ m.}$$

Example 14.18. *In a pair of overlapping vertical photographs, the mean distance between two principal points both of which lie on the datum is 6.375 cm. At the time of photography, the air-craft was 600 m above the datum. The camera has a focal length of 150 mm. In the common overlap, a tall chimney 120 m high with its base in the datum surface is observed. Determine difference of parallax for top and bottom of chimney.*

Solution.

$$s = \text{Scale of the photograph for datum elevation} = \frac{f}{H} = \frac{150/1000}{600} = \frac{1}{4000}$$

For the datum elevation, we have, from Eq. 14.39, $\frac{B}{b} = \frac{H}{f}$

or $$B = \frac{H}{f}\,b = s \times b = 4000 \times \frac{6.375}{100} = 255 \text{ m}$$

The parallaxes for the top and the bottom of the chimney are calculated from Eq. 14.35, *i.e.*

$$p = \frac{Bf}{H - h}$$

For the bottom of the chimney, $h = 0$ (since the bottom of the chimney is the datum), and hence

$$p_1 = \frac{255 \times 150 \text{ (mm)}}{600} = 63.75 \text{ mm}$$

For the top of the chimney, $h = 120$ m

$$\therefore \quad p_2 = \frac{255 \times 150 \text{ (mm)}}{(600 - 120)} = 79.69 \text{ mm}$$

Hence difference of parallax is given by

$$\Delta p = (p_2 - p_1) = 76.69 - 63.75 = \mathbf{15.94\ mm}$$

Check. From equation 14.40,

$$\Delta h = \frac{H \Delta p}{b + \Delta p} = \frac{600 \text{ m} \times 15.94 \text{ (mm)}}{63.75 \text{ (mm)} + 15.94 \text{ (mm)}} = 120.09 \text{ m} \approx 120 \text{ m}.$$

which is the same as the given height of the chimney.

Example 14.19. *A flag pole appears in two successive photographs taken at an altitude of 2000 m above datum. The focal length of the camera is 120 mm and the length of the air base is 200 m. The parallax for the top of the pole is 52.52 mm and for the bottom is 48.27 mm. Find the difference in elevation of top and bottom of the pole.*

Solution

The difference in elevation between two points is given by equation 14.37, *i.e.*

$$\Delta h = \left(\frac{p_2 - p_1}{p_1 p_2}\right) Bf = \left(\frac{52.52 - 48.27}{52.52 \times 48.27}\right) \times 200 \times 120 = \mathbf{44.2\ m.}$$

Example 14.20. *A pair of photographs was taken with an aerial camera from an altitude of 5000 m above m.s.l. The mean principal base measured is equal to 90 mm. The difference in parallax between two points is 1.48 mm. Find the difference in height between the two points if the elevation of the lower point is 500 m above datum.*

What will be the difference in elevation if the parallax difference is 15.5 mm ?

Solution

(*a*) $\Delta p = 1.48$ mm.

Since Δp is extremely small, Δh will also be small. Hence approximate formula [Eq. 14.42 (*a*)] can be used to calculate Δh.

Thus $dh = \frac{(H-h)^2}{hH} \cdot dp = \frac{(5000-500)^2 \times 1.48 \text{ (mm)}}{90 \text{ (mm)} \times 5000} = \frac{(4500)^2 \times 1.48}{450000}$

$= 66.60$ m

For more precise calculations, we have, from Eq. 14.41

$$\Delta h = \frac{(H-h)^2 \, \Delta p}{(H-h)\, \Delta p + bH} = \frac{(4500)^2 \times 1.48}{4500 \times 1.48 + 90 \times 5000} = \frac{(4500)^2 \times 1.48}{4500\,(1.48 + 100)}$$

$= \mathbf{65.6}$ **m.**

(*b*) $\Delta p = 15.5$ mm

$$\therefore \quad \Delta h = \frac{(H-h)^2 \, \Delta p}{(H-h)\, \Delta p + bH} = \frac{(4500)^2 \times 15.5}{4500 \times 15.5 + 90 \times 5000} = \frac{(4500)^2 \times 15.5}{4500\,(15.5 + 100)}$$

$= \mathbf{603.9}$ **m.**

14.28. MEASUREMENT OF PARALLAX : PARALLAX BAR

The principal point of one photograph can be transferred to the adjacent photographs by stereoscopic fusion. Thus the flight line joining the two principal points can be drawn on the photograph. This flight line becomes the x-axis of the photograph for the measurement of parallax. The y-axis is then drawn through the principal point, and perpendicular to the x-axis. The x and x' co-ordinates of any point (of which parallax is desired) can be measured with a line scale, and the parallax can be calculated by applying Eq. 14.33.

For greater refinement, the x-coordinates of the points can be found from a stereo-comparator. After identifying and transferring the principal points on the photographs, each photograph is oriented in the comparator in turn so that flight line is parallel to the x-axis of the comparator. The x readings of the principal point and those of other points (whose parallaxes are required) are taken, and the differences gives the x-coordinate of the point under consideration. Another photograph is then oriented and x coordinate found, and the parallax calculated from Eq. 14.33.

The difference of parallax is more commonly measured with the help of *parallax bar*.

Parallax Bar

A parallax bar, used to measure the parallax difference of two points, consists of a bar proper which holds a fixed plate of transparent material (plastic or glass) near the left end and a movable plate to the right end (Fig. 14.55). Each plate contains a tiny dot in its centre.

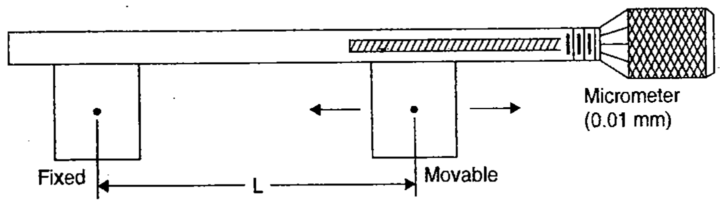


FIG. 14.55. PARALLAX BAR

The movable plate can be moved to the left or to the right by means of a micrometer screw which reads nearest to 0.01 mm, the total movement being about 25 mm. When these two dots are viewed properly under a stereoscope, they fuse into a single dot called *floating mark*. The marks are made to fuse by moving the right hand mark either to the left or to the right. After they have been fused, a slight movement of the movable mark will give the viewer the impression that the floating mark is moving up or down relative to the stereoscopic image. As the right hand mark is moved towards the left one, the floating mark appears to rise ; if it is moved to the right, the floating mark will appear to fall. These effects are due to the fact that the movement to the left increases the parallax of the marks, whereas the movement to the right decreases the parallax of the marks. Hence, if the floating mark is apparently placed on the ground at a known elevation, and the micrometer scale is read and is then moved to another point of unknown elevation, and the micrometer is turned until the floating mark again apparently rests on the ground surface, the difference in the two micrometer readings is a measure of Δp from which the difference in elevation can be calculated. This is the principle of the parallax bar.

Fig. 14.56 illustrates the principle of a parallax bar. On the left photograph, k is its principal point and k' is the conjugate principal point transferred from the next photograph. Similarly on the right hand photograph, k' is its principal point and k is the conjugate principal point transferred from the left photograph. Thus, kk' is the flight line on both the photographs. To orient them for stereoscopic observation, a fine straight line is drawn on a sheet of heavy drafting paper and the left hand photograph is placed on it in such a way that flight line is in exact coincidence with the line on the paper. This can be easily done by the laying a straight edge over the photograph and orienting it to the line. The separation of the two marks of the parallax bar is set to a distance L (measured within 1/2 mm), in such a way that it reads approximately the middle reading. The right hand photograph is then oriented by means of the flight line and is so placed as to cause a separation L between the principal point on one photograph and to corresponding position on the other photograph. The two photographs are then fused under a stereoscope (Fig. 14.48) and set so that their positions may not be altered.

Let it be required to measure the parallax difference between two points A and C whose images appear on both the photographs at (a, c) and (a', c') respectively. The left mark of the parallax bar is placed over a and the parallax bar

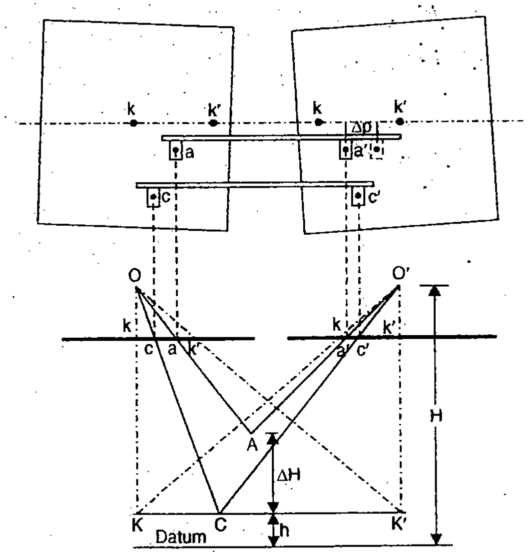


FIG. 14.56. PRINCIPLE OF A PARALLAX BAR.

is so oriented that it is parallel to the flight line. Move the right mark and make the fused dot to touch the ground point. Take the micrometer reading. Shift the bar bodily, put the left mark over the image c and move the right mark so that the fused mark again rests on the ground. Note the micrometer reading. The difference between the two readings gives the value Δp.

Thus in Fig. 14.56 when point a is fused, the separation of the marks is lesser and the point is higher as is clear from the two intersecting rays OaA and $O'a'A$ in the lower part of the diagram. Similarly, when c is fused, the separation of the marks is increased, and the point is lower as is clear from the two intersecting rays OcC and $O'c'C$.

The difference in elevation is then found by Equation 14.41, *i.e.*

$$\Delta H = \frac{(H-h)^2 \Delta p}{(H-h)\Delta p + b_m H} \quad ..(14.41)$$

where b_m is the mean principal base.

14.29. RECTIFICATION AND ENLARGEMENT OF PHOTOGRAPHS

Rectification is the process of rephotographing an aerial photograph so that the effects of tilt are eliminated. The rectification of tilted photograph taken from a given exposure station in space transforms the photograph into an *equivalent vertical photograph* taken from the same exposure station. Often the equivalent vertical photograph is enlarged or reduced as part of the process.

Fig. 14.57 (*a*) shows a photograph of an area taken with air camera vertical. The intersecting roads appear on the photo in their true positions. Fig. 14.57 (*b*) shows the distorted appearance of the roads on a titled photograph. Fig. 14.57 (*c*) shows the appearance of a rectified print. The roads are restored to their true shape, though the print is no longer square.

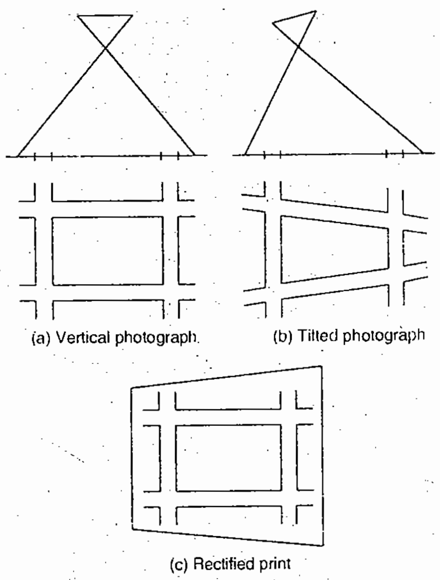


FIG. 14.57. RECTIFICATION OF TILTED PHOTOGRAPHS.

If the photograph is to be magnified, the principal distance of the photograph is changed so that the following equation is satisfied :

$$p = mf \quad ...(14.47)$$

where p = principal distance of the rectified photograph

f = focal length of the camera lens;

m = magnification factor.

If m is greater than unity, it denotes enlargement while if it is less than unity, it denotes an actual reduction of the photograph.

Various photographs are taken at different heights due to imperfect control in maintaining the aircraft perfectly at one altitude. The *purpose of magnification is to bring these to the same scale at a particular elevation*—either at the datum elevation or the average elevation of the terrain.

Fig. 14.58 shows a photograph with a tilt t at the exposure station O and flying height h. The negative and the photograph are parallel to each other. The rectified enlargement $b'\,k'\,a'$ is inclined at an angle t with the negative, and its principal distance is $p = mf$. The horizontal plane of the rectified photograph is known as the *easel plane*. It is to be noted that for the rectified enlargement of the photograph, the negative should be placed at a distance f from the lens. The lens will project the images from the negative in the proper direction, but since the distance from the lens to the negative is equal to the focal length of the lens, the projected bundles of the rays will be parallel to one another and they would never come to focus the enlargement plane. This is shown in Fig. 14.59 (*a*). This gives the condition that the entire negative must be placed behind the focal plane of the lens used in the rectifier.

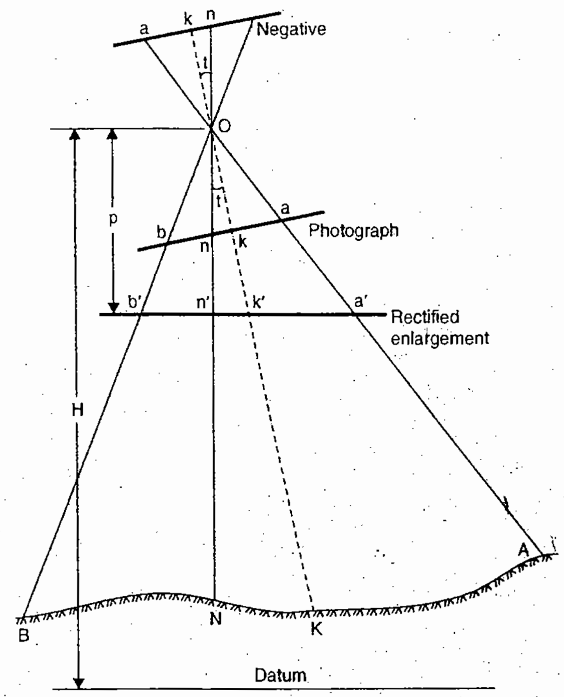


FIG. 14.58. RECTIFICATION AND ENLARGEMENT.

Scheimpflug Condition

As discussed in the previous paragraph if the negative plane is placed at the focal plane of the lens, the image cannot be focused. This is illustrated in Fig. 14.59 (*a*). If, however, the negative plane is placed beyond the focal plane, at a distance q from the lens and r is the corresponding position of the enlargement, the following two conditions are to be satisfied simultaneously :

$$\frac{1}{q} + \frac{1}{r} = \frac{1}{F} \quad \text{and} \quad r = mq \quad ...(14.48)$$

where F is the focal length of the lens of the rectifier and m is the magnification.

The relationships stated above are for a vertical photograph. However, these apply also for a tilted photograph. These conditions are shown in Fig. 14.59 (*c*). x and x' are the conjugate distances for the point a, while y and y' are conjugate distances for the point b. Hence, we have

$$\frac{1}{x} + \frac{1}{x'} = \frac{1}{F} \quad \text{and} \quad \frac{1}{y} + \frac{1}{y'} = \frac{1}{F}$$

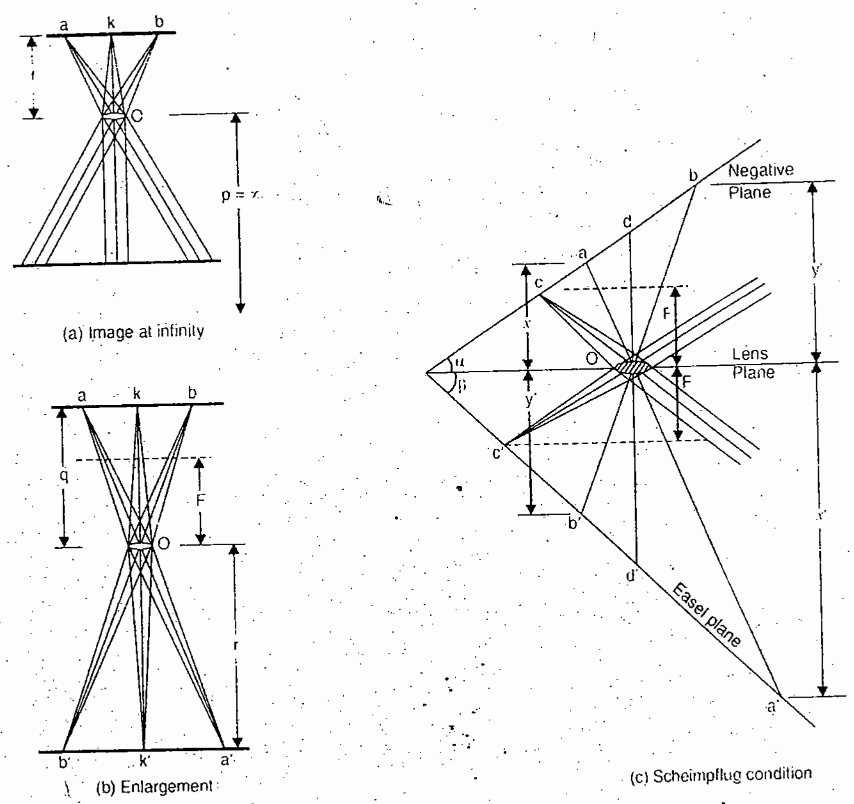


FIG. 14.59.

The negative plane makes an angle α with the lens plane, and the easel plane makes an angle β with the lens plane. It is to be noted that all the three planes intersect along one line. This is an important condition known as *Scheimpflug condition*.

The *Scheimpflug condition*, which must exist in order to produce sharp focus between the negative plane and the easel plane when these planes are not parallel, *states that the negative plane, the plane of the lens, and the easel plane must intersect along one line.*

In order to allow for a continuous range of tilt angles and magnification, there are in general, five independent elements necessary for rectification. These are :

(1) Variation of the projection distance.
(2) Tilt of the plane of projection about a horizontal axis.
(3) Rotation of the negative in its own plane (swing).
(4) Displacement of the negative in its own plane vertical to tilt axis.
(5) Displacement of negative in its own plane parallel to tilt axis.

An *automatic rectifier* is a rectifier so constructed that it automatically maintains the relationship between the object distance and the image distance, and at the same time fulfills the Scheimpflug condition. Fig. 14.60 shows the Wild E4 rectifier-enlarger introduced at

the 1964 Congress of Photogrammetry in Lisbon. The lens equation is automatically fulfilled by a cam inversor and the Scheimpflug condition is automatically fulfilled by an electronic simulator. Both cam inversor and electronic simulator are equipped with synchro systems. The instrument has enlargement ratios over a range from 0.8 X to 7 X and can be used for a largest negative size of 23 cm × 23 cm (9" × 9"). For further operational details, the reader is advised to see pamphlet P-1.302 *e* issued by M/s Wild Heerbrugg Ltd.

14.30. MOSACIS

Vertical photographs look so much like the ground that a set can be fitted together to form a *maplike photograph* of the ground. Such an assembly or getting of a series of overlapping photographs is called a *mosaic*. To a varying degree of accuracy, a mosaic is a map substitute. The mosaic has an over-all average scale comparable to the scale of a planimetric map.

Since they are taken at slightly varying altitudes and they contain tilts, they often do not fit each other very well. It is best to rephotograph them before they are used to bring them to desired scale and to eliminate some of the tilt.

A *controlled mosaic* is obtained when the photographs are carefully assembled so that the horizontal control points agree with their previously plotted positions. Making controlled mosaics is an art. A mosaic which is assembled without regard to any plotted control is called an *uncontrolled mosaic*.

The photographs are laid in such a sequence as to allow photo number and flight number of each photograph to appear on the finished assembly. This assembly is called an *index mosaic*. An index mosaic is a form of uncontrolled mosaic. A mosaic which is assembled from a single strip of photograph is called a *strip mosaic*.

The photographs used for preparing mosaics may consist of direct contact prints, of prints which have all been ratioed to a given datum scale in an enlarger, or of prints which have been fully rectified and ratioed in a rectifier.

A mosaic differs from a map in the following respects :

(1) A mosaic is composed of a series of perspective of the area, whereas a map is single orthographic projection.

(2) A mosaic contains local relief displacements, tilt distortions and non uniform scales, whiles a map shows the correct horizontal positions at a uniform scale.

(3) Various features appear as realistic photographic images on a mosaic, whereas they are portrayed by standard symbols on a map.

14.31. STEREOSCOPIC PLOTTING INSTRUMENTS

A stereoscopic plotting instrument is an optical instrument of high precision in which the spatial relationship of a pair of photographs at the instant of exposure is reconstructed. In such an instrument, the rays from the two photographs are projected and caused to intersect in its measuring space to form a theoretically perfect model of the terrain. A measured mark, visible to the operator is used to measure the stereoscopic model in all three dimensions. The horizontal movement of the measuring mark throughout the model is transmitted to a plotting pencil, which traces out the map position of the features appearing in the overlap area of two photographs forming the model.

A stereoscopic plotting instrument has four general components:

(1) a projection system (2) a viewing system
(3) a measuring system (4) a tracing system.

It is beyond the scope of the present book to illustrate fully the theory and working of the various plotting machines. However a brief description of the multiplex plotter is given below.

The Multiplex Plotter

The multiplex is probably the most widely used of any type of plotting machine. The equipment includes a reduction printer, a set of projectors mounted in series on a horizontal bar and a tracing table which provides both a floating mark and a tracing pencil to draw the map. The reduction printer produces reduced pictures on small glass plates. The 23 cm × 23 cm (or 9″ × 9″) size is thus reduced to a size 4 cm × 4 cm on the glass plates called *diapositives*.

Fig. 14.61 shows a pair of multiplex projectors forming a stereoscopic model. Each projector consists of a light source, a plate holder for the diapositive plate, and a lens which transmits the rays coming from the diapositive plate into the open space below the projector. The spatial model is obtained by projecting one photograph of an overlapping pair in red light and the other in blue-green light, and by observing the combination of colours through spectacles containing one red and one blue-green lens. This model, in fact, has three dimensions, and is not to be considered as virtual stereoscopic image as seen in a simple stereoscope. This method of viewing is called the *anaglyph system* of viewing reflected light and fulfills the condition of stereoscopic viewing.

Provision is made to move each of the projector in the directions of the *X, Y, Z co-ordinate axes, and also to rotate the projector about each of these axes. The X*-motion is parallel to the supporting bar, the *Y*-motion is perpendicular to the supporting bar and in the horizontal direction, while the Z-motion is perpendicular to the supporting bar and in the vertical direction. These six motions of each projector, independent of the others, make it possible to orient each projector in exactly in the same relation to the control points on the drawing table below, which the camera in the air had to the same corresponding actual ground points.

The *tracing table* contains a circular white disc with a pinhole in its centre. A light bulb below the disc provides a small pin point of light. This illuminated pin point is visible from the projectors and forms the measuring mark or floating mark in the spatial model. The disc can be raised or lowered so that the floating mark rests on the ground of the model. The tracing pencil point vertically below the floating mark gives position of the point on the map sheet. The tracing pencil traces pencil traces on the plotting sheet the horizontal movements of the floating mark. The disc is raised or lowered by means of a screw on the centre post at the back of the tracing stand. On the left post of the tracing stand is a millimetre scale on which is read the height of the disc above the drawing table which may be considered as the datum plane. The elevation of any point in the spatial model can be found by reading the vertical scale of the tracing stand after the floating mark has been set on the given point.

To trace a specific contour line on the map sheet, the disc is raised or lowered to give the correct reading (recorded in mm on the scale) corresponding to the desired elevation of the contour line. The operator moves the tracing table (with the pencil in the raised position) until the measuring mark comes into apparent contact with the model surface. and then lowers the pencil on the map sheet. By examining the stereoscopic view of the model slightly ahead of the measuring mark, the tracing table is moved in such a direction as to keep the measuring mark in contact with the surface of the model at all times. The line traced by pencil below the floating mark is the contour line since the floating mark was moving at a fixed elevation. When one contour line has been traced, the disc can be set to another height and the next contour can be traced in a similar manner.

The actual apparatus consists of a series of projectors (and not only two) mounted on a horizontal bar. The filters of the projectors are alternately red and blue-green. When the first true model has been placed in position, by viewing through the first two projectors and orienting them properly, the second true model is established by adjusting the third projector. The orientation and adjustments are done by means of ground-control points plotted on the drawing table. Thus each of the successive projectors can be oriented. This procedure is called *aerial triangulation* or *bridging* and is extended till other ground control appears. The proper adjustments are made throughout the series of projectors before drawing of the map is begun.

PROBLEMS

1. Define the following :
   (*i*) Air base, (*ii*) Tilt displacement.
   (*iii*) Principal point. (*iv*) Isocentre. (*v*) Isometric parallel.
2. Describe with sketches the field work of a survey with phototheodolite. Explain how you would plot the survey.
3. What is tilt distortion ? Prove that, in a tilted photograph, tilt distortion is radial from the isocentre.
4. Describe the various steps involved in the combination of vertical air photographs by the principal point radial line method.
5. Vertical photographs were taken from height of 3048 m, the focal length of the camera lens being 15.24 cm. If the prints were 22.86 × 22.86 cm and the overlap 60%, what was the length of the air base ? What would be the scale of the print ? (R.U.)
6. (*a*) Derive the parallax equation for determining heights from a pair of vertical photographs.
   (*b*) Two ground points *A* and *B* appear on a pair of overlapping photographs which have been taken from a height of 3650 m above mean sea-level. The base lines as measured on the two photographs are 89.5 mm and 90.5 mm respectively. The mean parallax bar readings for *A* and *B* are 29.32 mm and 30.82 mm. If the elevation of *A* above mean sea-level is 230.35 m, compute the elevation of *B*.
7. Two objects *A* and *B* whose elevations are 500 m and 1500 m respectively above mean sea-level are photographed from certain height with the axis of the camera vertical. The coordinates expressed in mm of the corresponding photo-images *a* and *b* are:

Point	*x co-ordinate*	*y co-ordinate*
a	+ 200	+ 150
b	− 320	− 300

The focal length = 200 mm and length *AB* = 44227 m. Find the height of the camera station (R.U.)

8. (a) Prove that on a tilted photograph height displacements are radial from the plumb point.
(b) Derive an expression for the height displacements in a vertical photograph.

9. (a) Explain with reference to aerial photographs, what is meant by end overlap and side overlap and why they are provided ?
(b) How do you determine the number of photographs necessary to cover a given area in an aerial survey ?

10. Write a note on radial line method of plotting.

11. Write short notes on the following :
(a) Stereoscopic vision. (b) Mirror stereoscope. (c) Crab and drift. (d) Parallax bar.

12. Describe, with the help of neat sketch a photo-theodolite.

13. (a) Explain how do you determine the focal length of the camera lens of a photo-theodolite.
(b) The distance from two points on a photographic print to the principal line are 42.36 mm to the left and 38.16 mm to the right. The angle between the points measured with a transit is 30° 45'. Determine the focal length of the lens.

14. (a) How do you determine the scale of an aerial photograph ? What do you understand by the terms 'datum scale' and 'average scale' ?

(b) A line PQ 2100 m long, lying at an elevation of 400 m measures 10.08 cm on a vertical photograph. If the focal length of the lens is 24 cm determine the scale of the photograph in an area, the average elevation which is 600 m.

15. A line AB lies on a terrain having an average elevation of 400 m above mean sea-level. It appears to be 8.72 cm on a photograph for which focal length is 24 cm. The same line measures 2.18 cm on a map which is to a scale of $\frac{1}{40000}$.
Calculate the flying altitude of the aircraft, above mean sea level. when the photograph was taken.

16. An object has an elevation of 400 m above mean sea-level. The distance from the principal point to the image of that point on the photograph is 4.86 cm. If the datum scale is $\frac{1}{12000}$ and focal length of the camera is 24 cm, determine the relief displacement of the point.

17. A tower AB is 40 m high, and the elevation of its bottom B is 800 m above mean sea-level. The distance of the image of the tower on a vertical photograph, taken at a flight altitude of 1800 m above mean sea-level, is 8.42 cm. Compute the displacement of the image of the top of the tower with respect to the image of its bottom.

18. A tower, lying on a flat area having an average elevation of 800 m above mean sea-level, was photographed with a camera having a focal length of 24 cm. The distance between the images of top and bottom of the tower measures 0.34 cm on the photograph. A line AB, 200 m long on the ground, measures 12.2 cm on the same photograph. Determine the height of the tower if the distance of the image of the top of the tower is 8.92 cm from the principal point.

19. The scale of an aerial photograph is 1 cm = 160 m, and the size of the photograph is 20 cm × 20 cm. If the longitudinal lap is 65% and side lap = 35%, determine the number of photographs required to cover an area of 232 sq. km.

ANSWERS

5.	1828·8 m ; $\frac{1}{20000}$.	6.	286·41 m.
7.	14030 m.	13.(b)	146·38 m.
14.(b)	1 cm = 200 m.	15.	2800 m.
16.	0·675 cm.	17.	0·34 cm
18.	60 m.	19.	100.

15

Electro-Magnetic Distance Measurement (EDM)

15.1. INTRODUCTION

There are three methods of measuring distance between any two given points :

1. Direct distance measurement (DDM), such as the one by chaining or taping.
2. Optical distance measurement (ODM), such as the one by tacheometry, horizontal subtense method or telemetric method using optical wedge attachments.
3. Electro-magnetic distance measurement (EDM) such as the one by geodimeter, tellurometer or distomat etc.

The method of direct distance measurement is unsuitable in difficult terrain, and some times impossible when obstructions occur. The problem was overcome after the development of optical distance measuring methods. But in ODM method also, the range is limited to 150 to 150 m and the accuracy obtained is 1 in 1000 to 1 in 10000. Electromagnetic distance measurement (EDM) enables the accuracies upto 1 in 10^5, over ranges upto 100 km.

EDM is a general term embracing the measurement of distance using electronic methods. In electro-magnetic (or electronic) method, distances are measured with instruments that rely on propagation, reflection and subsequent reception of either radio, visible light or infra-red waves. There are in excess of fifty different EDM systems available. However, we will discuss here the following instruments :

(i) Geodimeter (ii) Tellurometer (iii) Distomats.

15.2. ELECTROMAGNETIC WAVES

The EDM method is based on generation, propagation, reflection and subsequent reception of electromagnetic waves. The type of electromagnetic waves generated depends on many factors but principally, on the nature of the electrical signal used to generate the waves. The evolution and use of radar in the 1939-45 war resulted in the application of radio waves to surveying. However, this was suitable only for defence purposes, since it could not give the requisite accuracy for geodetic surveying. E. Bergestrand of the Swedish Geographical Survey, in collaboration with the manufacturers, Messrs. AGA of Sweden, developed a method based on the propagation of *modulated light waves* using instrument called *geodimeter*. Another instrument, called *tellurometer* was developed, using radio waves. Modern short and medium range EDM instruments (such as Distomats) commonly used in surveying use *modulated infra-red waves.*

Properties of electromagnetic waves

Electromagnetic waves, though extremely complex in nature, can be represented in the form of periodic sinusoidal waves shown in Fig. 15.1. It has the following properties:

1. The waves completes a *cycle* in moving from identical points *A* to *E* or *B* to *F* or *D* to *H*.

2. The number of times the wave completes a cycle in one second is termed as *frequency* of the wave. The frequency is represented by *f* hertz (*Hz*) where 1 hertz (*Hz*) is one cycle per second. Thus, if the frequency *f* is equal to 10^3 Hz, it means that the waves completes 10^3 cycles per second.

3. The length traversed in one cycle by the wave is termed as *wave length* and is denoted by λ (metres). Thus the *wave length* of a wave is the distance between two identical points (such as *A* and *E* or *B* and *F*) on the wave.

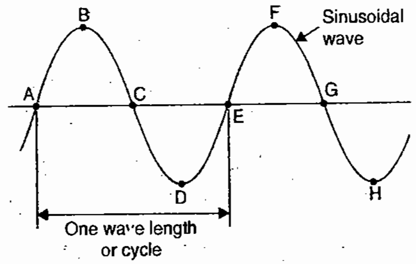


FIG 15.1 PERIODIC SINUSOIDAL WAVES.

4. The *period* is the time taken by the wave to travel through one cycle or one wavelength. It is represented by *T* seconds.

5. The *velocity* (*v*) of the wave is the distance travelled by in one second.

The frequency, wavelength and period can all vary according to the wave producing source. However, the velocity *v* of an electromagnetic wave depends upon the medium through which it is travelling. The velocity of wave in a vacuum is termed as *speed of light*, denoted by symbol *c*, the value of which is presently known to be 299792.5 km/s. For simple calculations, it may be assumed to be 3×10^8 m/s.

The above properties of an electromagnetic wave can be represented by the relation,

$$f = \frac{c}{\lambda} = \frac{1}{T} \qquad \text{...(15.1)}$$

Another property of the wave, known as *phase* of the wave, and denoted by symbol φ, is a very convenient method of identifying fraction of a wavelength or cycle, in EDM. One cycle or wave-length has a phase ranging from 0° to 360°. Various points *A*, *B*, *C* etc. of Fig. 15.1 has the following phase values :

Point →	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*
Phase φ°	0	90	180	270	360 (or 0)	90	180	270

Fig. 15.2 gives the electromagnetic spectrum. The type of electromagnetic wave is known by its wavelength or its frequency. However, all these travel with a velocity *approximately* equal to 3×10^8 m/s. *This velocity forms the basis of all electromagnetic measurements.*

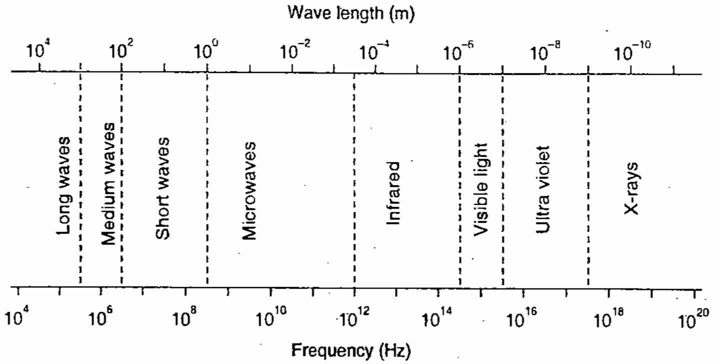


FIG. 15.2 ELECTROMAGNETIC SPECTRUM.

Measurement of transit times

Fig. 15.3 (*a*) shows a survey line *AB*, the length *D* of which is to be measured using EDM equipment placed at ends *A* and *B*. Let a transmitter be placed at *A* to propagate electromagnetic waves towards *B*, and let a receiver be placed at *B*, along with a timer. If the timer at *B starts* at the instant of transmission of wave from *A*, and *stops* at the instant of reception of incoming wave at *B*, the *transit time* for the wave from *A* and *B* in known.

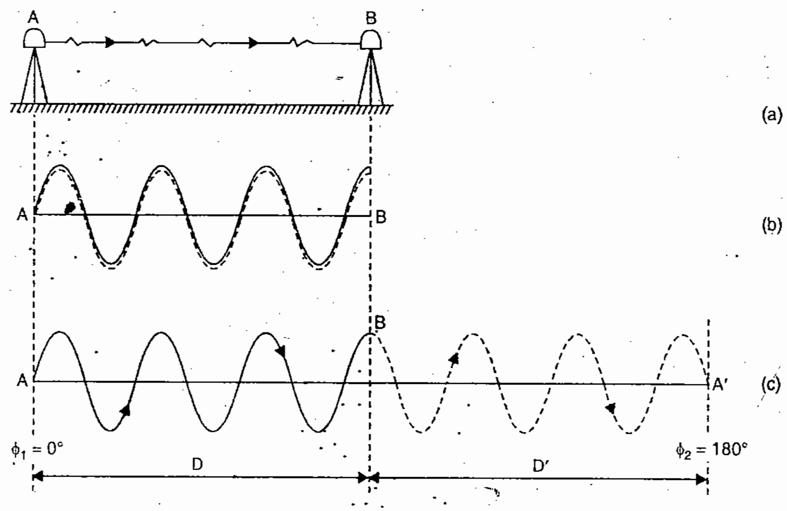


FIG. 15.3. MEASUREMENT OF TRANSIT TIME.

From this transit time, and from the known velocity of propagation of the wave, the distance *D* between *A* and *B* can be easily computed. However, this transit time is of the order of 1×10^{-6} s which requires very advanced electronics. Also it is extremely difficult to start the timer at *B* when the wave is transmitted at *A*. Hence a reflector is placed at B instead of a receiver. This reflector reflects the waves back towards A, where they are received (Fig. 15.3 (b)]. Thus the equipment at A acts both as a transmitter as well as receiver. The *double transit time* can be easily measured at A. This will require EDM timing devices with an accuracy of $\pm 1 \times 10^{-9}$ s.

Phase Comparison

Generally, the various commercial EDM systems available do not measure the transit time directly. Instead, the distance is determined by measuring the phase difference between the transmitted and reflected signals. This phase difference can be expressed as fraction of a cycle which can be converted into units of time when the frequency of wave is known. Modern techniques can easily measure upto $\frac{1}{1000}$ part of a wavelength.

In Fig. 15.3 (b), the wave transmitted from A towards B is instantly reflected from B towards A, and is then received at A, as shown by dotted lines. The same sequence is shown in Fig. 15.3 (c) by opening out the wave, wherein A and A' are the same. The distance covered by the wave is

$$2D = n\lambda + \Delta\lambda \qquad ...(15.2)$$

where d = distance between A and B

λ = wavelength

n = whole number of wavelengths travelled by the wave

$\Delta\lambda$ = fraction of wavelength travelled by the wave.

The measurement of component $\Delta\lambda$ is known as *phase comparison* which can be achieved by electrical phase detectors.

Let φ_1 = phase of the wave as it is transmitted at A

φ_2 = phase of the wave as it is received at A'

Then $$\Delta\lambda = \frac{\text{phase difference in degrees}}{360°} \times \lambda \quad \text{or} \quad \Delta\lambda = \frac{(\varphi_2 - \varphi_1)°}{360°} \times \lambda \qquad ...(15.3)$$

The determination of other component $n\lambda$ of equation 15.2 is referred to as *resolving the ambiguity of the phase comparison*, and this can be achieved by any one of the following methods.

(i) by increasing the wavelength manually in multiples of 10, so that a coarse measurement of D is made, enabling n to be deduced.

(ii) by measuring the line AB using three different (but closely related) wavelengths, so as to form three simultaneous equations of the form

$$2D = n_1\lambda_1 + \Delta\lambda_1 \;;\; 2D = n_2\lambda_2 + \Delta\lambda_2 \;;\; 2D = n_3\lambda_3 + \Delta\lambda_3$$

The solution of these may give the value of D.

In the latest EDM equipment, this problem is solved automatically, and the distance D is displayed.

For example, let λ for the wave of Fig. 15.3 (c) be 20 m. From the diagram, $n = 6$, $\varphi_1 = 0°$ and $\varphi_2 = 180°$.

$$\therefore \quad 2D = n\lambda + \Delta\lambda = n\lambda + \frac{\varphi_2 - \varphi_1}{360°} \times \lambda$$

or $$2D = (6 \times 20) + \frac{180 - 0}{360} \times 20$$

$$\therefore \quad D = 65 \text{ m.}$$

This measurement of distance by EDM is analogous to the measurement of AB by taping. wherein

$$D = ml + \Delta l$$

where l = length of tape = 20 m (say)

m = whole No. of tapes = 3

Δl = remaining length of the tape in the end bay

Hence the recording in the case of taping will be $D = 3 \text{ m} \times 20 + 5 = 65$ m.

15.3. MODULATION

As stated above, EDM measurements involve the measurement of fraction $\Delta\lambda$ of the cycle. Modern phase comparison techniques are capable of resolving to better than $\frac{1}{1000}$ part of a wavelength. Assume $\pm$ 10 mm to be the accuracy requirement for surveying equipment, this must represent $\frac{1}{1000}$ of the measuring wavelength. This means that $\lambda = 10 \times 1000$ mm = 10 m, which is a maximum value. However, by use of modern circuitory, λ can be increased to 40 m, which corresponds to $f = 7.5 \times 10^6$ Hz. Thus, the lowest value of f that can be used is 7.5×10^6 Hz. At present, the range of frequencies that can be used in the measuring process is limited to approximately 7.5×10^6 to 5×10^8 Hz.

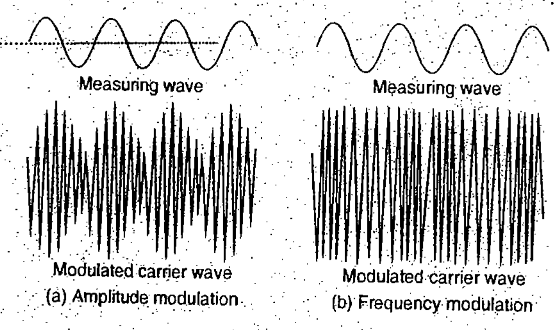


FIG. 15.4. MODULATION

In order to increase the accuracy, it is desirable to use an extremely high frequency of propagation. However, the available phase comparison techniques cannot be used at frequencies greater than 5×10^8 Hz which corresponds to a wavelength $\lambda = 0.6$ m. On the other hand, the lower frequency value in the range of 7.5×10^6 to 5×10^8 Hz is not suitable for direct transmission through atmosphere because of the effects of interference, reflection, fading and scatter.

The problem can be overcome by the technique of *modulation* wherein the measuring wave used for phase comparison is superimposed on a *carrier wave* of much higher frequency. EDM uses two methods of modulating the carrier wave :

(a) Amplitude modulation. (b) Frequency modulation.

In *amplitude modulation*, the carrier wave has constant frequency and the modulating wave (the measuring wave) information is conveyed by the amplitude of the carrier waves. In the *frequency modulation*, the carrier wave has constant amplitude, while its frequency varies in proportion to the amplitude of the modulating wave. *Frequency modulation* is used in all microwave EDM instruments while *amplitude modulation* is done in *visible light instruments* and infrared instruments using higher carrier frequencies.

15.4. TYPES OF EDM INSTRUMENTS

Depending upon the type of carrier wave employed, EDM instruments can be classified under the following three heads :

(*a*) Microwave instruments

(*b*) Visible light instruments

(*c*) Infrared instruments.

For the corresponding frequencies of carrier waves, reader may refer back to Fig. 15.2. It is seen that all the above three categories of EDM instruments use short wavelengths and hence higher frequencies.

1. Microwave instruments

These instruments come under the category of long range instruments, where in the carrier frequencies of the range of 3 to 30 GHz (1 GHz = 10^9) enable distance measurements upto 100 km range. *Tellurometer* come under this category.

Phase comparison technique is used for distance measurement. This necessitates the erection of some form of *reflector* at the remote end of the line. If *passive reflector* is placed at the other end, a weak signal would be available for phase comparison. Hence an electronic signal is required to be erected at the reflecting end of the line. This instrument, known as *remote instrument* is identical to the *master instrument* placed at the measuring end. The *remote instrument* receives the transmitted signal, amplifies it and transmits it back to the master in exactly the phase at which it was received. This means that microwave EDM instruments require two instruments and two operators. Frequency modulation is used in most of the microwave instruments. The method of varying the measuring wavelength in multiplies of 10 is used to obtain an unambiguious measurement of distance. The microwave signals are radiated from small aerials (called *dipoles*) mounted in front of each instrument, producing directional signal with a beam of width varying from 2° to 20°. Hence the alignment of master and remote units is not critical. Typical maximum ranges for microwave instruments are from 30 to 80 km, with an accuracy of ± 15 mm to ± 5 mm/km.

2. Visible light instruments

These instruments use visible light as carrier wave, with a higher frequency, of the order of 5×10^{14} Hz. Since the transmitting power of carrier wave of such high frequency falls off rapidly with the distance, the range of such EDM instruments is lesser than those of microwave units. A *geodimeter* comes under this category of EDM instruments.

The carrier, transmitted as light beam, is concentrated on a signal using lens or mirror system, so that signal loss does not take place.

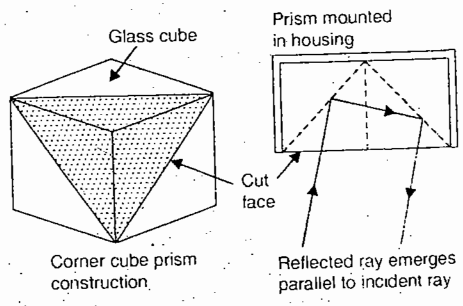


FIG. 15.5. CORNER CUBE PRISM.

Since the beam divergence is less than 1°, accurate alignment of the instrument is necessary. *Corner-cube prisms*, shown in Fig. 15.5 are used as reflectors at the remote end. These prisms are constructed from the corners of glass cubes which have been cut away in a plane making an angle of 45° with the faces of the cube.

The light wave, directed into the cut-face is reflected by highly silvered inner surfaces of the prism, resulting in the reflection of the light beam along a parallel path. This is obtainable over a range of angles of incidence of about 20° to the normal of the front face of the prism. *Hence the aligment of the reflecting prism towards the main EDM instrument at the receiver (or transmitting) end is not critical.*

The advantage of visible light EDM instruments, over the microwave EDM instruments is that only one instrument is required, which work in conjunction with the inexpensive corner cube reflector. *Amplitude modulation* is employed, using a form of electro-optical shutter. The line is measured using three different wavelengths, using the same carrier in each case. The EDM instrument in this category have a range of 25 km, with an accuracy of ± 10 mm to ± 2 mm/km. The recent instruments use pulsed light sources and highly specialised modulation and phase comparison techniques, and produce a very high degree of accuracy of ± 0.2 mm to ± 1 mm/km with a range of 2 to 3 km.

3. Infrared instruments

The EDM instruments in this group use near infrared radiation band of wavelength about 0.9 μ m as carrier wave which is easily obtained from gallium arsenide (Ga As) infrared emitting diode. These diodes can be very easily directly *amplitude modulated* at high frequencies. Thus, modulated carrier wave is obtained by an inexpensive method. Due to this reason, there is predominance of infrared instruments in EDM. Wild Distomats fall under this category of EDM instruments.

The power output of the diodes is low. Hence the range of these instruments is limited to 2 to 5 km. However, this range is quite sufficient for most of the civil engineering works. The EDM instruments of this category are very light and compact, and these can be theodolite mounted. This enables angles and distances to be measured simultaneously at the site. A typical combination is Wild DI 1000 infra-red EDM with Wild T 1000 electronic theodolite ('Theomat'). The accuracy obtainable is of the order of ± 10 mm, irrespective of the distance in most cases.

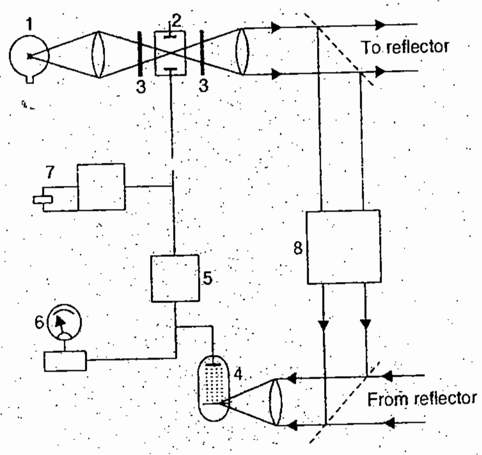
The carrier wavelength in this group is close to the visible light spectrum. Hence infrared source can be transmitted in a similar manner to the visible light system using geometric optics, a lens/mirror system being used to radiate a highly collimated beam of angular divergence of less than 15'. Corner cube prisms are used at the remote end, to reflect the signal.

Electronic tacheometer, such as Wild TC 2000 'Tachymat' is a further development of the infrared (and laser) distance measurer, which combines theodolite and EDM units. Microprocessor controlled angle measurement give very high degree of accuracy, enabling horizontal and vertical angles, and the distances (horizontal, vertical, inclined) to be automatically displaced and recorded.

15.5. THE GEODIMETER

The method, based on the propagation of *modulated light waves*, was developed by E. Bergestrand of the Swedish Geographical Survey in collaboration with the manufacturer, M/s AGA of Sweden. Of the several models of the geodimeter manufacured by them, model 2-A can be used only for observations made at night while model-4 can be used for limited day time observations.

Fig. 15.6 shows the schematic diagram of the geodimeter. Fig. 15.7 shows the photograph of the front panel of model-4 geodimeter mounted on the tripod. The main instrument is stationed at one end of the line (to be measured) with its back facing the other end of the line, while a reflector (consisting either of a spherical mirror or a reflex prism system) is placed at the other end of the line.



1. Incandescent lamp
2. Kerr cell
3. Nicol's prisms
4. Photo tube
5. Variable electrical delay unit
6. Null indicator
7. Crystal controlled oscillator
8. Variable light delay unit.

FIG. 15.6. SCHEMATIC DIAGRAM OF THE GEODIMETER.

The light from an incandescent lamp (1) is focused by means of an achromatic condenser and passed through a Kerr cell (2). The Kerr cell consist of two closely spaced conducting plates, the space between which is filled with nitrobenzene. When ~~high voltage is applied~~ to the plates of the cell and a ray of light is focused on it, the ray is split into two parts, each moving with different velocity. Two Nicol's prisms (3) are placed on either side of the Kerr cell. The light leaving the first Nicol's prisms is plane polarised. The light is split into two (having a phase difference) by the Kerr cell. On leaving the Kerr cell, the light is recombined. However, because of phase difference, the resulting beam is elliptically polarised. Diverging light from the second polariser can be focused to a parallel beam by the transmitter objective, and can then be reflected from a mirror lens to a large spherical concave mirror.

On the other end of the line being measured is put a reflex prism system or a spherical mirror, which reflects the beam of light back to the geodimeter. The receiver system of the geodimeter consists of spherical concave mirror, mirror lens and receiver objective. The light of variable intensity after reflection, impinges on the cathode of the photo tube (4). In the photo tube, the light photons impinge on the cathode causing a few primary electrons to leave and travel, accelerated by a high frequency voltage, to the first dynode, where the secondary emission takes place. This is repeated through a further eight dynodes. The final electron current at the anode is some hundreds of thousand times greater than that at the cathode. The sensitivity of the photo tube is varied by applying the high frequency-Kerr cell voltage between the cathode and the first dynode. The low frequency vibrations are eliminated by a series of electrical chokes and condensers. The passages of this modulating voltage through the instrument is delayed by means of an adjustable

electrical delay unit (5). The difference between the photo tube currents during the positive and negative bias period is measured on the *null indicator* (6) which is a sensitive D.C. moving coil micro-ammeter. In order to make both the negative and positive current intensities equal (*i.e.* in order to obtain null-point), the phase of the high frequency voltage from the Kerr cell must be adjusted ± 90° with respect to the voltage generated by light at the cathode.

Thus, the light which is focused to a narrow beam from the geodimeter stationed at one end to the reflector stationed at the other end of the line, is reflected back to the photo multiplier. The variation in the intensity of this reflected light causes the current from the photo multiplier to vary where the current is already being varied by the direct signal from the crystal controlled oscillater (7). The phase difference between the two pulses received by the cell are a measure of the distance between geodimeter and the reflector (*i.e.*, length of the line).

The distance can be measured at different frequencies. On Model-2A of the geodimeter, three frequencies are available. Model-4 has four frequencies. Four phase positions are available on the *phase position indicator*. Changing phase indicates that the polarity of the Kerr cell terminals of high and low tension are reversed in turn. The 'fine' and 'coarse' delay switches control the setting of the electrical delay between the Kerr cell and the photo multiplier. The power required is obtained from a mobile gasoline generator. Model-4 has a night range of 15 meters to 15 km, a daylight range of 15 to 800 metres and an average error of ± 10 mm ± five millionth of the distance. It weighs about 36 kg without the generator.

15.6. THE TELLUROMETER

In the Tellurometer, high frequency radio waves (or microwaves) are used intead of light waves. It can be worked with a light weight 12 or 24 volt battery. Hence the instrument is highly portable. Observations can be taken both during day as well as night, while in the geodimeter, observations are normally restricted in the night. However, two such Tellurometres are required, one to be stationed at each end of the line, with two highly skilled persons, to take observations. One instrument is used as the *master set or control set* while the other instrument is used as the *remote set* or *slave set*. In Model MRA-2 (manufactured by M/s. Cooke, Troughton and Simms Ltd), each set can either be used as the master set or remote set by switching at 'master' and 'remote' positions respectively. Fig. 15.8 shows the photograph of Tellurometer (Model MRA-2).

Fig. 15.9 shows the block diagram of the Tellurometer, first designed by Mr. T.L. Wadley of the South African Council for Scientific and Industrial Research. Radio waves are emitted by the master instrument at a frequency of 3000 Mc.s. (3×10^9 c.p.s.) from a klystron and have superimposed on them a crystal controlled frequency of 10 Mc.s. The high frequency wave is termed as *carrier wave*. Waves at high frequencies can be propagated in straight line paths other than long distance much more readily. The low frequency wave is known as the *pattern wave* and is used for making accurate measurements. The light frequency pattern wave is thus said to be frequency modulated (F.M.) by low frequency pattern wave. This modulated signal is received at the remote station where a second klystron is generating another carrier wave at 3033 Mc.s. The difference between the two high

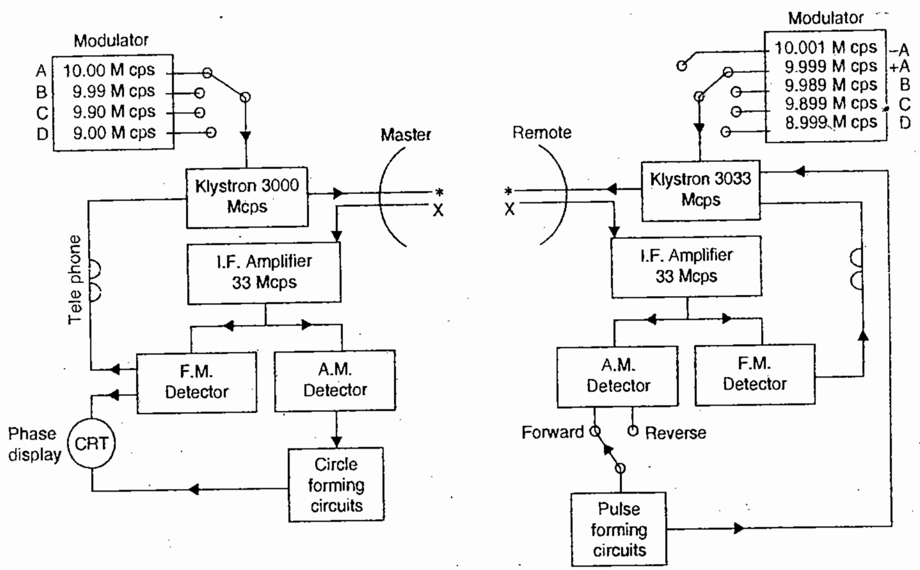


FIG. 15.9 BLOCK DIAGRAM OF THE TELLUROMETER SYSTEM.

frequencies, *i.e.* 3033 – 3000=33 Mc.s. (known as *intermediate frequency*) is obtained by an electrical 'mixer', and is used to provide sufficient sensitivity in the internal detector circuits at each instrument. In addition to the carrier wave of 3033 Mc.s., a crystal at the remote station is generating a frequency of 9.999 Mc.s. This is *heterodyned* with the incoming 10 Mc.s. to provide a 1 k c.p.s. signal. The 33 Mc.s. intermediate frequency signal is amplitude modulated by 1 k c.p.s. signal. The amplitude modulated signal passes to the amplitude demodulator, which detects the 1 k c.p.s. frequency. At the pulse forming circuit, a pulse with a repitition frequency of 1 k c.p.s. is obtained. This pulse is then applied to the klystron and frequency modulates the signal emitted, *i.e.*, 3033 Mc.s. modulated by 9.999 Mc.s. and pulse of 1 k c.p.s. This signal is received at the master station. A further compound heterodyne process takes place here also, where by the two carrier frequencies subtract to give rise to an intermediate frequency of 33 Mc.s. The two *pattern frequencies* of 10 and 9.999 Mc.s. also subtract to provide 1 k c.p.s *reference frequency* as amplitude modulation. The *change in the phase between this and the remote 1 k c.p.s. signal is a measure of the distance*. The value of phase delay is expressed in time units and appear as a *break* in a circular trace on the oscilloscope cathode ray tube.

Four low frequencies (*A, B, C* and *D*) of values 10.00, 9.99, 9.90 and 9.00 Mc.p.s. are employed at the master station, and the values of phase delays corresponding to each of these are measured on the oscilloscope cathode ray tube. The phase delay of *B, C* and *D* are subtracted from *A* in turn. The *A* values are termed as 'fine readings' and the *B, C, D* values as 'coarse readings'. The oscilloscope scale is divided into 100 parts. The wavelength of 10 Mc.s. pattern wave as approximately 100 ft. (30 m) and hence

each division of the scale represents 1 foot on the two-way journey of the waves or approximately 0.5 foot on the length of the line. The final readings of *A*, *A* – *B*, *A* – *C* and *A* – *D* readings are recorded in millimicro seconds (10^{-9} seconds) and are converted into distance readings by assuming that the velocity of wave propagations as 299,792.5 km/sec. It should be noted that the success of the system depends on a property of the *heterodyne process*, that the phase difference between two heterodyne signal is maintained in the signal that results from the mixing.

15.7. WILD 'DISTOMATS'

Wild Heerbrugg manufacture EDM equipment under the trade name 'Distomat', having the following popular models :

1. Distomat DI 1000
2. Distomat DI 5S
3. Distomat DI 3000
4. Distomat DIOR 3002
5. Tachymat TC 2000 (Electronic tacheometer)

1. Distomate DI 1000

Wild Distomat DI 1000 is very small, compact EDM, particularly useful in building construction, civil engineering construction, cadastral and detail survey, particularly in popplated areas where 99% of distance measurements are less than 500 m. It is an EDM that makes the tape redundant. It has a range of 500 m to a single prism and 800 m to three prisms (1000 m in favourable conditions), with an accuracy of 5 mm + 5 ppm. It can be fitted to all Wild theodolites, such as T 2000, T 2000 S, T 2 etc.

The infra-red measuring beam is reflected by a prism at the other end of the line. In the five seconds that it takes, the DI 1000 adjusts the signal strength to optimum level, makes 2048 measurements on two frequencies, carries out a full internal calibration, computes and displays the result. In the tracking mode 0.3 second updates follow the initial 3- second measurement. The whole sequence is automatic. One has to simply point to the reflector, touch a key and read the result.

The Wild modular system ensures full compatibility between theodolites and Distomats. The DI 1000 fits T 1, T 16 and T 2 optical theodolites, as shown in Fig. 15.10 (*a*). An optional key board can be used. It also combines with Wild T 1000 electronic theodolite and the Wild T 2000 informatics theodolite to form fully electronic *total station* [Fig. 15.10 (*b*)]. Measurements, reductions and calculations are carried out automatically. The DI 1000 also connects to the GRE 3 data terminal [Fig. 15.10 (*c*)]. If the GRE 3 is connected to an electronic theodolite with DI 1000, all information is transferred and recorded at the touch of a single key. The GRE can be programmed to carry out field checks and computations.

When DI 1000 distomat is used separately, it can be controlled from its own key board. There are only three keys on the DI 1000, each with three functions, as shown Fig. 15.11. Colour coding and a logical operating sequence ensure that the instrument is easy to use. The keys control all the functions. There are no mechanical switches. The liquid-crystal display is unusually large for a miniaturized EDM. Measured distances are presented clearly and unambiguously with appropriate symbols for slope, horizontal distance, height and setting out. In test mode, a full check is provided of the display, battery power and return signal strength. An audible tone can be activated to indicate return of signal. Scale (ppm) and additive constant (mm) settings are displayed at the start of each measurement.

Input of ppm takes care of any atmospheric correction, reduction to sea level and projection scale factor. The mm input corrects for the prism type being used. The microprocessor permanently stores ppm and mm values and applies them to every measurement. Displayed heights are corrected for earth curvature and mean refraction.

DI 1000 is designed for use as the standard measuring tool in short range work. A single prism reflector is sufficient for most tasks. For occasional longer distance (upto 800 m), a three prism reflector can be used. The power is fed from NiCd rechargeable batteries.

2. Distomat DI 5S

Wild DI 5S is a medium range infra-red EDM controlled by a small powerful microprocessor, It is multipurpose EDM. The 2.5 km range to single prism covers all short-range requirements: detail, cadastral, engineering, topograhic survey, setting out, mining, tunnelling etc. With its 5 km range to 11 prisms, it is ideal for medium-range control survey : traversing, trigonometrical heighting, photogrammetric control, breakdown of triangulation and GPS networks etc. Finely tuned opto-electronics, a stable oscillator, and a microprocessor that continuously evaluates the results, ensure the high measuring accuracy of 3 mm + 2 ppm standard deviation is standard measuring mode and 10 m + 2 ppm standard deviation in tracking measuring mode.

Fig. 15.12 shows the view of DI 5S. It has three control keys, each with three functions. There are no mechanical switches. A powerful microprocessor controls the DI 5S. Simply touch the DIST key to measure. Signal attenuation is fully automatic. Typical measuring time is 4 seconds. In tracking mode, the measurement repeats automatically every second. A break in the measuring beam due to traffic etc., does not affect the accuracy. A large, liquid-crystal display shows the measured distance clearly and unambiguously throughout the entire measuring range of the instrument. Symbols indicate the displayed values. A series of dashes shows the progress of the measuring cycle. A prism constant from – 99 mm to + 99 mm can be input for the prism type being used. Similarly, ppm values from – 150 ppm to + 150 ppm can be input for automatic compensation for atmospheric conditions, height above sea level and projection scale factor. These values are stored until replaced by new values. The microprocessor corrects every measurement automatically.

DI 5S can be also fitted to Wild electronic theodolites T 1000 and T 2000 [Fig. 15.13 (*a*)] or to Wild optical theodolites T 1, T 16, T 2, [Fig. 15.13 (*b*)]. The infra-red measuring beam is parallel to the line of signal. Only a single pointing is needed for both angle and distance measurements. When fitted to an optical theodolite, an optional key board [Fig. 15.13 (*b*] covert it to efficient low cost effective total station. The following parameters are directly obtained for the corresponding input values (Fig. 15.14):

(*a*) Input the vertical angle for
  (*i*) Horizontal distance
  (*ii*) Height difference corrected for earth curvature and mean refraction.
(*b*) Input the horizontal angle for
  (*i*) Coordinate differences ΔE and ΔN.
(*c*) Input the distance to be set out for
  (*i*) ΔD, the amount by which the reflector has to be moved forward or back.

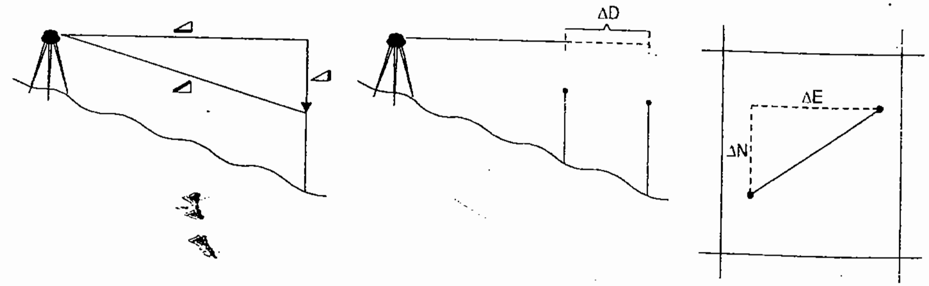


FIG. 15.14

When fitted to an electronic theodolite (T 1000 or T 2000) DI 5S transfers the slope distance to the theodolite. The following reductions (Fig. 15.15) are carried out in the theodolite microprocessor.

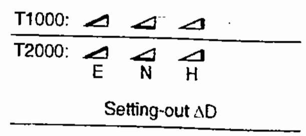


FIG. 15.15

The DI 5S can also be connected to GRE 3 data terminal for automatic data acquisition. The EDM is powered from a NiCd rechargeable battery. When used on a Wild electronic theodolite, DI 5S is powered from the theodolites' internal battery.

3. Distomats DI 3000 and DI 3002

Wild DI 3000 distomat is a long range infra-red EDM in which infra-red measuring beam is emitted from a *laser diode*. Class I laser products are inherently safe ; maximum permissible exposure cannot be exceeded under any condition, as defined by International Electrotechnical Commission.

The DI 3000 is a *time-pulsed* EDM. The time needed for a pulse of infra-red light to travel from the instrument to the reflector and back is measured. The displayed result is the mean of hundreds or even thousands of time-pulsed measurements. The pulse technique has the following important advantages :

(*i*) **Rapid measurement**. It provides 0.8 second rapid measurement for detail surveys, tacheometry, setting out etc. It is advantageous for long range measurements in turbulent atmospheric conditions.

(*ii*) **Long range**. Its range is 6 km to 1 prism in average conditions and 14 km to 11 prisms in excellent conditions.

(*iii*) **High accuracy**. Accuracy is 5 mm + 1 ppm standard deviation. A calibrated quartz crystal ensures 1 ppm frequency stability throughout the temperature range – 20° C to + 60° C. In tracking mode, accuracy is 10 mm + 1 ppm.

(*iv*) **Measurement to moving targets**. For measuring to moving targets, the times-pulse measuring technique is very advantageous. There are practically no limits to the speed at which an object may move. For this purpose, a reflector should be suitably attached to the object or vehicle to which measurements have to be made. The distomat can be (*a*) manually controlled, (*b*) connected to Wild GRE 3 data terminal for automatic recording or (c) connected on-line to a computer for remote control and real-time processing results. The following important operations can be achieved on moving objects:

(a) ***Offshore surveys***. DI 3000 can be mounted on electronic theodolite for measuring to ships, dredgers and pipe laying barges, positioning oil rigs, controlling docking manoeuvres etc. (Fig. 15.16).

(b) ***Controlling objects on rails.*** DI 3000 can be connected on-line to computer for controlling the position of cranes, gantries, vehicles, machinery on rails, tracked equipments etc. (Fig. 15.17).

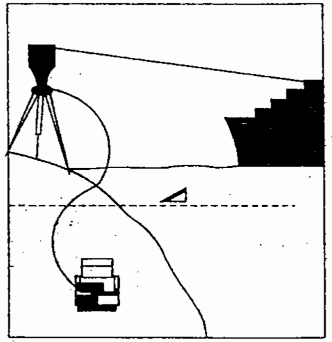


FIG. 15.16.

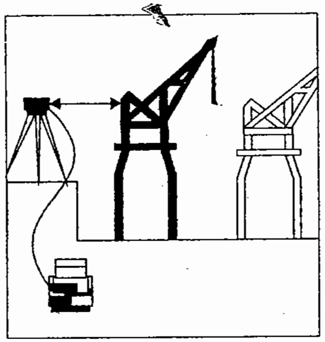


FIG. 15.17

(c) ***Monitoring movements in deformation surveys.*** DI 3000 can be connected with GRE 3 or computer for continuous measurement to rapidly deforming structures, such as bridges undergoing load tests (Fig. 15.18).

(d) ***Positioning moving machinery.*** DI 3000 can be mounted on a theodolite for continuous determination of the position of mobile equipment. (Fig. 15.19).

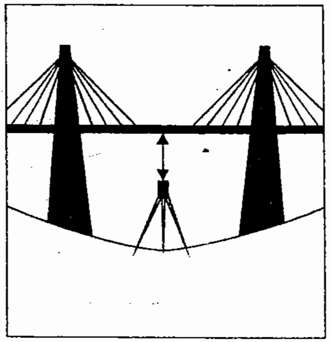


FIG. 15.18.

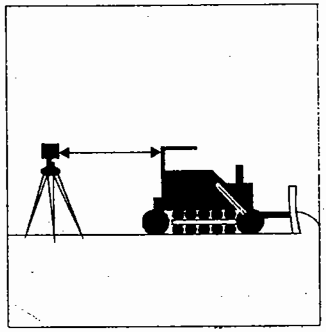


FIG. 15.19

The DI 3000 is also ideal all-round EDM for conventional measurements in surveying and engineering : control surveys, traversing, trigonometrical heighting, breakdown of GPS

networks, cadastral, detail and topographic surveys, setting out etc. It combines with Wild optical and electronic theodolites. It can also fit in a yoke as stand-alone instrument.

Fig. 15.20 shows a view of DI 3000 distomat, with its control panel, mounted on a Wild theodolite. The large easy to read LCD shows measured values with appropriate signs and symbols. An acoustic signal acknowledges key entries and measurement. With the DI 3000 on an optical theodolite, reductions are via the built in key board. For cadastral, detail, engineering and topographic surveys, simply key in the vertical circle reading. The DI 3000 displays slope and horizontal distance and height difference. For traversing with long-range measurements, instrument and reflector heights can be input the required horizontal distance. The DI 3000 displays the amount by which the reflector has to be moved forward or back. All correction parameters are stored in the non-volatile memory and applied to every measurements. Displayed heights are corrected for earths curvature and mean refraction.

4. Distomat DIOR 3002

The DIOR 3002 is a special version of the DI 3000. It is designed specifically for distance measurement without reflector. Basically, DIOR 3002 is also time pulsed Infra-red EDM. When used without reflectors, its range varies from 100 m to 250 m only, with a standard deviation of 5 mm to 10 mm. The interruptions of beam should be avoided. However, DIOR 3002, when used with reflectors have a range of 4 km to 1 prism, 5 km to 3 prisms and 6 km to 11 prisms.

Although, the DIOR 3002 can fitted on any of the main Wild theodolites, the T 1000 electronic theodolite is the most suitable. When used without reflectors, it can carry the following operation.

(i) ***Profile and cross-sections*** (Fig. 15.21). DIOR 3002 with an electronic theodolite, can be used for measuring tunnel profiles and cross-sections, surveying stopes, caverns, interior of storage tanks, domes etc.

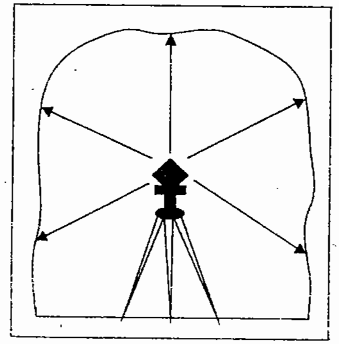


FIG. 15.21

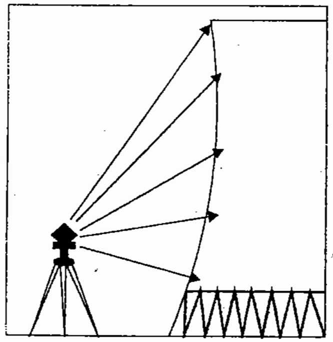


FIG. 15.22

(ii) ***Surveying and monitoring buildings, large objects quarries, rock faces, stock piles*** (Fig. 15.22). DIOR 3002 with a theodolite and data recorder can be used for measuring and monitoring large objects, to which access is difficult, such as bridges, buildings, cooling towers, pylons, roofs, rock faces, towers, stock piles etc.

***(iii)* Checking liquid levels, measuring to dangerous or touch sensitive surfaces** (Fig. 15.23). DIOR 3002 on line to a computer can be used for controlling the level of liquids in storage tanks, determining water level in docks and harbours, measuring the amplitude of waves around oil rigs etc., also for measuring to dangerous surfaces such as furnace linings, hot tubes, pipes and rods.

***(iv)* Landing and docking manoeuvres** (Fig. 15.24). It can be used for measuring from helicopters to landing pads and from ships to piers and dock walls.

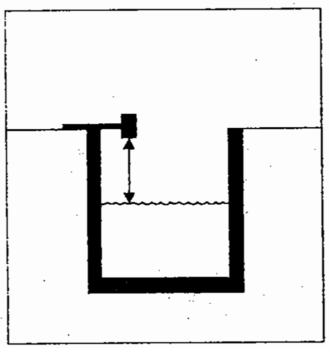


FIG. 15.23

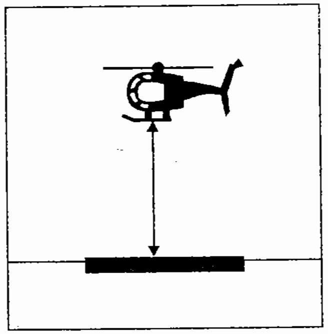


FIG. 15.24

5. WILD 'TACHYMAT' TC 2000

Wild TC 2000 (Fig. 15.25) is a fully integrated instrument. It combines in one instrument the advantages of the T 2000 informatics theodolite with the distance measuring capabilities of Wild distomats. For applications where distances and angles are always required, and instrument with built-in EDM is particularly useful. Wild TC 2000 having built-in EDM is a single package *total station* which can be connected to Wild GRE 3 data terminal. The same telescope is used for observing and distance measurement. The infra-red measuring beam coincides with the telescope line of sight.

The telescope is panfocal, magnification and field of view vary with focusing distance. When focusing to distant targets, the magnification is 30 X. Over shorter distances, the field widens and the magnification is reduced for easy pointing to the prism. the telescope with coarse and fine focusing is used for both angle and distant measurement.

The whole unit, theodolite and built-in EDM, is operated from the key board. Angles and distances can be measured in both telescope positions. Single attenuation and distance measurements are fully automatic. Normal distance measurement takes 6.5 seconds with a standard deviation of 3 mm ± 2 ppm. In tracking mode, the display updates at 2.5 seconds intervals and the standard deviation is 10 mm to 20 mm. The 2 km range to a single prism covers all short range work. Maximum range is about 4 km in average atmospheric conditions.

Key board control. The entire equipment—angle and distance measuring and recording—is controlled from the key board. The multifunctional capability of the instrument makes it suitable for almost any task.

Pair of displayed values. The panel directly displays angles, distances, heights and co-ordinates of the observed point where the signal (reflector prism) is kept (Fig. 15.26). Height above datum and station co-ordinates can be entered and stored.

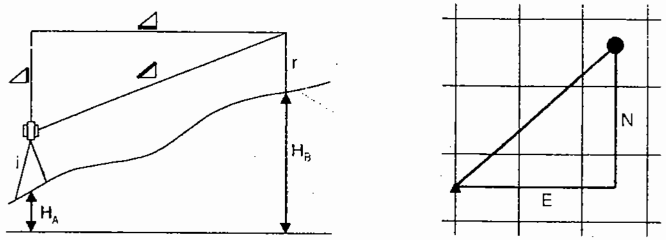


FIG. 15.26.

The following pairs are displayed :

(i)	Hz circle	V circle
(ii)	Hz circle	Horizontal distance
(iii)	Height difference	Height above datum
(iv)	Slope distance	V circle
(v)	Easting	Northing.

Remote object height (ROH). The direct height readings of inaccessible objects, such as towers and power lines, the height difference and height above datum changes with telescope. However, both the pairs of values are displayed automatically. The microprocessor applies the correction for earth curvature and mean refraction. Corrected heights are displayed.

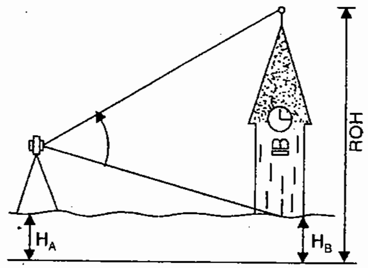


FIG. 15.27. DETERMINATION OF ROH.

Traversing program. The coordinates of the reflector and the bearing on the reflector can be stored for recall at the next set-up. Thus, traverse point coordinates are available in the field and closures can be verified immediately.

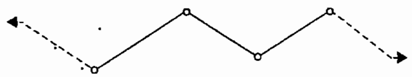


FIG. 15.28. TRAVERSING.

Setting out for direction, distance and height. The required direction and horizontal distance can be entered. The instrument displays:

(i) The angle through which the theodolite has to be turned.

(ii) The amount by which the reflector has to be moved.

And by means of remote object height (ROH) capability, markers can be placed at the required height above datum.

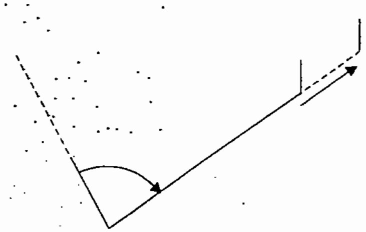


FIG. 15.29. SETTING OUT.

Setting out can be fully automated with GRE 3 data terminal. The bearings and distances to the points to be set out are computed from the stored coordinates and transferred automatically to the TC 2000 *total station.*

Differences in Hz and V. For locating targets and for real time comparisons of measurements in deformation and monitoring surveys, it is advantageous to display angular differences in the horizontal and vertical planes between a required direction and the actual telescope pointing.

15.8. TOTAL STATION

A total station is a combination of an electronic theodolite and an electronic distance meter (EDM). This combination makes it possible to determine the coordinates of a reflector by aligning the instruments cross-hairs on the reflector and simultaneously measuring the vertical and horizontal angles and slope distances. A micro-processor in the instrument takes care of recording, readings and the necessary computations. The data is easily transferred to a computer where it can be used to generate a map. Wild, 'Tachymat' TC 2000, described in the previous article is one such *total station* manufactured by M/s Wild Heerbrugg.

As a teaching tool, a total station fulfills several purposes. Learning how to properly use a total station involves the physics of making measurements, the geometry of calculations, and statistics for analysing the results of a traverse. In the field, it requires team work, planning, and careful observations. If the total station is equipped with data-logger it also involves interfacing the data-logger with a computer, transferring the data, and working with the data on a computer. The more the user understands how a total station works, the better they will be able to use it.

Fundamental measurements : When aimed at an appropriate target, a total station measures three parameters (Fig. 15.31)

1. The *rotation* of the instrument's optical axis from the instrument north in a horizontal plane : i.e. *horizontal angle*

2. The *inclination* of the optical axis from the local vertical i.e. *vertical angle.*

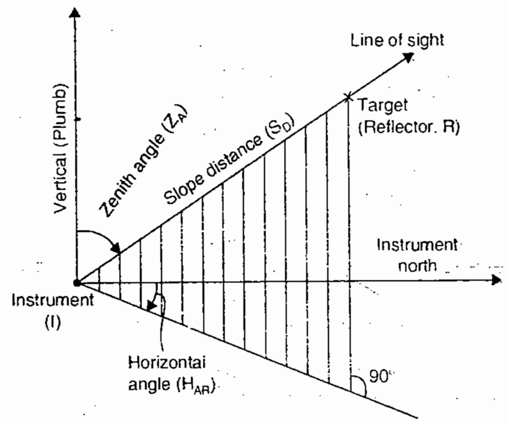


FIG. 15.31. FUNDAMENTAL MEASUREMENTS MADE BY A TOTAL STATION

3. The distance between the instrument and the target i.e. *slope distance*

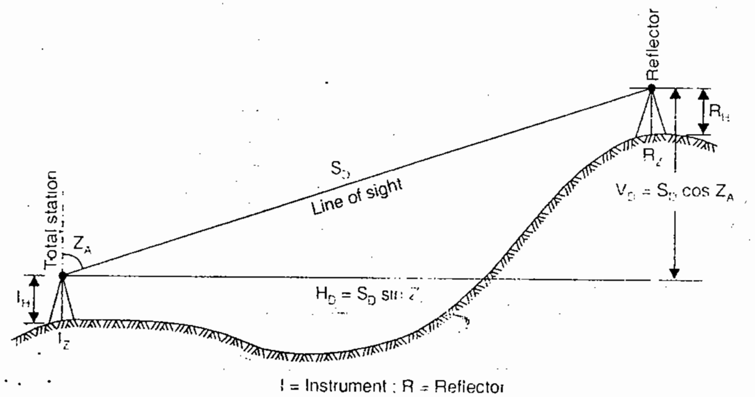
All the numbers that may be provided by the total station are derived from these three *fundamental measurements*

1. Horizontal Angle

The horizontal angle is measured from the zero direction on the horizontal scale (or *horizontal circle*). When the user first sets up the instrument the choice of the zero direction is made — this is *Instrument North.* The user may decide to set zero (North) in the direction of the long axis of the map area, or choose to orient the instrument approximately to True, Magnetic or Grid North. The zero direction should be set so that it can be recovered if the instrument was set up at the same location at some later date. This is usually done by sighting to another benchmark, or to a distance recognizable object. Using a magnetic compass to determine the orientation of the instrument is not recommended and can be very inaccurate. Most total stations can measure angle to at least 5 seconds, or 0.0013888°. The best procedure when using a Total Station is to set a convenient "north" and carry this through the survey by using backsights when the instrument is moved.

2. Vertical Angle : The vertical angle is measured relative to the local vertical (plumb) direction. The vertical angle is usually measured as a *zenith angle* (0 ° is vertically up, 90° is horizontal, and 180° is vertically down), although one is also given the option of making 0 ° horizontal. The zenith angle is generally easier to work with. The telescope will be pointing downward for zenith angles greater than 90 ° and upward for angles less than 90 °.

Measuring vertical angles requires that the instrument be exactly vertical. It is very difficult to level an instrument to the degree of accuracy of the instrument. Total stations contain an internal sensor (the vertical compensator) that can detect small deviations of the instrument from vertical. Electronics in the instrument then adjust the horizontal and



S_D = slope distance; V_D = Vertical distance between telescope and reflector; H_D = Horizontal distance; Z_A = Zenith angle; I_H = Instrument height; R_H = Reflector height; I_Z = Ground elevation of total station; R_Z = Ground elevation of reflector.

FIG 15.32 GEOMETRY OF THE INSTRUMENT (TOTAL STATION) AND REFLECTOR.

vertical angles accordingly. The compensator can only make small adjustments, so the instrument still has to be well leveled. If it is too far out of level, the instrument will give some kind of "tilt" error message.

Because of the compensator, the instrument has to be pointing exactly at the target in order to make an accurate vertical angle measurement. If the instrument is not perfectly leveled then as you turn the instrument about the vertical axis (i.e., change the horizontal angle) the vertical angle displayed will also change.

3. Slope Distance : The instrument to reflector distance is measured using an Electronic Distance Meter (EDM). Most EDM's use a Gallium Aresnide Diode to emit an infrared light beam. This beam is usually modulated to two or more different frequencies. The infrared beam is emitted from the total station, reflected by the reflector and received and amplified by the total station. The received signal is then compared with a reference signal generated by the instrument (the same signal generator that transmits the microwave pulse) and the phase-shift is determined. This phase shift is a measure of the travel time and thus the distance between the total station and the reflector.

This method of distance measurement is not sensitive to phase shifts larger than one wavelength, so it cannot detect instrument-reflector distances greater than 1/2 the wave length (the instrument measures the two-way travel distance). For example, if the wavelength of the infrared beam was 4000 m then if the reflector was 2500 m away the instrument will return a distance of 500 m.

Since measurement to the nearest millimeter would require very precise measurements of the phase difference, EDM's send out two (or more) wavelengths of light. One wavelength may be 4000 m, and the other 20 m. The longer wavelength can read distances from 1 m to 2000 m to the nearest meter, and then the second wavelength can be used to measure distances of 1 mm to 9.999 m. Combining the two results gives a distance accurate to millimeters. Since there is overlap in the readings, the meter value from each reading can be used as a check.

For example, if the wavelengths are $\lambda_1 = 1000$ m and $\lambda_2 = 10$ m, and a target is placed 151.51 metres away, the distance returned by the λ_1 wavelength would be 151 metres, the λ_2 wavelength would return a distance of 1.51 m. Combining the two results would give a distance of 151.51 m.

Basic calculations

Total Stations only measure three parameters : *Horizontal Angle, Vertical Angle,* and *Slope Distance.* All of these measurements have some error associated with them, however for demonstrating the geometric calculations. we will assume the readings are without error.

Horizontal distance

Let us use symbol *I* for instrument (total station) and symbol *R* for the reflector. In order to calculate coordinates or elevations it is first necessary to convert the slope distance to a horizontal distance. From inspection of Fig. 15.32 the horizontal distance (H_D) is

$$H_D = S_D \cos(90° - Z_A) = S_D \sin Z_A \quad \text{...(1) ...(15.4)}$$

where S_D is the slope distance and Z_A is the zenith angle. The horizontal distance will be used in the coordinate calculations.

Vertical distance

We can consider two vertical distances. One is the *Elevation Difference (dZ)* between the two points on the ground. The other is the *Vertical Difference* (V_D) between the tilting axis of the instrument and the tilting axis of the reflector. For elevation difference calculation we need to know the height of the tilting-axis of the instrument (I_H), that is the height of the center of the telescope, and the height of the center of the reflector (R_H)

The way to keep the calculation straight is to imagine that you are on the ground under the instrument (Fig. 15.32). If you move up the distance I_H, then travel horizontally to a vertical line passing through the reflector then up (or down) the vertical distance (V_D) to the reflector, and then down to the ground (R_H) you will have the elevation difference dZ between the two points on the ground. This can be written as

$$dZ = V_D + (I_H - R_H) \quad \text{...(2) ...(15.5)}$$

The quantities I_H and R_H are measured and recorded in the field. The vertical difference V_D is calculated from the vertical angle and the slope distance (see Fig. 15.32)

$$V_D = S_D \sin(90° - Z_A) = S_D \cos Z_A \quad \text{...(3) ...(15.6)}$$

Substituting this result (3) into equation (2) gives

$$dZ = S_D \cos Z_A + (I_H - R_H) \quad \text{...(4) ...(15.7)}$$

where dZ is the change in elevation with respect to the ground under the total station. We have chosen to group the instrument and reflector heights. Note that if they are the same then this part of the equation drops out. If you have to do calculations by hand it is convenient to set the reflector height the same as the instrument height.

If the instrument is at a known elevation, I_Z, then the elevation of the ground beneath the reflector, R_Z, is

$$R_Z = I_Z + S_D \cos Z_A + (I_H - R_H) \quad \text{...(5) ...(15.8)}$$

Coordinate calculations

So far we have only used the vertical angle and slope distance to calculate the elevation of the ground under the reflector. This is the Z-coordinate (or elevation) of a point. We

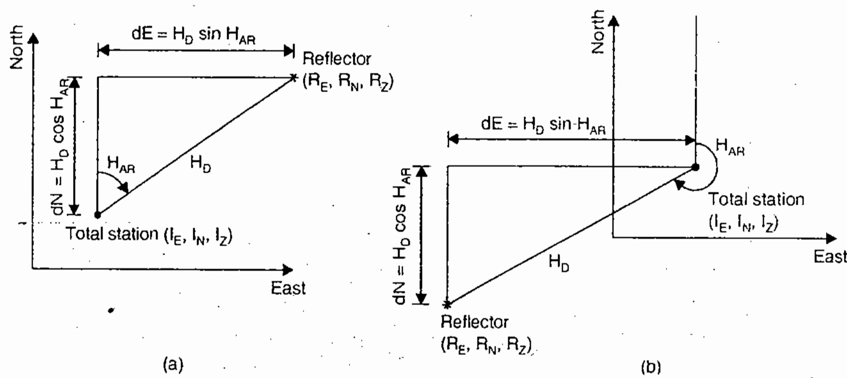


FIG. 15.33. COMPUTATION OF EAST AND NORTH COODINATES OF THE REFLCTOR

now want to calculate the X- (or East) and Y- (or North) coordinates. The zero direction set on the instrument is instrument north. This may not have any relation on the ground to true, magnetic or grid north. The relationship must be determined by the user. Fig. 15.33 shows the geometry for two different cases, one where the horizontal angle is less than 180° and the other where the horizontal angle is greater than 180°. The sign of the coordinate change [positive in Figure 15.33 (*a*) and negative in Fig. 15.33 (*b*)] is taken care of by the trigonometric functions, so the same formula can be used in all cases. Let us use symbol E for easting and N for northing, and symbol I for the instrument (i.e total station) and R for the reflector. Let R_E and R_N be the easting and northing of the reflector and I_E and I_N be the easting and northing of the instrument (i.e. total station)

From inspection of Fig. 15.33 the coordinates of the relector relative to the total station are

dE = Change in Easting $= H_D \sin H_{AR}$

dN = Change in Northing $= H_D \cos H_{AR}$

where H_D is the horizontal distance and H_{AR} is the horizontal angle measured in a clockwise sense from instrument north. In terms of fundamental measurments (i.e. equation 1) this is the same as

$$dE = S_D \sin Z_A \sin H_{AR} \quad ...(15.9)$$

$$dN = S_D \cos (90° - Z_A) \cos H_{AR} = S_D \sin Z_A \cos H_{AR} \quad ...(15.10)$$

If the easting and northing coordinates of the instrument station are known (in grid whose north direction is the same as instrument north) then we simply add the instrument coordinates to the change in easting and northing to get the coordinates of the reflector. The coordinates of the ground under the reflector, in terms of fundamental measurments are :

$$R_E = I_E + S_D \sin Z_A \sin H_{AR} \quad ...(15.11)$$

$$R_N = I_N + S_D \sin Z_A \cos H_{AR} \quad ...(15.12)$$

$$R_Z = I_Z + S_D \cos Z_A + (I_H - R_H) \quad ...(15.13)$$

where I_E, I_N, and I_Z are the coordinates of the total station and R_E, R_N, R_Z are the coordinates of the ground under the reflector. These calculations can be easily done in a spreadsheet program.

All of these calculations can be made within a total station, or in an attached electronic notebook. Although it is tempting to let the total station do all the calculations, it is wise to record the three fundamental measurements. This allows calculations to be checked, and provides the basic data that is needed for a more sophisticated error analysis.

16

Remote Sensing

16.1. INTRODUCTION

Remote sensing is broadly defined as science and art of collecting information about objects, area or phenomena from distance without being in physical contact with them. In the present context, the definition of remote sensing is restricted to mean the process of acquiring information about any object without physically contacting it in any way regardless of whether the observer is immediately adjacent to the object or millions of miles away. Human eye is perhaps the most familiar example of a remote sensing system. In fact, sight, smell and hearing are all rudimentary forms of remote sensing. However, the term remote sensing is restricted to methods that employ electromagnetic energy (such as light, heat, microwave) as means of detecting and measuring target characteristics. Air craft and satellites are the common *platforms* used for remote sensing. Collection of data is usually carried out by highly sophisticated *sensors* (i.e. camera, multispectral scanner, radar etc.). The information carrier, or communication link is the electromagnetic energy. Remote sensing data basically consists of wave length intensity information by collecting the electromagnetic radiation leaving the object at specific wavelength and measuring its intensity. Photo interpretation can at best be considered as the *primitive form* of remote sensing. Most of the modern remote sensing methods make use of the reflected infrared bands, thermal infrared bands and microwave portion of the electromagnetic spectrum.

Classification of remote sensing

Remote sensing is broadly classified into two categories

(*i*) Passive remote sensing and (*ii*) Active remote sensing

Passive remote sensing : It uses sun as a source of EM energy and records the energy that is naturally radiated and/or reflected from the objects.

Active remove sensing : It uses its own source of EM energy, which is directed towards the object and return energy is measured.

16.2. HISTORICAL SKETCH OF REMOTE SENSING

Remote sensing became possible with the invention of camera in the nineteenth century. Astronomy was one of the first fields of science to exploit this technique. Although, it was during the first World War that free flying aircrafts were used in a remote sensing role, but the use of remote sensing for environmental assessment really became established after the second World War. It not only proved the value of aerial photography in land reconnaissance and mapping, but had also driven technological advances in air borne camera design, film characteristics and photogrammetic analysis.

However, upto early 1960's air borne missions were one of the expensive surveys, providing data for relatively small area at a single instant of time. Moreover, all the photographs were black and white. Colour photography came into existence after the invention of infrared films in 1950. From about 1960, remote sensing underwent a major development when it extended to space and sensors began to be placed in space. From 1970's started the new era of remote sensing. The first designated earth resources satellite was launched in July 1972, originally named as ERTS-1 which is now referred as Landstat-1. It was designed to acquire data from earth surface as systematic, repetitive and multi-spectral basic. The first Radar remote sensing satellite, SEASAT, was launched in 1978.

Prior to mid 1980's, the majority of satellites had been deployed by USA and USSR. France launched first of SPOT series in 1985 and in 1988, first Indian Remote Sensing Satellite (IRS-1A) was put into orbit. Satellites launched by Japan include JERS (Japanese Earth Resources Satellite) and MOS (Marine Observation Satellite). Radar satellites have been launched in 1991 and 1995 by European Consortium (ERS) and by Canada in 1995 (RADARSAT).

16.3. IDEALIZED REMOTE SENSING SYSTEM

An idealised remote sensing system consists of the following stages (Fig. 16.1)

1. Energy source.
2. Propagation of energy through atmosphere
3. Energy interaction with earth's surface features.
4. Airborne/space borne sensors receiving the reflected and emitted energy
5. Transmission of data to earth station and generation of data produce.
6. Multiple-data users

1. The energy source : The uniform energy source provides energy over all wave lengths. The passive RS system relies on sun as the strongest source of EM energy and measures energy that is either reflected and or emitted from the earth's surface features. However, active RS systems use their own source of EM energy.

2. Propagation of energy from the atmosphere : The EM energy, from the source pass through the atmosphere on its way to earth's surface. Also, after reflection from the earth's surface, it again pass through the atmosphere on its way to sensor. The atmosphere modifies the wave length and spectral distribution of energy to some extent, and this modification varies particularly with the wave length.

3. Interaction of energy with surface features of the earth : The interaction of EM energy, with earth's surface features generates reflected and/or emitted *signals* (spectral response patterns or signatures). The spectral response patterns play a central role in detection, identification and analysis of earth's surface material.

4. Air borne/space borne sensors : Sensors are electromagnetic instruments designed to receive and record retransmitted energy. They are mounted on satellites, aeroplanes or even balloons. The sensors are highly sensitive to wave lengths, yielding data on the absolute brightness from the object as a function of wavelength.

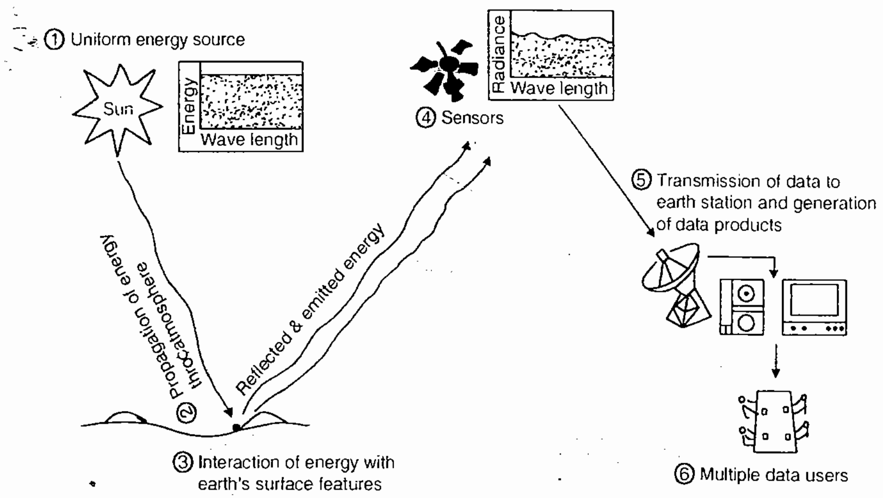


FIG. 16.1 IDEALISED REMOTE SENSING SYSTEM

5. Transmission of data to earth station and data product generation : The data from the sensing system is transmitted to the ground based earth station along with the telemetry data. The real-time (instantaneous) data handling system consists of high density data tapes for recording and visual devices (such as television) for quick look displays. The *data products* are mainly classified into two categories :

(*i*) Pictorial or Photographic product (analogue)

and (*ii*) Digital product

6. Multiple data users : The multiple data users are those who have knowledge of great depth, both of their respective disciplines as well as of remote sensing data and analysis techniques. The same set of *data* becomes various forms of *information* for different users with the understanding of their field and interpretation skills.

16.4. BASIC PRINCIPLES OF REMOTE SENSING

Remote sensing employ electromagnetic energy and to a great extent relies on the interaction of electromagnetic energy with the matter (object). It refers to the sensing of EM radiation, which is reflected, scattered or emitted from the object.

16.4.1. ELECTROMAGNETIC ENERGY

It is a form of energy that moves with the velocity of light (3×10^8 m/sec) in a harmonic pattern consisting of sinusoidal waves, equally and repetitively spaced in time. It has two fields : (*i*) electrical field and (*ii*) magnetic field, both being orthogonal to each other. Fig. 16.2 show the electromagnetic wave pattern, in which the electric components are vertical and magnetic components are horizontal.

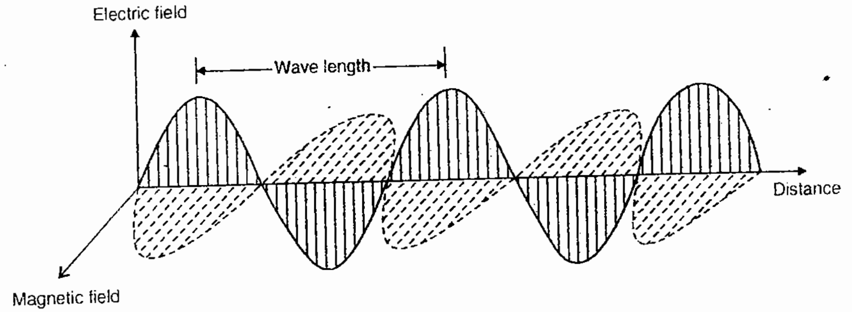


FIG. 16.2

Electromagnetic energy consists of *photons* having particle like properties such as energy and momentum. The EM energy is characterised in terms of velocity c ($\approx 3 \times 10^8$ m/s), wave length λ and frequency f. These parameters are related by the equation :

$$\lambda = \frac{c}{f} \quad ...(16.1)$$

where λ = wave length, which is the distance between two adjacent peaks. The wave lengths sensed by many remote sensing systems are extremely small and are measured in terms of micro meter (μm or 10^{-6} m) or nanometer (nm or 10^{-9} m)

f = frequency, which is defined as the number of peaks that pass any given point in one second and is measured in Hertz (Hz).

The *amplitude* is the maximum value of the electric (or magnetic) field and is a measure of the amount of energy that is transported by the wave.

Wave theory concept explains how EM energy propagates in the form of a wave. However, this energy can only be detected when it interacts with the matter. This interaction suggests that the energy consists of many discrete units called *photons* whose energy (Q) is given by :

$$Q = h.f = \frac{h.c}{\lambda} \quad ...(16.2)$$

where h = Plank's constant $= 6.6252 \times 10^{-34}$ J-s

The above equation suggests that shorter the wave length of radiation, more is the energy content.

16.4.2. ELECTROMAGNETIC SPECTRUM

Although *visible light* is the most obvious manifestation of EM radiation, other forms also exist. EM radiation can be produced at a range of wave lengths and can be categorised according to its position into discrete regions which is generally referred to *electro-magnetic spectrum*. Thus the electromagnetic spectrum is the continuum of energy that ranges from meters to nano-meters in wave length (Fig. 16.3) travels at at the speed of light and propagates through a vacuum like the outer space (Sabine, 1986). All matter radiates a

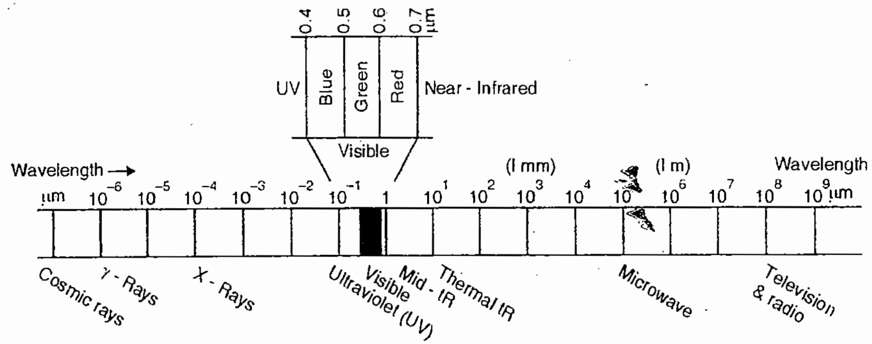


FIG. 16.3. ELECTROMAGNETIC SPECTRUM

range of electromagnetic energy, with the peak intensity shifting toward progressively shorter wave length at an increasing temperature of the matter. In general, the wave lengths and frequencies vary from shorter wavelength-high frequency cosmic waves to long wave length-low frequency radio waves (Fig. 16.3 and Table 16.1).

TABLE 16.1. ELECTROMAGNETIC SPECTRAL REGIONS (SABINE, 1987)

Region	*Wave length*	*Remarks*
1. Gamma ray	< 0.03 nm	Incoming radiation is completely absorbed by the upper atmosphere and is not available for remote sensing
2. X-ray	0.03 to 3.0 nm	Completely absorbed by atmosphere. Not employed in remote sensing
3. Ultraviolet	0.3 to 0.4 μ m	Incoming wavelengths less than 0.3 μ m are completely absorbed by ozone in the upper atmosphere
4. Photo graphic UV band	0.3 to 0.4 μm	Transmitted through atmosphere. Detectable with film and photodetectors, but atmospheric scattering is severe
5. Visible	0.4 to 0.7 μm	Images with film and photo detectors. Includes reflected energy peak of earth at 0.5 μm.
6. Infrared	0.7 to 1.00 μm	Interaction with matter varies with wave length. Atmospheric transmission windows are separated.
7. Reflected IR band	0.7 to 3.0 μm	Reflected solar radiation that contains information about thermal properties of materials. The bands from 0.7 to 0.9 μm is detectable with film and is called the photographic IR band.
8. Thermal IR	3 to 5 μ m	Principal atmospheric windows in the 8 to 14 μm thermal region. Images at these wavelengths are acquired by optical mechanical scanners and special vidicon systems but not by film. Microwave 0.1 to 30 cm longer wavelength can penetrate clouds, fog and rain. Images may be acquired in the active or passive mode
9. Radar	0.1 to 30 cm	Active form of microwave remote sensing. Radar images are acquired at various wavelength bands.
10. Radio	> 30 cm	Longest wavelength portion of electromagnetic spectrum. Some classified radars with very long wavelengths operate in this region.

Earths atmosphere absorbs energy in Gamma ray, X-ray and most of the ultra-violet region. Therefore, these regions are not used for remote sensing. *Remote sensing deals with energy in visible, infrared, thermal and microwave regions.* These regions are further subdivided into bands such as blue, green, red (in visible region), near infrared, mid-infrared, thermal and microwave etc. It is important to realize that significant amount of remote sensing performed within infrared wave length is not related to heat. It is photographic infrared at a slightly longer wave length (invisible to human eye) than red. Thermal infrared remote sensing is carried out at longer wave lengths.

16.4.3. WAVE LENGTH REGIONS AND THEIR APPLICATIONS IN REMOTE SENSING

Fig 16.3 shows the EM spectrum which is divided into discrete regions on the basis of wavelength. Romote sensing mosthy deals with energy in visible (Blue, green, red) infrared (near-infrared, mid-infrared, thermal-infrared) regions Table 16.2 gives the wave length region along with the principal applications in remote sensing. Energy reflected from earth during daytime may be recorded as a function of wavelength. The maximum amount of energy is reflected at 0.5 μ m, called the *reflected energy peak.* Earth also radiates energy both during day and night time with maximum energy radiated at 9.7 μ m, called *radiant energy peak.*

TABLE 16.2. WAVE LENGTH REGIONS AND THEIR APPLICATIONS IN REMOTE SENSING

Region	*Wave length* (μm)	*Principal Applications*
(a) Visible Region		
1. Blue	0.45 – 0.52	Coastal morphology and sedimentation study, soil and vegetation differentiation, coniferous and deciduous vegetation discrimination.
2. Green	0.52 – 0.60	Vigor assessment, Rock and soil discrimination, Turbidity and bathymetry studies.
3. Red	0.63 – 0.69	Plant species differentiation
(b) Infrared Region		
4. Near Infrared	0.76 – 0.90	Vegetation vigour, Biomass, delineation of water features, land forms/geomorphic studies.
5. Mid-infrared	1.55 – 1.75	Vegetation moisture content, soil moisture content, snow and cloud differentiation
6. Mid-infrared	2.08 – 2.35	Differentiation of geological materials & soils
7. Thermal IR	3.0 – 5.0	For hot targets, i.e. Fires and volcanoes
8. Thermal IR	10.4 – 12.5	Thermal sensing, vegetation discrimination, volcanic studies.

16.4.4. CHARACTERISTICS OF SOLAR RADIATION

All objects above 0°K emit EM radiation at all wavelengths due to conversion of heat energy into EM energy. All stars and planets emit radiation. Our chief star, the Sun, is almost a spherical body with a diameter of 1.39×10^6 km. The continuous conversion of hydrogen to helium which is the main constituent of the Sun, generates the energy that is radiated from the outer layers. Passive remote sensing uses Sun as its source of EM radiation. Sun is the strongest source of radiant energy and can be approximated by a body source of temperature 5750 – 6000° K. Although Sun produces EM radiation across a range of wave lengths, the amount of energy it produces is not evenly distributed along this range. Approximately 43% is radiated within the visible wavelength (0.4 to 0.7 μm), and 48% of the energy is transmitted at wave length greater than 0.7 μ m, mainly within infrared range.

If the energy received at the edge of earth's atmosphere were distributed evenly over the earth, it would give an average incident flux density of 1367 W/m². This is known as the *solar constant.* Thirty five percent of incident radiant flux is reflected back by the earth. This includes the energy reflected by clouds and atmosphere. Seventeen percent of it is absorbed by the atmosphere while 48% is absorbed by the earth's surface materials (Mather, 1987).

16.4.5. BASIC RADIATION LAWS

Stefan-Boltzmann law

All bodies above temperature of 0° K emit EM radiation and the energy radiated by an object at a particular temperature is given by

$$M = \sigma T^4 \qquad ...(16.3)$$

where M = total *spectral exitance* of a black body, W/m²

σ = Stefan-Boltzmann constant = 5.6697×10^{-11} W/m²/K⁴

T = absolute temperature

A black body is a hypothetical ideal radiator that totally absorbs and remits all energy incident upon it. The distribution of spectral exitance for a black body at 5900° K closely approximates the sun's spectral exitance curve (Mather 1987), while the earth can be considered to act like a black body with a temperature of 290 °K.

Wien's displacement law

The wave length at which a black body radiates its maximum energy is inversely proportional to temperature and is given by

$$\lambda_m = \frac{A}{T} \qquad ...(16.4)$$

λ_m = wave length of maximum spectral exitance

A = Wien's constant = 2.898×10^{-3} mK

T = temperature of the body

As the temperature of the black body increases, the dominant wave length of the emitted radiation shifts towards shorter wave length.

3. *Plank's law*

The total energy radiated in all directions by unit area in unit time in a spectral band for a given by is given by

$$M_\lambda = \frac{C_1}{\lambda^5 \cdot e^{(C_2/\lambda T)-1}} \qquad ...(16.5)$$

where M_λ = Spectral exitance per unit wave length

C_1 = First radiation constant $= 3.742 \times 10^{16}$ W/m^2

C_2 = Second radiation constant $= 1.4388 \times 10^{2}$ mK

It enables to assess the proportion of total radiant exitance within selected wave length.

16.5. EM RADIATION AND THE ATMOSPHERE

In remote sensing, EM radiation must pass through atmosphere in order to reach the earth's surface and to the sensor after reflection and emission from earth's surface features. The water vapour, oxygen, ozone, CO_2, aerosols, etc. present in the atmosphere influence EM radiation through the mechanism of (*i*) scattering, and (*ii*) absorption.

Scattering

It is unpredictable diffusion of radiation by molecules of the gases, dust and smoke in the atmosphere. Scattering reduces the image contrast and changes the spectral signatures of ground objects. Scattering is basically classified as (*i*) selective, and (*ii*) non-selective, depending upon the size of particle with which the electromagnetic radiation interacts. The *selective scatter* is further classified as (*a*) *Rayleigh's scatter*, and (*b*) *Mies scatter*.

Rayleigh's scatter: In the upper part of the atmosphere, the diameter of the gas molecules or particles is much less than the wave length of radiation. Hence haze results on the remotely sensed imagery, causing a bluish grey cast on the image, thus reducing the contrast. Lesser the wave length, more is the scattering.

Mie's scatter : In the lower layers of atmosphere, where the diameter of water vapour or dust particles approximately equals wave length of radiation, Mie's scatter occurs.

Non-selective scatter : Non-selective scattering occurs when the diameter of particles, is several times more (approximately ten times) than radiation wavelength. For visible wave lengths, the main sources of non-selective scattering are pollen grains, cloud droplets, ice and snow crystals and raindrops. It scatters all wave length of visible light with equal efficiency. It justifies the reason why cloud appears white in the image.

Absorption

In contrast to scattering, atmospheric absorption results the effective loss of energy as a consequence of the attenuating nature of atmospheric constituents, like molecules of ozone, CO_2 and water vapour. Oxygen absorbs in the ultraviolet region and also has an absorption band centered on 6.3 μ m. Similarly CO_2 prevents a number of wave lengths reaching the surface. Water vapour is an extremely important absorber of EM radiation within infrared part of the spectrum.

Atmospheric windows

The amount of scattering or absorption depends upon (*i*) wave length, and (*ii*) composition of the atmosphere. In order to *minimise* the effect of atmosphere, it is essential to choose the regions with high *transmittance*.

The wavelengths at which EM radiations are partially or wholly transmitted through the atmosphere are known as atmospheric windows and are used to acquire remote sensing data.

Typical *atmospheric windows* on the regions of EM radiation are shown in Fig. 16.4.

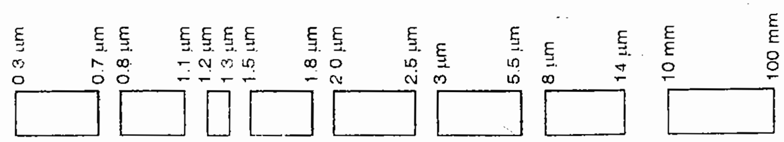


FIG. 16.4 ATMOSPHERIC WINDOWS

The sensors on remote sensing satellites must be designed in such a way as to obtain data within these well defined *atmospheric windows*.

16.6. INTERACTION OF EM RADIATION WITH EARTH'S SURFACE

EM energy that strikes or encounters matter (object) is called *incident radiation*. The EM radiation striking the surface may be (*i*) reflected/scattered, (*ii*) absorbed, and/or (*iii*) transmitted. These processes are not mutually exclusive — EM radiations may be partially reflected and partially absorbed. Which processes actually occur depends on the following factors (1) wavelength of radiation (2) angle of incidence, (3) surface roughness, and (4) condition and composition of surface material.

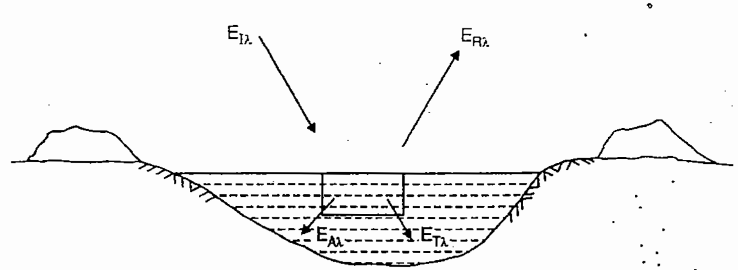


FIG. 16.5. INTERACTION MECHANISM

Interaction with matter can change the following properties of incident radiation: (*a*) Intensity (*b*) Direction (*c*) Wave length (*d*) Polarisation, and (*e*) Phase.

The science of remote sensing detects and records there changes.

The energy balance equation for radiation at a given wave length (λ) can be expressed as follows.

$$E_{I\lambda} = E_{R\lambda} + E_{A\lambda} + E_{T\lambda} \quad \text{...(16.6)}$$

where $E_{I\lambda}$ = Incident energy; $E_{R\lambda}$ = Reflected energy

$E_{A\lambda}$ = Absorbed energy; $E_{T\lambda}$ = Transmitted energy.

The proportion of each fraction ($E_{R\lambda}/E_{A\lambda}/E_{T\lambda}$) will vary for different materials depending upon their composition and condition. Within a given features type, these proportions will vary at different wave lengths, thus helping in discrimination *of different objects*. Reflection, scattering, emission are called surface phenomenon because these are determined by the properties of surface, viz. colour, roughness. Transmission and absorption are called volume phenomena because these are determined by the internal characteristics of the matter, viz. density and condition.

Modification of basic equation : In remote sensing, the amount of reflected energy ($E_{R\lambda}$) is more important than the absorbed and transmitted energies. Hence it is more convenient to rearrange the terms of Eq. 16.6 as follows

$$E_{R\lambda} = E_{I\lambda} - [E_{A\lambda} + E_{T\lambda}] \quad ...(16.7)$$

Eq. 16.7 is known as the *balance equation*.

Dividing all the terms by $E_{I\lambda}$, we get

$$\frac{E_{R\lambda}}{E_{I\lambda}} = 1 - \left[\frac{E_{A\lambda}}{E_{I\lambda}} + \frac{E_{T\lambda}}{E_{I\lambda}}\right]$$

or $$\rho_\lambda = 1 - [\alpha_\lambda + \gamma_\lambda] \quad ...(16.8)$$

where $\rho_\lambda = \frac{E_{R\lambda}}{E_{I\lambda}}$ = Reflectance ; $\alpha_\lambda = \frac{E_{A\lambda}}{E_{I\lambda}}$ = Absorbance ; $\gamma_\lambda = \frac{E_{T\lambda}}{E_{I\lambda}}$ = Transmittance

Since almost all earth surface features are very opaque in nature, the transmittance (γ_λ) can be neglected. Also, according to Kirchoffs' law of physics, the absorbance (α_λ) is taken as *emissivity* (ζ). Hence Eq. 16.8 becomes

$$\rho_\lambda = 1 - \zeta_\lambda \quad ...(16.9)$$

Eq. 16.9 is the fundamental equation by which the conceptual design of remote sensing technology is built.

If $\zeta_\lambda = 0$, then ρ_λ (i.e. reflectance) is equal to one; this means that the total energy incident on the object is reflected and recorded by sensing system. The classical example of this type of object is snow (i.e. white object).

If $\zeta_\lambda = 1$, then $\rho_\lambda = 0$, indicating that whatever the energy incident on the object, is completely absorbed by that object. Black body such as lamp smoke is an example of this type of object. Hence it is seen that reflectance varies from zero for the black body to one for white body.

16.7. REMOTE SENSING OBSERVATION PLATFORMS

Two types of platforms have been in use in remote sensing. :

(*i*) Air borne platforms (*ii*) Space based platforms

1. Air borne platforms

Remote sensing of the surface of the earth has a long history, dating from the use of cameras carried by balloons and pigeons in the eighteenth and nineteenth centuries. Later, air craft mounted systems were developed for military purposes during the early part of 20th century. Air borne remote sensing was the well known remote sensing method used in the initial years of development of remote sensing in 1960's and 1970's. Air crafts were mostly used as RS platforms for obtaining photographs. Aircraft carrying the RS equipment should have maximum stability, free from vibrations and fly with uniform speed. In India, three types of aircrafts are currently used for RS operations: Dakota, AVRO and Beach-craft Superking Air 200. The RS equipments available in India are multi-spectral scanner, ocean colour radiometer, aerial cameras for photography in B/W, colour & near infrared etc.

But the air craft operations are very expensive and moreover for periodical monitoring of constantly changing phenomena like crop growth, vegetation cover etc. Air craft based platform cannot provide cost and time effective solutions.

2. Space based platforms

Space borne remote sensing platforms, such as a satellite, offer several advantages over airborne platforms. It provides *synoptic view* (i.e. observation of large area in a single image), systematic and repetitive coverage. Also, platforms in space are very less affected by atmospheric drag, due to which the orbits can be well defined. Entire earth or any designated portion can be covered at specified intervals synoptically, which is immensely useful for management of natural resources.

Satellite : It is a platform that carries the sensor and other payloads required in RS operation. It is put into earth's orbit with the help of *launch vehicles*.

The space borne platforms are broadly divided into two classes :

(*i*) Low altitude near-polar orbiting satellites. (*ii*) High altitude Geo-stationary satellites

Polar orbiting satellites

These are mostly the remote sensing satellites which revolve around earth in a Sun synchronous orbit (altitude 700-1500 km) defined by its fixed inclination angle from the earth's NS axis. The orbital plane rotates to maintain precise pace with Sun's westward progress as the earth rotates around Sun. Since the position in reference to Sun is fixed, the satellite crosses the equator precisely at the same local solar time.

Geo-stationary satellites

These are mostly communication/meteorological satellites which are stationary in reference to the earth. In other words, their velocity is equal to the velocity with which earth rotates about its axis. Such satellites always cover the fixed area over earth surface and their altitude is about 36000 km.

Landstat Satellite Programme

National Aeronautical and Space Administration (NASA) of USA planned the launching of a series of Earth Resources Technology Satellites (ERTS), and consequently ERTS-1 was launched in July 1972 and was in operation till July 1978. Subsequently, NASA renamed ERTS programme as "Landstat" programme, and ERTS-1 was renamed retrospectively as Landstat-1. Five Landstat satellites have been launched so far. Landstat images have found a large number of applications such as agriculture, botany, cartography, civil engineering, environmental monitoring, forestry, geography, geology, land resources analysis, land use planning, oceanography and water quality analysis.

SPOT Satellite programme

France, Sweden and Belgium joined together and pooled up their resources to develop an earth observation satellite programme known as System Pourl Observation dela Terre, abbreviated as SPOT. The first satellite of the series, SPOT-1 was launched in Feb. 1988. The high resolution data obtained from SPOT sensors, namely, Thematic Mapper (TM) and High Resolution Visible (HRV) have been extensively used for urban planning, urban growth assessment, transportation planning, besides the conventional application related to natural resources.

Indian Remote Sensing Satellites (IRS)

1. Satellite for Earth Observation (SEO-I), now called Bhaskara-I was the first Indian remote sensing satellite launched by a soviet launch vehicle from USSR in June, 1979, into a near circular orbit.

2. SEO-II, (Bhaskara II) was launched in Nov. 1981 from a Soviet cosmodrome.

3. India's first semi-operational remote sensing satellite (IRS) was launched by the Soviet Union in Sept. 1987.

4. The IRS series of satellites launched by the IRS mission are : IRS IA, IRS IB, IRS IC, IRS ID and IRS P4.

16.8. SENSORS

Remote sensing sensors are designed to record radiations in one or more parts of the EM spectrum. Sensors are electronic instruments that receive EM radiation and generate an electric signal that correspond to the energy variation of different earth surface features. The signal can be recorded and displayed as numerical data or an image. The strength of the signal depends upon (*i*) Energy flux, (*ii*) Altitude, (*iii*) Spectral band width, (*iv*) Instantaneous field of view (IFOV), and (*v*) Dwell time.

A *scanning system* employs detectors with a narrow field of view which sweeps across the terrain to produce an image. When photons of EM energy radiated or reflected from earth surface feature encounter the detector, an electrical signal is produced that varies in proportion to the number of photons.

Sensors on board of Indian Remote sensing satellites (IRS)

***1. Linear Imaging and Self Scanning Sensor* (LISS I)**

This payload was on board IRS 1A and 1B satellites. It had four bands operating in visible and near IR region.

***2. Linear Imaging and Self Scanning Sensor* (LISS II)**

This payload was on board IRS 1A and 1B satellites. It has four bands operating in visible and near IR region.

***3. Linear Imaging and Self Scanning Sensor* (LISS III)**

This payload is on board IRS 1C and 1D satellites. It has three bands operating in visible and near IR region and one band in short wave infra region.

***4. Panchromatic Sensor* (PAN)**

This payload is on boards IRS 1C and 1D satellites. It has one band.

***5. Wide Field Sensor* (WiFS)**

This payload is on boards IRS 1C and 1D satellites. It has two bands operating in visible and near IR region.

***6. Modular Opto-Electronic Scanner* (MOS)**

This payload is on board IRS P3 satellite.

***7. Ocean Colour Monitor* (OCM)**

This payload is on board IRS-P4 satellite. It has eight spectral bands operating in visible and near IR region.

***8. Multi Scanning Microwave Radiometer* (MSMR)**

This payload is on board IRS 1D satellite. This is a passive microwave sensor.

16.9. APPLICATIONS OF REMOTE SENSING

Remote sensing affords a practical means for accurate and continuous monitoring of the earth's natural and other resources and of determining the impact of man's activities on air, water and land. The launch of IRS 1C satellite (Dec. 1995) with state of art sensors provided a new dimension and further boosted the applications of space-base remote sensing technology for natural resources management. With the unique combinations of payload, the IRS-1C has already earned the reputation as the '*Satellite for all applications*' IRS-1C/1D carry three imaging sensors (LISS-III, PAN and WiFS) characterised by different resolutions and coverage capabilities. These three imaging sensors provide image data for virtually all levels of applications ranging from cadastral survey to regional and national level mapping. The LISS-III data with 21.2- 23.5 m resolution has significantly improved separability amongst various crops and vegetation types, leading to identification of small fields and better classification accuracy. The frequent availability of data from WiFS payload has helped in monitoring *dynamic phenomena* like vegetation, floods, droughts, forest fire etc.. A major benefit of the multi-sensor IRS-1C/1D payload is the capability to merge the multi spectral LISS-III data, with high resolution PAN imagery. This merger of multispectral and high resolution data facilitates detailed land cover classification and delineation of linear and narrow roads/lanes, structures, vegetation types and parcels of land.

A summary of RS applications is given below, discipline wise.

1. Agriculture

(*i*) Early season estimation of total cropped area
(*ii*) Monitoring crop condition using crop growth profile.
(*iii*) Identification of crops and their coverage estimation in multi-cropped regions.
(*iv*) Crop yield modelling
(*v*) Cropping system/crop rotation studies
(*vi*) Command area management
(*vii*) Detection of moisture stress in crops and quantification of its effect on crop yield
(*viii*) Detection of crop violations
(*ix*) Zoom cultivation—desertification

2. Forestry

(*i*) Improved forest type mapping
(*ii*) Monitoring large scale deforestation, forest fire
(*iii*) Monitoring urban forestry
(*iv*) Forest stock mapping
(*v*) Wild life habitat assessment

3. Land use and soils

(*i*) Mapping land use/cover (level III) at 1 : 25000 scale or better

(*ii*) Change detection
(*iii*) Identification of degraded lands/erosion prone areas
(*iv*) Soil categorisation

4. *Geology*
(*i*) Lithological and structural mapping
(*ii*) Geo morphological mapping
(*iii*) Ground water exploration
(*iv*) Engineering geological studies
(*v*) Geo-environmental studies
(*vi*) Drainage analysis
(*vii*) Mineral exploration
(*viii*) Coal fire mapping
(*ix*) Oil field detection

5. *Urban Land use*
(*i*) Urban land use level IV mapping
(*ii*) Updating of urban transport network
(*iii*) Monitoring urban sprawl
(*iv*) Identification of unauthorised structures.

6. *Water resources*
(*i*) Monitoring surface water bodies frequently and estimation of their spatial extent
(*ii*) Snow-cloud discrimination leading to better delineation of snow area.
(*iii*) Glacier inventory

7. *Coastal Environment*
(*i*) More detailed inventory of coastal land use on 1:25000 scale
(*ii*) Discrimination of coastal vegetation types.
(*iii*) Monitoring sediment dynamics
(*iv*) Siting of coastal structures

8. *Ocean Resources*
(*i*) Wealth of oceans /explorations/productivity
(*ii*) Potential fishing zone
(*iii*) Coral reef mapping
(*iv*) Low tide/high tide marking

9. *Watershed*
(*i*) Delineation of watershed boundaries/partitioning of micro watershed
(*ii*) Watershed characterisation at large scale (size, shape, drainage, landuse/cover)
(*iii*) Siting of water harvesting structures
(*iv*) Monitoring watershed development
(*v*) Major river valley projects.

10. *Environment*
(*i*) Impact assessment on vegetation, water bodies.
(*ii*) Siting applications
(*iii*) Loss of biological diversity/biosphere reserves/ecological hot spot areas /wet land environment.

11. *Street network-based applications*
(*i*) Vehicle routing and scheduling
(*ii*) Location analysis—site selection—evacuation plans.

12. *Land parcel-based applications*
(*i*) Zoning, sub division plan review.
(*ii*) Land acquisition
(*iii*) Environmental management
(*iv*) Water quality management
(*v*) Maintenance of ownership

13. *Natural resources based applications*
(*i*) Management of wild and scenic rivers, recreation resources, flood plains, wet lands, agricultural lands, aquifers, forest, wild life etc..
(*ii*) Environmental Impact Analysts (EIA)
(*iii*) View shed analysis
(*iv*) Hazardous or toxic facility siting
(*v*) Ground water modelling and contamination tracking
(*vi*) Wild life analysis, migration routes planning.

14. *Facilities management*
(*i*) Locating underground pipes, cables
(*ii*) Balancing loads in electrical networks
(*iii*) Planning facility maintenance
(*iv*) Tracking energy use.

15. *Disasters*
(*i*) Mapping flood inundated area, damage assessment
(*ii*) Disaster warning mitigation

16. *Digital elevation models*
(*i*) Contours (> 10 m)
(*ii*) Slope /Aspect analysis
(*iii*) Large scale thematic mapping upto 1:25000 scale.

PROBLEMS

1. What do you understand by remote sensing ? Differentiate between active and passive remote sensing.
2. Explain, with the help of a neat sketch, an idealized remote sensing system
3. Write a detailed note on electro-magnetic energy used for remote sensing.

4. What do you understand by electro-magnetic spectrum ? State the wave length regions, along with their uses, for remote sensing applications.
5. Explain the interaction mechanism of EM radiation with earth's surface, stating the basic interaction equation.
6. Write a note on remote sensing observation platforms
7. Write a note on various types of sensors used for remote sensing in India.
8. Write a detailed note on applications of remote sensing.

Appendix - A

ADDITIONAL EXAMPLES USEFUL FOR COMPETETIVE EXAMINATIONS

Example A-1. *What are the elements of a simple circular curve ? Two straight lines PQ and QR intersect at chainage (375 + 12), the angle of intersection being 110°. Calculate the chainage of tangent points of a right handed circular curve of 400 m radius.*

(U.P.S.C. Engg. Services Exam, 1983)

Solution

Various elements of simple circular curve are : (*i*) length of the curve (*ii*) tangent length (*iii*) length of long chord, (*iv*) apex distance, and (*v*) mid-ordinate.

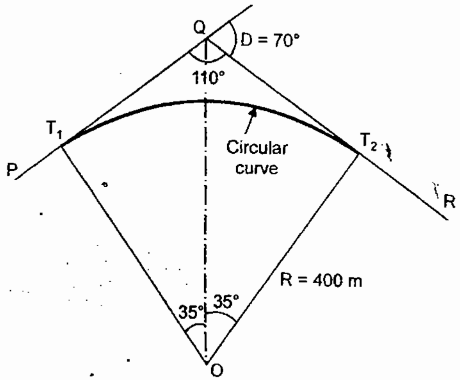


FIG. A-1.

Angle of deflection

$$\Delta = 180° - 110° = 70°$$

Tangent length $T = R \tan \Delta/2$

$$= 400 \tan 70°/2 = 280.08 \text{ m}$$

Length of circular curve, $l = \frac{\pi R \Delta}{180°}$

$$= \frac{\pi (400)(70°)}{180°} = 488.69 \text{ m}.$$

Taking the length of chain as 30 m and length of link as 0.2 m,

Chainage of point of intersection Q

$$= 375 \times 30 + 12 \times 0.2 = 11252.40$$

$\therefore$ Chainage of point of curve, $T_1 = 11252.40 - 280.08 = 10972.32$ m

Chainage of point of tangency, $T_2 = 10972.32 + 488.69 =$ **11461.01 m**

Example A-2. *Two parallel railway lines are to be connected by a reverse curve, each section having the same radius. If the centre lines are 8 m apart, and the maximum distance between tangent points is 32 m, find the maximum allowable radius that can be used.*

(U.P.S.C. Engg. Services Exam. 1985)

Solution : Given Distance $T_1 T_2 = L = 32$ m ; $v = 8$ m

We have the special case of $R_1 = R_2 = R$.

Hence from Eq. 2.37 (*a*), $L^2 = 4 R v$

$$\therefore \quad R = \frac{L^2}{4 v} = \frac{(32)^2}{4 \times 8} = \mathbf{32 \ m}$$

Example A-3. *A right hand circular curve is to connect two straights PQ and QR, the bearings of which are 60° 30′ and 120° 42′ respectively. The curve is to pass through a point S such that QS is 79.44 m and the angle PQS is 34° 36′. Determine the radius of the curve.*

If the chainage of the intersection point is 2049.20 m, determine the tangential angles required to set out the first two pegs on curve at through chainage of 20 m.

(U.P.S.C. Engg. Services Exam., 1986)

Solution

Given : $\angle PQS = \alpha = 34° 36'$

Length $QS = z = 79.44$ m

Bearings of $PQ = 60° 30'$

Bearing of $QR = 120° 42'$

∴ Deflection angle Δ

$= 120° 42' - 60° 30' = 60° 12'$

$\Delta/2 = 30° 6'$

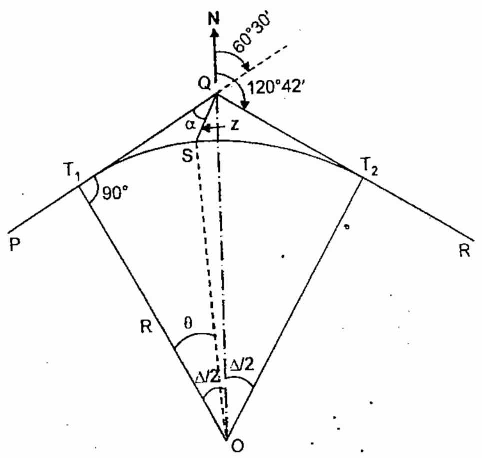


FIG. A-2

From Eq. 1.18,

$$\cos(\alpha + \theta) = \frac{\cos(\alpha + \Delta/2)}{\cos \Delta/2}$$

$$\therefore \cos(\alpha + \theta) = \frac{\cos(34° 36' + 30° 6')}{\cos 30° 6'} = 0.49397$$

$$\therefore \alpha + \theta = 60°.398 = 60° 23'.89$$

$$\therefore \theta = 60° 23'.89 - 34° 36' = 25° 47'.89$$

Again, from Eq. 1.19,

$$R = \frac{z \sin \alpha}{1 - \cos \theta} = \frac{79.44 \sin 34° 36'}{1 - \cos 25° 47'.89} = \mathbf{452.60\ m}$$

Tangent length $T_1Q = R \tan \Delta/2 = 452.60 \tan 30° 6' = 262.36$ m

Given : Chainage of point $Q = 2049.20$ m

∴ Chainage of point $T_1 = 2049.20 - 262.36 = 1786.84$ m $= (89 \times 20 + 6.84)$ m

∴ Length of first subchord = 6.84 m

Now, in general, tangential angle $\delta = \frac{1718.9}{R} C$, where C is the chord length

∴ For first chord, $\delta_1 = \frac{1718.9}{452.60} \times 6.84 = 0° 25' 58.6''$

For second chord $\delta_2 = \frac{1718.9}{452.60} \times 20 = 75'.957 = 1° 15' 57''.4$

The total tangential angles are :

$$\Delta_1 = \delta_1 = 0° 25' 58''.6$$

$$\Delta_2 = \Delta_1 + \delta_2 = 0° 25' 58''.6 + 1° 15' 57''.4 = 1° 41' 56''$$

Example A-4. *Calculate the chainage at the beginning and at the end of a B.G. railway track when it deflects through an angle of 30° with a centre line radius of 300 m. Given*

(i) *The rate of radial gain of acceleration is 0.3 m/sec^2*

(ii) *The designed speed of the train is 60 k.m.p.h.*

(iii) *The chainage of intersection point is 1400 m*

(U.P.S.C. Civil Engg. Services Exam. 1987)

Solution : Given $\alpha = 0.3\ /sec^2/sec$; $\Delta = 30°$

$V = 60$ km/sec ; $R = 300$ m

The length of transition curve at each end of circular curve is given by

$$L \simeq \frac{V^3}{14R} = \frac{(60)^3}{14 \times 300} = 51.43 \text{ m}$$

$$\Delta_s = \frac{1719\ L}{R} \text{ minutes} = \frac{1719 \times 51.43}{300} = 294.69 \text{ minutes} = 4° 54' 41''$$

$$\text{Shift } s = \frac{L^2}{24\ R} = \frac{(51.43)^2}{24 \times 300} = 0.37 \text{ m}$$

$$\text{Total tangent length } TV = (R + s) \tan \frac{\Delta}{2} + \frac{L}{2}$$

$$= (300 + 0.37) \tan \frac{30°}{2} + \frac{51.43}{2} = 106.20 \text{ m}$$

Central angle for circular curve, $\Delta_c = \Delta - 2\,\Delta_s = 30° - 2\,(4° 54' 41'') = 20°.177$

$$\text{Length of circular curve} = \frac{\pi\, R\, \Delta_c}{180°} = \frac{\pi\,(300)\,(20°.177)}{180°} = 105.65 \text{ m}$$

∴ Total length of composite curve = 51.43 + 105.65 + 51.43 = 208.51 m

Now chainage of P.I = 1400 m

Deduct total tangent lengh = 106.20 m

Chainage of point T = 1293.80

Add total length of composite curve = 208.51

Chainage of point T' = 1502.31 m

Hence chainage at the beginning = **1293.80 m**

Chainage at the end = **1502.31 m**

Example A-5. *A 1.5% gradient meets a − 0.5 gradient at a chainage of 1000 m and reduced level of 75 m. The sight distance is 300 m. Determine the length of the* *vertical curve and the R.L. of the tangent points. Assume that the eye level of the driver is 1.125 m above the road surface.*

(U.P.S.C. Engg. Services Exam. 1989)

Solution : Given $g_1 = +1.5\%$; $g_2 = -0.5\%$; $S = 300$ m ; $h_1 = 1.125$ m

Let us assume h_2 = height of obstruction = 0.1 m

$$L = \frac{S^2 (g_1 - g_2)}{200 (\sqrt{h_1} + \sqrt{h_2})^2} = \frac{(300)^2 (1.5 + 0.5)}{200 (\sqrt{1.125} + \sqrt{0.1})^2} = 474.73 \simeq \mathbf{475\ m}\ \text{(say)}$$

R.L. of summit = 75 m

$\therefore$ R.L. of point of commencement $= 75 - \frac{1.5}{100}\left(\frac{475}{2}\right) = \mathbf{71.44\ m}$

R.L. of point of tangency $= 75 - \frac{0.5}{100}\left(\frac{475}{2}\right) = \mathbf{73.81\ m}$

Example A-6. *Calculate the sun's azimuth and hour angle at sunset at a place in latitude 40° N when its declination is 20° N.*

(U.P.S.C. Engg. Services Exam, 1990)

Solution

Consider astronomical triangle ZPM, where M is the position of the sun at horizon and P is the north pole.

ZP = co-latitude $= 90° - \theta = 90° - 40° = 50°$

$ZM = 90°$ since the sun is at horizon at its setting

$MP = 90° - \delta = 90° - 20° = 70°$

From cosine formula,

$$\cos A = \frac{\cos PM - \cos MZ \cos ZP}{\sin MZ \sin ZP}$$

$$= \frac{\cos 70° - \cos 90° \cos 50°}{\sin 90° \sin 50°}$$

$$= \frac{\cos 70°}{\sin 50°} = 0.4464756$$

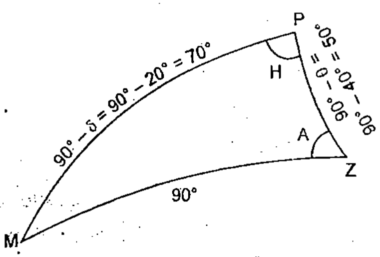


FIG. A-3

$\therefore$ $A = 63°.482215 = \mathbf{63° 28' 56''}$

Also $\cos H = \frac{\cos MZ - \cos MP \cos ZP}{\sin MP \sin ZP} = \frac{\cos 90° - \cos 70° \cos 50°}{\sin 50° \sin 70°} = -0.3054073$

or $H = 170°.78267 = \mathbf{107° 46' 58''}$

Example A-7. *Two tangents intersect at chainage 1200 m, the deflection angle being 40°. Compute the data for setting out a 400 m radius curve by deflection angles and offsets. Take 30 m chord lengths in the general reach.*

(U.P.S.C. Engg. Services Exam 1990)

Solution : Given $\Delta = 40°$; $R = 400$ m.

Length of back tangent $= R \tan \frac{\Delta}{2} = 400 \tan 20° = 145.59$ m

Length of the curve $= \frac{\pi R \Delta}{180°} = \frac{\pi (400)(40°)}{180°} = 279.25$ m

Chainage of point of intersection = 1200 m (given)

$\therefore$ Chainage of tangent $T_1 = 1200 - 145.59 = 1054.41$ m

Chainage of tangent $T_2 = 1054.41 + 279.25$ m $= 1333.66$

$\therefore$ Length of first subchord $= 1080 - 1054.41 = 25.59$ m

Length of last subchord $= 1333.66 - 1320 = 13.66$ m

Length of regular chord = 30 m

Number of full chords $= \frac{279.25 - (25.59 + 13.66)}{30} = 8$

Total No. of chords $= 1 + 8 + 1 = 10$

(1) Computation of Rankine's deflection angles

In general, $\delta = 1718.9 \frac{C}{R}$ minutes

$\therefore$ $\delta_1 = 1718.9 \times \frac{25.59}{400} = 109.97$ minutes $= 1°49'58''$

δ_2 to $\delta_9 = 1718.9 \times \frac{30}{400} = 128.92$ minutes $= 2°8'55''$

$\delta_{10} = 1718.9 \times \frac{13.66}{400} = 58.7$ minutes $= 0° 58' 42''$

$\therefore$ $\Delta_1 = \delta_1 = 1°49'58''$

$\Delta_2 = \Delta_1 + \delta_2 = 1° 49' 58'' + 2° 08' 55'' = 3° 58' 53''$

$\Delta_3 = \Delta_2 + \delta_3 = 3° 58' 53'' + 2° 08' 55'' = 6° 07' 48''$

$\Delta_4 = \Delta_3 + \delta_4 = 6° 07' 48'' + 2° 08' 55'' = 8° 16' 43''$

$\Delta_5 = \Delta_4 + \delta_5 = 8° 16' 43'' + 2° 08' 55'' = 10° 25' 38''$

$\Delta_6 = \Delta_5 + \delta_6 = 10° 25' 38'' + 2° 08' 55'' = 12° 34' 33''$

$\Delta_7 = \Delta_6 + \delta_7 = 12° 34' 33'' + 2° 08' 55'' = 14° 43' 28''$

$\Delta_8 = \Delta_7 + \delta_8 = 14° 43' 28'' + 2° 08' 55'' = 16° 52' 23''$

$\Delta_9 = \Delta_8 + \delta_9 = 16° 52' 23'' + 2° 08' 55'' = 19° 01' 18''$

$\Delta_{10} = \Delta_9 + \delta_{10} = 19° 01' 18'' + 0° 58' 42'' = 20° 00' 00''$ (check)

(2) Computation of offsets from chords produced

In general, $O_0 = \frac{C_n}{2R}(C_{n-1} + C_n)$ ...(2.14 c)

$\therefore$ $O_1 = \frac{C_1}{2R}(0 + C_1) = \frac{C_1^2}{2R} = \frac{(25.59)^2}{2 \times 400} = 0.82$ m

$$O_2 = \frac{C_2}{2R}(C_1 + C_2) = \frac{30}{2 \times 400}(25.59 + 30) = 2.08 \text{ m}$$

$$O_3 \text{ to } O_9 = \frac{C(C + C)}{2R} = \frac{30(30 + 30)}{2 \times 4000} = 2.25 \text{ m}$$

$$O_{10} = \frac{C_{10}}{2R}(C_9 + C_{10}) = \frac{13.66(30 + 13.66)}{2 \times 400} = 0.75 \text{ m}$$

Example A-8. *Two straights AB and BC meet at an inaccessible point B and are to be connected by simple curve of 600 m radius. Two points P and Q were selected in AB and BC respectively, and the following data were obtained :*

$\angle APQ = 150°$, $\angle CQP = 160°$; $PQ = 150.0$ m

Make the necessary calculations for setting out the curve by the method of tangential angles, given that the chainage of P = 1600.00 m take unit chord of 30 m length.

(U.P.S.C. Engg. Services, Exam, 1991)

Solution : Given $\angle APQ = 150°$; $\angle CQP = 160°$; $PQ = 150.0$ m ; $R = 600$ m

$\therefore \quad \angle BPQ = \alpha = 180° - 150° = 30°$

$\angle BQP = \beta = 180° - 160° = 20°$

$\therefore$ Deflection angle $\Delta = \alpha + \beta = 30° + 20° = 50°$

$$\text{Tangent length} = R \tan\frac{\Delta}{2} = 600 \tan 25° = 279.78 \text{ m}$$

$$\text{Length of the curve} = \frac{\pi R \Delta}{180°} = \frac{\pi(600)(50°)}{180°} = 523.60 \text{ m}$$

From triangle BPQ, $BP = PQ \times \dfrac{\sin 20°}{\sin 130°}$

$$= 150 \times \frac{\sin 20°}{\sin 130°} = 66.97 \text{ m}$$

Chainage of P = 1600.00 m (given)

Add PB = 66.97 m

$\therefore$ Chainage of B = 1666.97 m

Subtract tangent length = 279.78 m

$\therefore$ Chainage of T_1 = 1387.19 m

Add length of curve = 523.60 m

$\therefore$ Chainage of T_2 = 1910.79 m

$\therefore$ Length of first subchord = 1410 − 1387.19 = 22.81 m

Length of last subchord = 1910.79 − 1890 = 20.79 m

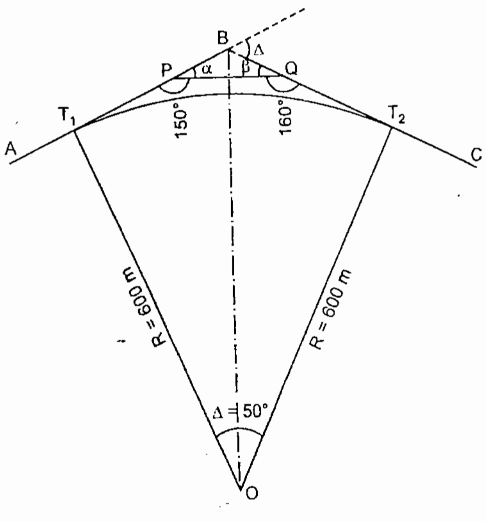


FIG. A-4.

Length of regular chord = 30 m

$$\text{Number of regular chords} = \frac{523.60 - (22.81 + 20.79)}{30} = 16$$

Total Number of chords = 1 + 16 + 1 = 18

$$\delta_1 = \frac{1718.9\, c_1}{R} = \frac{1718.9 \times 22.81}{600} = 65.35 \text{ minutes} = 1°5'\,21''$$

$$\delta_2 \text{ to } \delta_{17} = \frac{1718.9 \times 30}{600} = 85.945 \text{ minutes} = 1°\,25'\,57''$$

$$\delta_{18} = \frac{1718.9 \times 20.79}{600} = 59.5598 \text{ minutes} = 0°\,59'\,34''$$

The values of tangential angles and total deflection angles are tabulated below.

S.N.	*Tangential angle* (δ)	*Total deflection angle* (Δ)	S.N.	*Tangential angle* (δ)	*Total deflection angle* (Δ)
1.	1° 05′ 21″	1° 05′ 21″	10.	1° 25′ 57″	13° 58′ 54″
2.	1° 25′ 57″	2° 31′ 18″	11.	1° 25′ 57″	15° 24′ 51″
3.	1° 25′ 57″	3° 57′ 15″	12.	1° 25′ 57″	16° 50′ 48″
4.	1° 25′ 57″	5° 23′ 12″	13.	1° 25′ 57″	18° 16′ 45″
5.	1° 25′ 57″	6° 49′ 09″	14.	1° 25′ 57″	19° 42′ 42″
6.	1° 25′ 57″	8° 15′ 06″	15.	1° 25′ 57″	21° 08′ 39″
7.	1° 25′ 57″	9° 41′ 03″	16.	1° 25′ 57″	22° 34′ 36″
8.	1° 25′ 57″	11° 07′ 00″	17.	1° 25′ 57″	24° 00′ 33″
9.	1° 25′ 57″	12° 32′ 57″	18.	0° 59′ 34″	25° 00′ 07″ (≏ 25° 00′ 00″)

Example A-9. *A vertical parabolic curve is to be used under a grade separation structure. The minus grade left to right is 4% and the plus grade is 3%. The intersection of two grades is at 435 m and at an elevation of 251.48 m. The curve passes through its fixed point at a chainage of 460 m and R.L. of 260 m. Find the length of the curve. State the criteria which should be considered in setting minimum length of sag vertical curve on highways.*

(U.P.S.C. Engg. Services Exam. 1995)

Solution (Fig. A-5)

Point P is having greater chainage, and hence it will lie to the right hand side of point of intersection B.

$$x_0 = \text{chainage of } P - \text{ chainage of } B = 460 - 435 = 25 \text{ m}$$

Tangent elevation of P on the tangent AB = Ele. of $B - \dfrac{4}{100} \times 25$

$= 251.48 - 1 = 250.48 =$ Ele. of point P_1

Tangent elevation of P on the tangent BC

$= \text{Ele. of } B + \dfrac{3}{100} \times 25 = 251.48 + \dfrac{3}{100} \times 25 = 252.23$ (Ele. of P_2)

Let $\quad y_1 = PP_1 = 260 - 250.48 = 9.52$ m

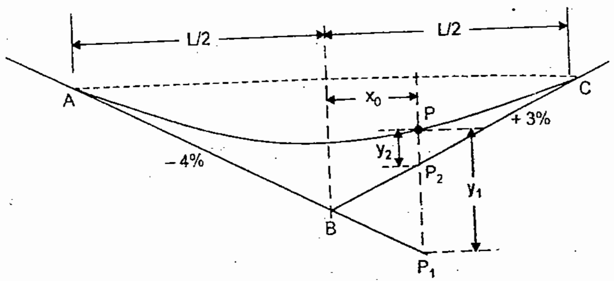


FIG. A-5

$$y_2 = PP_2 = 260 - 252.23 = 7.77 \text{ m}$$

Now $$\frac{y_1}{y_2} = \frac{\left(\frac{L}{2} + x_0\right)^2}{\left(\frac{L}{2} - x_0\right)^2} \quad \text{or} \quad \frac{9.52}{7.77} = \left(\frac{\frac{L}{2} + 25}{\frac{L}{2} - 25}\right)^2$$

or $$\frac{\frac{L}{2} + 25}{\frac{L}{2} - 25} = 1.1069 \quad \text{or} \quad \frac{L}{2} + 25 = 1.1069\frac{L}{2} - 27.672$$

or $$0.1069\frac{L}{2} = 52.6725 \quad \text{From which} \quad \frac{L}{2} = 492.73 \quad \text{or} \quad L \simeq \mathbf{985.5\ m}$$

Example A-10. *A vertical photograph was taken from 3200 m above mean sea level with a camera of focal length 120 mm. It contained two points 'a' and 'b' corresponding to ground points A and B. Calculate the horizontal length AB, as well as the average scale along line ab from the following data :*

Photo points	*Elevation above msl (m)*	*Photo coordinates*	
		x (mm)	*y (mm)*
a	640	*+ 19.50*	*– 14.60*
b	780	*+ 26.70*	*+ 10.80*

(U.P.S.C.Engg. Services Exam., 1997)

Solution

The ground co-ordinates are given by

$$X_a = \frac{H - h_a}{f} x_a = \frac{3200 - 640}{120} \times (+19.50) = +416 \text{ m}$$

$$Y_a = \frac{H - h_a}{f} y_a = \frac{3200 - 640}{120} \times (-14.60) = -311.47 \text{ m}$$

$$X_b = \frac{H - h_b}{f} x_b = \frac{3200 - 780}{120} \times (+26.70) = +538.45 \text{m}$$

$$Y_b = \frac{H - h_b}{f} y_b = \frac{3200 - 780}{120} \times (+10.80) = +217.80 \text{ m}$$

$\therefore$ Length $$AB = \sqrt{(X_a - X_b)^2 + (Y_a - Y_b)^2}$$

$$= \sqrt{(416 - 538.45)^2 + (-311.47 - 217.80)^2} = \mathbf{543.25\ m}$$

Average scale $$S_{av} = \frac{f}{H - h_{av}} \quad \text{where} \quad h_{av} = \frac{640 + 780}{2} = 710 \text{ m}$$

$$\therefore \quad S_{av} = \frac{120 \text{ mm}}{(3200 - 710) \text{ m}} = \frac{1 \text{ mm}}{20.75 \text{ m}}$$

$\therefore$ Average scale is **1 mm = 20.75 m**

Example A-11. *Two tangents interest at chainage 50.60 (50 chains and 60 links), the deflection angle being 61°. Calculate the necessary data for setting out a circular highway curve of 20 chains radius to connect the two tangents by the method of offsets from the chord. Take peg interval equal to 100 links with length of the chain being 20 metres (100 links).*

(U.P.S.C. Engg. Services Exam, 1998)

Solution :

Given $R = 20 \times 20 = 400$ m; $\Delta = 61°$

Chainage of point of intersection V

$= 50.60$ chains

$= 50 \times 20 + 0.6 \times 20 = 1012$ m

Length of tangent $T = R \tan \Delta/2$

$= 400 \tan 30.5° = 235.62$ m.

$\therefore$ Chainage of $T_1 = 1012 - 235.62$

$= 776.38$ m

Length of curve

$$= \frac{\pi R \Delta}{180°} = \frac{\pi (400)\, 61°}{180°} = 425.86 \text{ m}$$

$\therefore$ Chainage of T_2

$= 776.38 + 425.86 = 1202.24$ m

Let us set out the curve by means of offsets from the long chord.

Length of long chord $T_1 T_2 = 2R \sin \Delta/2$

$= 2 \times 400 \sin 30.5° = 406.03$ m

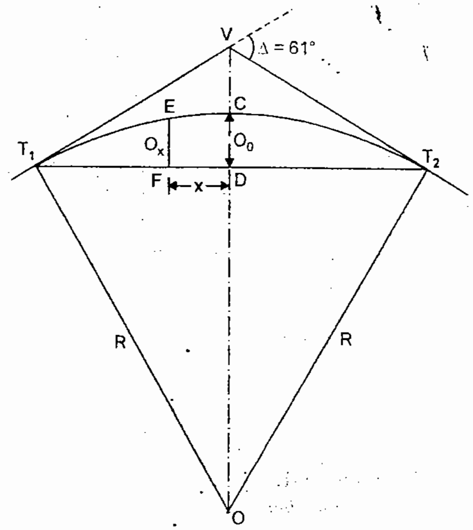


FIG. A-6

Central ordinate $$O_0 = R - \sqrt{R^2 - (L/2)^2} = 400 - \sqrt{(400)^2 - (406.03/2)^2} = 55.35 \text{ m}$$

$$O_x = \sqrt{R^2 - x^2} - (R - O_0) = \sqrt{(400)^2 - x^2} - (400 - 55.35)$$

or $$O_x = \sqrt{160000 - x^2} - 344.65 \text{ m}$$

Taking x at a peg interval of 20 m, values of O_x can be computed as tabulated below.

x	O_x	x	O_x
0	55.35	120	36.93
20	54.85	140	30.05
40	53.34	160	21.96
60	50.82	180	12.56
80	47.27	200	1.76
100	42.65	203.015	0.00

Example A-12. *A section line AB 300 m long on a flat terrain measures 102.4 mm on the vertical photograph. A radio tower also appears on the photograph. The distance measured from the principal points to the image of the bottom and top of the radio tower was found to be 7 cm and 8 cm respectively. The average elevation of the terrain was 553 m. Determine the height of the tower. Take $f = 152.4$ mm.*

(U.P.S.C. Engg. Services Exam., 2001)

Solution : Refer Example 14.12 and Fig. 14.24.

Scale of photographs, $S = \frac{102.40 \text{ mm}}{300 \text{ m}}$

But $S = \frac{f}{H - h_a} = \frac{152.4 \text{ m}}{(H - 553) \text{ m}}$

$\therefore \quad \frac{102.40}{300} = \frac{152.4}{H - 553}$

or $\quad H = \frac{152.4 \times 300}{102.40} + 553 = 999.48$ m

Also, height of the tower above the base is given by

$$h = \frac{d \cdot H}{r} \quad \text{...(14.20)}$$

where $\quad d = 8 - 7 = 1$ cm $= 10$ mm

r = radial distance of top of the tower = 8 cm = 80 mm

$\therefore \quad h = \frac{10 \times 999.48}{80} = \mathbf{124.94\ m}$

Example A-13. *An ascending gradient 1 in 60 meets a descending gradient of 1 in 50. Find the length of vertical curve for a stopping sight distance of 180 m.*

(U.P.S.C. Engg. Services Exam., 2002)

Solution

$$g_1 = +\frac{1}{60} \times 100 = +1.667\%; \quad g_2 = -\frac{1}{50} \times 100 = -2\%$$

Let us assume that $L > S$. Then

$$L = \frac{S^2 (g_1 - g_2)}{200 (\sqrt{h_1} + \sqrt{h_2})^2}$$

For stopping sight distance, taking $h_1 = 1.37$ m and $h_2 = 0.1$ m,

$$L = \frac{S^2 (g_1 - g_2)}{442} = \frac{(180)^2 (1.667 + 2)}{442} = \mathbf{268.8\ m}$$

Since L comes out to be greater than S, our assumption is correct.

Example A-14. *A tape of nominal length '30 m' is standarised in catenary at 42 N tension and found to be 29.9820 m. If the mass of the tape is 0.016 kg/m, calculate the horizontal length of a span recorded as 16.7262 m.*

Solution : Given : $m = 0.016$ kg/m ; $P = 42$ N

Standardised chord length = 29.9820 m

Sag correction $\quad C_{s1} = \frac{w^2 l_1^3}{24 P^2} = \frac{(mg)^2 l_1^3}{24 P^2} = \frac{(0.016 \times 9.81)^2 (30)^3}{24 (42)^2} = 0.0157$ m

$\therefore$ Standard arc length = 29.9820 + 0.0157 = 29.9977

Standardisation error per 30 m = 29.9977 − 30 = − 0.0023 m

Now recorded arc length = 16.7262 m

Standardisation error $= -16.7262 \times \frac{0.0023}{30} = -0.0013$

$\therefore$ Standard arc length = 16.7262 − 0.0013 = 16.7249 m

$$\text{Sag correction} = C_{s1} \left(\frac{16.7262}{30} \right)^3 = 0.0157 \left(\frac{16.7262}{30} \right)^3 = -0.0027 \text{ m}$$

$\therefore$ Standarised chord length = 16.7249 − 0.0027 = **16.7222**

Example A-15. *A copper transmission line 12 mm diameter is stretched between two points 300 m apart at the same level, with a tension of 5×10^3 N when the temperature is 32° C. It is necessary to define its limiting positions when the temperature varies. Making use of the correction for sag, temperature, and ~~elasticity normally applied to~~ base line measurements by tape in catenary, find the tension at a temperature at − 12° C and sag in the two cases. For copper, Young's modulus = 7×10^4 N/mm², density 9×10^3 kg/m³ and coefficient of linear expansion = 1.7×10^{-5}/° C.*

Solution

Mass per unit length is $m = \pi r^2 \rho = \pi (0.006)^2 \times 9 \times 10^3 = 1.0179$ kg/m

$$C_{s1} = \frac{(mg)^2 l^3}{24 P^2} = \frac{(1.0179 \times 9.81)^2 (300.000)^3}{24 (5 \times 10^3)^2} = 4.487 \text{ m}$$

Length of wire needed at 32° C = 300.000 + C_{s1} = 300.000 + 4.487 = 304.487 m

The approximate length of wire is thus 304.49 m and this value may be used in the above expression for C_{s1} to given a better approximation.

Thus, $\quad C_{s1} = \frac{(1.0179 \times 9.81)^2 (304.49)^3}{24 (5000)^2} = 4.692$ m

Hence better length = 300.000 + 4.692 = 304.692 m

Amount of sag $= h_1 = \frac{w\, l_1\, d_1}{8P} = \frac{mg\, l_1\, d_1}{8\, P}$

or $$h_1 = \frac{1.0179 \times 9.81\ (304.692)\ (300.000)}{8 \times 5000} = \mathbf{22.819\ m}$$

When the temperature falls to $-12°$ C,

Contraction of wire, $\delta l = l\, \alpha\, \Delta T = 304.692 \times 1.7 \times 10^{-5}\ [32 - (-12)] = 0.228$ m

$\therefore$ Adjusted length of wire $= 304.692 - 0.228 = 304.464$ m

$\therefore$ Now sag $$h_2 = \frac{1.0179 \times 9.81\ (304.464)\ (300.00)}{8 \times 5000} = \mathbf{22.802\ m}$$

Since $h \propto \frac{1}{P}$, $$P_2 = P_1 \left(\frac{h_1}{h_2}\right) = 5000 \left(\frac{22.819}{22.802}\right) = \mathbf{5003.7\ N}$$

Note : A change of 3.7 N in tension will expand the tape by

$$\delta l = \frac{l\, \Delta P}{AE} = \frac{304.692 \times 3.7}{\pi\, (6)^2 \times 7 \times 10^4} = 0.0001 \text{ m which is negligible.}$$

Example A-16. *The details given below refer to the measurement of the first 30 m bay of a base line. Determine the correct length of the bay reduced to mean sea level.*

With the tape hanging in catenary at a tension of 95 N and at a mean temperature of 13° C the recorded length was 29.9821 m. The difference in height between the ends was 0.40 m and the site was 500 m above m.s.l.

The tape had previously been standardised in catenary at a tension of 70 N and at a temperature of 15° C and the distance between the zeros was 29.9965 m. Take the following values : R = 6367.3 km; mass of tape = 0.0191 kg/m; sectional area of tape = 3.63 mm²; E = 2.1 × 10⁵ N/mm² and temperature coefficient of expansion of tape = 12 × 10⁻⁶ per °C.

Solution

1. ***Correction for standardisation***

The tape is 29.9965 m at 70 N and 15° C

$c = (29.9965 - 30.000) = -0.0035$ per 30 m

2. ***Correction for temperature***

$$c_t = \alpha\,(T_m - T_0)\, l = 12 \times 10^{-6}\,(13 - 15) \times 30 = -0.0007 \text{ m}$$

3. ***Correction tension***

$$c_p = \frac{(P - P_0)\, l}{A\,E} = \frac{(95 - 70).30}{3.63 \times 2.1 \times 10^5} = +0.0010 \text{ m}$$

4. ***Correction for slope***

$$c_v = -\frac{h^2}{2L} = -\frac{(0.40)^2}{2 \times 30} = -0.0027 \text{ m}$$

5. ***Correction for sag***

$$c_s = -\frac{(mg)^2\, l^3}{24}\left[\frac{1}{P^2} - \frac{1}{P_0^2}\right] = -\frac{(0.0191 \times 9.81)^2\,(30)^3}{24}\left[\frac{1}{(95)^2} - \frac{1}{(70)^2}\right] = +0.0037 \text{ m}$$

6. ***Correction for reduction to m.s.l***

$$c_{msl} = -\frac{lh}{R} = -\frac{30 \times 500}{6367 \times 10^3} = -0.0024 \text{ m}$$

$\therefore$ Correct length $= 29.9821 - 0.0035 - 0.0007 + 0.0010 - 0.0027 + 0.0037 - 0.0024$

$= \mathbf{29.9775\ m}$

Example A-17. *A steel tape has the following specifications.*

(i) Mass = 0.5 kg *(ii) cross-sectional area = 2 mm²*

(iii) Young's modulus = 20 × 10¹⁰ N/m² (iv) length at 20° C and 50 N = 30.005 m

(v) Coefficient of linear expansion = 11 × 10⁻⁶ per °C

It is to be used in catenary but in order to reduce the number of corrections to be applied to the measured lengths, it is suggested that

(a) the standard temperature be adjusted so that the actual length is equal to the nominal length of 30.000 m

(b) the tape be used at a tension such that the effects of sag and tension will be compensating.

(N.B. : The acceptable tension will be in the region of 100 N)

Solution

(a) ***New standard temperature*** : Desired $c_t = 30.005 - 30.000 = 0.005$ m

But $$c_t = \alpha\,(T_m - T_0)\, l$$

$\therefore$ $$\Delta T = \frac{c_t}{l\,\alpha} = \frac{0.005}{30 \times 11 \times 10^{-6}} = 15.2°\ C$$

Hence to contract the tape by 0.005 m, the temperature would be needed to be reduced by 15.2° C.

$\therefore$ Near standard temperature $= T_0 - 15.2° = 20° - 15.2° = \mathbf{4.8°\ C}$

(b) ***Normal tension*** (P_n)

$$c_p = \frac{P_n - P_0}{AE}\, l$$

$$c_s = \frac{(mg\, l)^2\, l}{24\, P_n^2}$$

Equating the two, $$\frac{(mgl)^2\, l}{24\, P_n^2} = \frac{(P_n - P_0)\, l}{AE}$$

or $$\frac{AE\,(mgl)^2}{24} = P_n^3 - P_n^2\, P_0$$

or $$P_n^3 - 50\, P_n^2 - \frac{(0.5 \times 9.81)^2 \times 2 \times 2 \times 10^5}{24} = 0$$

or $$P_n^3 - 50\, P_n^2 = 400984$$

Let us solve this by trial and error, in the tabular form shown below :

Trial No.	P_n	L.H.S.	R.H.S.
1.	100	500000	400984
2.	98	460992	400984
3.	96	423936	400984
4.	95	406125	400984
5.	94.7	400874	400984

Hence $P_n = 94.7 \approx$ **95 N** (Say)

Example A-18. *A length AD is measured in three bays with ground distances recorded as follows : AB : 42.361 m, BC = 25.734 m and CD = 52.114.*

From a theodolite station at A, instrument height 1.46 m, staff reading were taken with a mean vertical angle of – 3° 24′30″ to B 1.58, to C 0.96 and D 0.96 m. Calculate the horizontal length of line AD.

Solution : Given $\alpha = -3° 24' 30''$ throughout In general $\delta\alpha'' = \dfrac{206265\,(h_1 - h_2)\cos\alpha}{l}$

For line AB : $h_1 = 1.46$ m; $h_2 = 1.58$ m

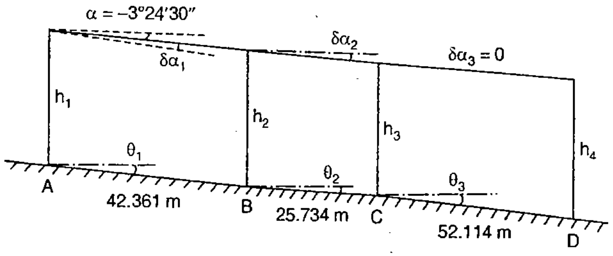


FIG. A-7

$$\delta\alpha_1'' = \frac{206265\,(1.46 - 1.58)\cos(-3° 24' 30'')}{42.361} = -583'' = -0° 9' 43''$$

Then $\theta_1 = 3° 24' 30'' - 0° 9' 43'' = -3° 34' 13''$

∴ $L_1 = l_1 \cos\theta_1 = 42.361 \cos 3° 34' 13'' = 42.279$ m

Line BC : $h_2 = 1.58$ m , $h_3 = 0.96$ m

∴ $$\delta\alpha_2'' = \frac{206265\,(1.58 - 0.96)\cos(-3° 24' 30'')}{25.734} = 4961'' = 1° 22' 41''$$

∴ $\theta_2 = \alpha + \delta\alpha_2 = -3° 24' 30'' + 1° 22' 41'' = -2° 01' 49''$

∴ $L_2 = l_2 \cos\theta_2 = 25.734 \cos 2° 01' 49'' = 25.718$ m

Line CD $h_3 = 0.96$ m ; $h_4 = 0.96$ m

∴ $$\delta\alpha_3 = \frac{206265\,(0.96 - 0.96)\cos(-3° 24' 30'')}{52.114} = 0$$

∴ $\theta_3 = \alpha + \delta\alpha_3 = -3° 24' 30''$

and $L_3 = l_3 \cos\theta_3 = 52.114 \cos 3° 24' 30'' = 52.022$ m

Total horizontal length of the line $AD = 42.279 + 25.718 + 52.022 =$ **120.019 m**

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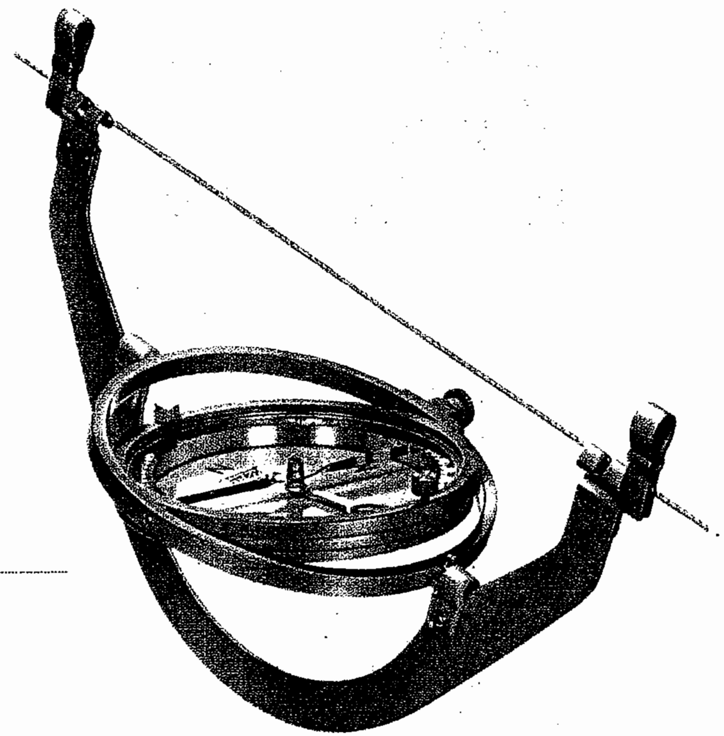

FIG. 7.9. KESSEL TYPE MINING COMPASS

(TO FACE PAGE 208)