An Orthogonal Polynomial Approach Towards the Design of the Linear phase Image Processing Filter

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CERTIFICATE

This is to certify that the work entitled "An Orthogonal Polynomial Approach towards the Design of the Linear phase Image Processing Filter" submitted by Meenu Jain and Rohini Mukhopadhyay in partial fulfillment for the award of degree of Bachelor of Technology, of Jaypee University of Information Technology, in 2007 has been carried out under my supervision. This work has not been submitted partially or wholly to any other university or institute of award of this or any other degree or diploma.

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Prof. Sunil V. Bhooshan

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Meenu Jain

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LIST OF ABBREVIATIONS

FIR – Finite Impulse Response
IIR – Infinite Impulse Response
DFT – Discrete Fourier Transform
IDFT – Inverse Discrete Fourier Transform
i.e. – that is

ABSTRACT

This project describes a method for designing Linear Phase Image Processing Filters based on the orthogonal polynomial approach. The design of Linear Phase Image Processing Filter is a field where an immense amount of work had been carried out since a long period of time and numerous distinctive approaches had been taken to carry out designs of different types of Image Processing Filters. Here we will design a linear phase FIR image processing filter using orthogonal polynomials in the cylindrical co-ordinate system; i.e., in *(ρ, Φ, z)*.

I. INTRODUCTION

The design of Linear Phase Image Processing Filter is a field where an immense amount of work had been carried out since a long period of time and numerous distinctive approaches had been taken to carry out designs of different types of Image Processing Filters. Here we will design a linear phase FIR image processing filter using orthogonal polynomials in the cylindrical co-ordinate system; i.e., in (ρ, Φ, z).

Definition: Two functions $g_1(x)$ and $g_2(x)$ are said to be orthogonal over a certain interval $x_1 \leq x \leq x_2$, if:

$$\int_{x_1}^{x_2} g_1(x)\, g_2(x)\, dx = 0 \qquad \ldots\ldots\ldots\ldots(1)$$

That is, when we multiply two different functions and then integrate over the interval from x_1 to x_2 the result is zero. A set of functions which has this property is described as being orthogonal over the interval from x_1 to x_2. The term "orthogonal" is employed here in correspondence to a similar situation which is encountered in dealing with vectors. The scalar product of two vectors V_i and V_j (also referred to as the dot product or as the inner product) is a scalar quantity defined as:

$$V_{ij} = |V_i||V_j|\cos(V_i, V_j) = V_{ji} \quad \ldots\ldots(2)$$

In the above equation $|V_i|$ and $|V_j|$ are the magnitudes of the respective vectors and $cos(V_i, V_j)$ is the cosine of the angle between the vectors. If it should turn out that $V_{ij} = 0$ then (ignoring the trivial cases in which $V_i = 0$ and/or $V_j = 0$) V_i and V_j are perpendicular (i.e. , orthogonal) to one another. Thus vectors whose scalar product is zero are physically orthogonal to one another and, in correspondence, functions whose integral product, as in equation (1) is zero are also orthogonal to one another. Physically, it means that the two signals represented by $g_1(x)$ and $g_2(x)$ do not have any common energy between the interval x_1 to x_2.

Suppose, we are interested in a function $f(x)$ over a certain interval then it can be expanded as a linear sum of infinite orthogonal functions, each function having a certain weight.

$$f(x) = c_1 g_1(x) + c_2 g_2(x) + \ldots\ldots + c_n g_n(x) + \ldots$$

Where,

$g_n(x)$ is a member of orthogonal set of functions, and

c_n is the weight of the function.

A **finite impulse response** (FIR) filter is a type of a digital_filter. It is 'finite' because its response to an impulse ultimately settles to zero. This is in contrast to infinite impulse response filters which have internal feedback and may continue to respond indefinitely.

Properties

A FIR filter has a number of useful properties which sometimes make it preferable to an infinite impulse response filter. FIR filters:

- Are inherently stable. This is due to the fact that all the poles are located at the origin and thus are located within the unit circle.
- Require no feedback. This means that any rounding errors are not compounded by summed iterations. The same relative error occurs in each calculation.
- Can have linear phase

The Infinite Impulse Response (IIR) system has an infinite number of non zero terms, i.e., its impulse response sequence is of infinite duration. IIR structures are usually implemented using structures having feedback (recursive structures-poles and zeros) and FIR filters are usually implemented using structures with no feedback (non-recursive structures-all zeros). The response of the FIR filter depends only on the present and the past input samples, whereas for IIR filters, the present response is a function of present and past N values of excitation as well as past values of response.

Linear phase is a property of a filter, where the phase response of the filter is a linear function of frequency, excluding the possibility of wraps at $\pm\pi$. In a causal system, perfect linear phase may only be achieved with a discrete-time FIR filter.

FIR filters are employed in filtering problems where linear phase characteristics within the passband of the filter is required. If this is not required, either an IIR or an FIR may be employed. An IIR has lesser number of side lobes in the stop band than an FIR with the same number of parameters. For this reason if some phase distortion is tolerable, an IIR filter is preferable. Also, the implementation of an IIR filter involves fewer parameters, less memory requirements a lower computational complexity.
Filters can have a linear or non-linear phase depending upon the delay functions, namely the phase delay and the group delay. The phase and group delays of a filter are given by

$$\tau_p = -\frac{\phi(\omega)}{\omega} \quad \text{and} \quad \tau_g = -\frac{d\phi(\omega)}{d\omega}, \text{ respectively.}$$

Linear phase filters are those filters in which the phase delay and group delay are constant, i.e., independent of frequency. Linear phase filters are also called constant time delay filters. Both these delays of the linear phase FIR filters are equal and constant over the frequency band.

Since the group delay of a filter that has linear phase (or generalized linear phase) property is constant, all frequency components have equal delay times. So the distortion due to the frequency selective delays are not present which is mostly a desired feature.

However, a filter with *non-linear phase* has a group delay that varies with frequency, resulting in phase distortion.

Some examples of filters having linear and non-linear phases are given below for comparison purposes.

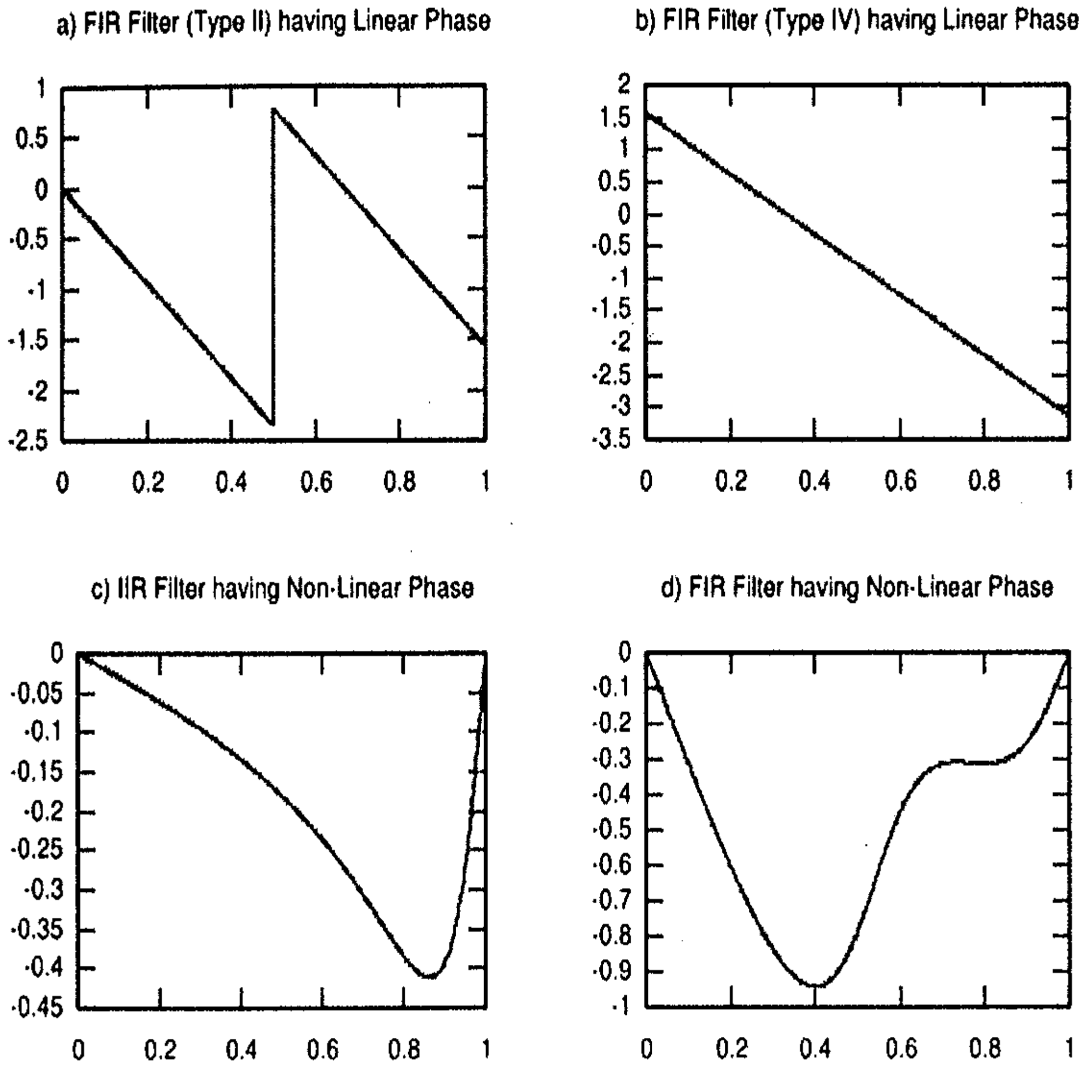

Fig. 1

About Digital Image Processing:

Interest in digital image processing methods stem from two principal application areas: improvement of pictorial information for human interpretation, and, processing for image data for storage, transmission and representation for autonomous machine perception.
An image may be defined as a two dimensional function, f(x,y), where x and y are spatial (plane) coordinates, and the amplitude f at any pair of coordinates (x,y) is called the intensity or gray level of the image at that point. When x, y and the amplitude values of f are all finite, discrete quantities, we call the image a digital image. The field of digital image processing refers to processing digital images by means of a digital computer. Note that a digital image is composed of a finite number of elements, each of which has a particular location and value. These elements are referred to as pixels.
The filter that we shall design would be in the domain of frequency. So, the image must also be converted into the frequency domain so that it can pass through the filter. The one dimensional discreet fourier transform and its inverse are well known. Extension of one-dimensional DFT and its inverse to two dimensions is straight forward. The DFT of a function (image) f(x,y) of size M×N is given by the equation

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}.$$

As in the 1-D case, this expression must be computed for values of u = 0, 1, 2..........., M-1, and also for v = 0, 1, 2,.........N-1. similarly, given F(u,v), we obtain f(x,y) via the inverse Fourier transform, fiven by the expression

$$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$$

These two equations comprise the two-dimensional, discrete Fourier transform (DFT) pair. The variables u and v are the transform or frequency variables, and x and y are the spatial or image variables.
Filtering in the frequency domain is straightforward. It consists of the following steps:

1. Multiply the input image by $(-1)^{x+y}$ to center the transform.
2. Compute F(u,v), the DFT of the image from (1).
3. Multiply F(u,v) by a filter function H(u,v).
4. Compute the inverse DFT of the result in (3).
5. Obtain the real part of the result in (4).
6. Multiply the result in (5) by $(-1)^{x+y}$.

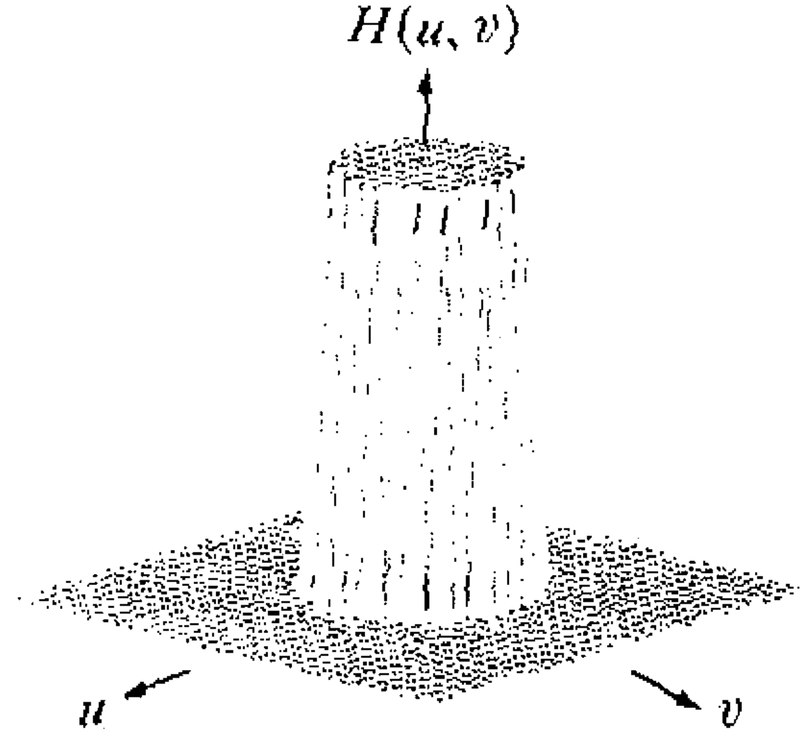

Fig. 2

The figure above shows a three dimensional perspective of H(u,v), the filter function of an ideal low pass filter, as a function of u and v. The term ideal filter indicates that frequencies below the cut off frequency are passed with no attenuation, whereas, all frequencies above it are completely attenuated. The low pass filter that we have considered is radially symmetric about the origin. Sharp cut off frequencies of an ideal low pass filter cannot be realized with electronic components, although they can certainly be implemented in a computer. A low pass filter causes blurring of the input image. Blurring in the image is a clear indication that most of the sharp detail information in the picture is removed by the filter. Blurring reduces as we increase the cut off frequency of the filter. The advantage of adding a low pass filter is that it will remove all noise above the cut off frequency. The image thus blurred can be sharpened using various image processing techniques

II. PROCEDURE:

Here we have developed a set of two dimensional orthogonal polynomials which are in the cylindrical co-ordinate system. All polynomials are orthogonal because they follow the following property:

$$\int_0^{2\pi}\int_0^1 P_i(\rho)P_j(\rho)\rho d\rho d\phi = 0 \quad \ldots\ldots\ldots(3)$$

Where, ρ represents the frequency component and Φ is the angle.

The procedure to design an Image Processing Filter with linear phase includes the following steps:

Step 1:
Let the set of orthogonal polynomials is represented as follows:

$P_1(\rho) = 1$

$P_2(\rho) = 1+a_2\rho^2$

.

.

.

$P_{10}(\rho) = 1+a_{10}\rho^2+b_{10}\rho^4+c_{10}\rho^6+d_{10}\rho^8+e_{10}\rho^{10}+f_{10}\rho^{12}+g_{10}\rho^{14}+h_{10}\rho^{16}+i_{10}\rho^{18}$

.

.

.

$P_n(\rho) = 1+a_n\rho^2+b_n\rho^4+\ldots\ldots$

Where, $a_n, b_n, c_n, _{etc}$. are constant numeric coefficients that are to be calculated.

Step2:
From equation (3), we get

$$2\pi\int_0^1 P_i(\rho)P_j(\rho)\rho d\rho = 0$$

i.e. $$\int_0^1 P_i(\rho)P_j(\rho)\rho d\rho = 0 \text{-------- } (4)$$

In order to find the constant numeric coefficients of any polynomial $P_n(\rho)$, we have to multiply $P_n(\rho)$ by all the polynomials from $P_1(\rho)$ to $P_{n-1}(\rho)$ individually and apply it in equation (4). By solving these equations we get a set of simultaneous linear equations only in terms of constant numeric coefficients of $P_n(\rho)$, since the coefficients for polynomials up to $P_{n-1}(\rho)$ have already been calculated. This is illustrated as follows-

Polynomial $P_1(\rho)$ does not have any constant numeric coefficients.

$P_1(\rho)=1$

The constant numeric coefficients for the next polynomial, $P_2(\rho)$ is calculated as follows:

$$\int_0^1 P_1(\rho)P_2(\rho)\rho d\rho = 0$$

$P_1(\rho)=1,\ P_2(\rho)= 1-a_2\rho^2$

i.e., $$\int_0^1 1\times(1+a_2\rho^2)\rho d\rho = 0$$

i.e., $$\frac{1}{2}+\frac{a_2}{4}=0$$

i.e., $a_2=-2$, the constant numeric coefficient of $P_2(\rho)$

Therefore, $P_2(\rho)=1-2\rho^2$

The next polynomial, $P_3(\rho)=1+a_3\rho^2+b_3\rho^4$

Taking $P_1(\rho)$ and $P_3(\rho)$

$$\int_0^1 P_1(\rho)P_3(\rho)\rho d\rho = 0$$

i.e. $$\int_0^1 1\times(1+a_3\rho^2+b_3\rho^4)\rho d\rho = 0$$

i.e., $$\frac{1}{2}+\frac{a_3}{4}+\frac{b_3}{6}=0 \quad \ldots\ldots\ldots\ldots\text{(i)}$$

Taking $P_2(\rho)$ and $P_3(\rho)$

$$\int_0^1 P_2(\rho)P_3(\rho)\rho d\rho = 0$$

i.e., $$\int_0^1 (1-2\rho^2)\times(1+a_3\rho^2+b_3\rho^4)\rho d\rho = 0$$

i.e., $$a_3+b_3=0 \quad \ldots\ldots\ldots\ldots\text{(ii)}$$

Solving the equations (i) and (ii), we get,

$a_3 = -6$

$b_3 = 6$

Where, a_3 and b_3 are the constant numeric coefficient of $P_3(\rho)$

Hence, $P_3(\rho)=1-6\rho^2+6\rho^4$

Likewise we follow the same procedure for the rest of the polynomials and get their coefficients, and the polynomials up to $P_{10}(\rho)$ as calculated, are as follows:

$P_4(\rho) = 1-(192/31)\rho^2+(190/31)\rho^4+(20/31)\rho^6$

$P_5(\rho) = 1-20\rho^2+90\rho^4-140\rho^6+70\rho^8$

$P_6(\rho) = 1-30\rho^2+210\rho^4-560\rho^6+630\rho^8-252\rho^{10}$

$P_7(\rho) = 1-42\rho^2+420\rho^4-1680\rho^6+3150\rho^8-2772\rho^{10}+924\rho^{12}$

$P_8(\rho) = 1-56\rho^2+756\rho^4-4200\rho^6+11550\rho^8-16632\rho^{10}+12012\rho^{12}-3432\rho^{14}$

$P_9(\rho) = 1-72\rho^2+1260\rho^4-9240\rho^6+34650\rho^8-72072\rho^{10}+84084\rho^{12}-51480\rho^{14}-12870\rho^{16}$

$P_{10}(\rho) = 1-(113461974/9448465)\rho^2+(4369887180/145506361)\rho^4-(409164420/20786623)\rho^6-(88976385/41573246)\rho^8+(53757117/9448465)\rho^{10}-(6353367/726805)\rho^{12}+(8413128/1017527)\rho^{14}-(17877897/4070108)\rho^{16}+(1986433/2035054)\rho^{18}$

Step 3:

Now we shall calculate the approximate function for the filter.

Suppose the ideal filter function is given by the following equation in frequency domain:

$$f(\rho) = \begin{cases} 0 & 0<\rho<\rho_s \\ 1000 & \text{else} \end{cases} \qquad \text{---------------(5)}$$

Where, ρ_s is the cutoff frequency.

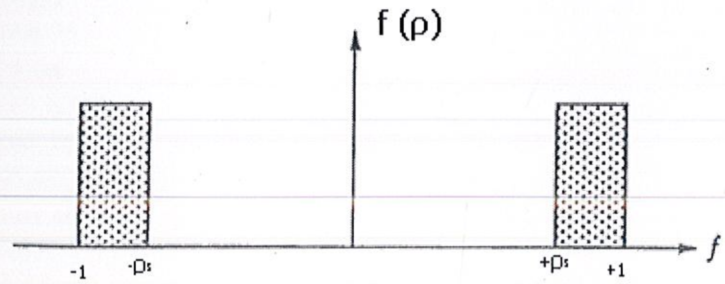

Fig. 3

We will approximate our filter function by a linear sum of finite number of weighted orthogonal polynomials; i.e. ,

$$f_a(\rho)=w_1P_1(\rho)+ w_2P_2(\rho)+ w_3P_3(\rho)+\ldots\ldots\ldots+ w_{10}\,P_{10}\,(\rho) \qquad \text{------(6)}$$

where, w_n = weight of the nth polynomial

To calculate w_n, we multiply equation (6) by $\rho\,P_n(\rho)$, then integrate it over limits 0 to 1 with respect to ρ. Hence the terms in equation (6) that do not contain w_n get eliminated because of the property stated in equation (4). Solving the remaining equation we get the value of the weight of that polynomial.

$$\int_0^1 f(\rho)P_n(\rho)\,\rho d\rho = \int_0^1 P_n(\rho)\,[w_1P_1(\rho)+ w_2P_2(\rho) + \ldots + w_nP_n(\rho) + \ldots + w_{10}\,P_{10}(\rho)]\,\rho d\rho$$

On solving this equation, we get the weight of the nth polynomial as,

$$w_n = \frac{\int_0^1 f(\rho)\,P_n(\rho)\rho d\rho}{\int_0^1 P_n^2(\rho)\rho d\rho}$$

For $P1$, $w1$ would be,

$$w_1 = \frac{\int_0^{0.5} f(\rho)\,P_1(\rho)\rho d\rho + \int_{0.5}^1 f(\rho)\,P_1(\rho)\rho d\rho}{\int_0^1 P_1^2(\rho)\rho d\rho} = \frac{0+1000\int_{0.5}^1 1\times\rho d\rho}{\int_0^1 1^2\times\rho d\rho}$$

i.e.,w_1=750

Likewise this method is used to calculate the weights for the rest of the orthogonal polynomials and are as follows:

w_2 = -562.5
w_3 = -468.75
w_4 = 59.04
w_5 = 263.6712
w_6 = 306.152
w_7 = 66.65048
w_8 = -198.4407
w_9 = 0.073
w_{10}=-81.89

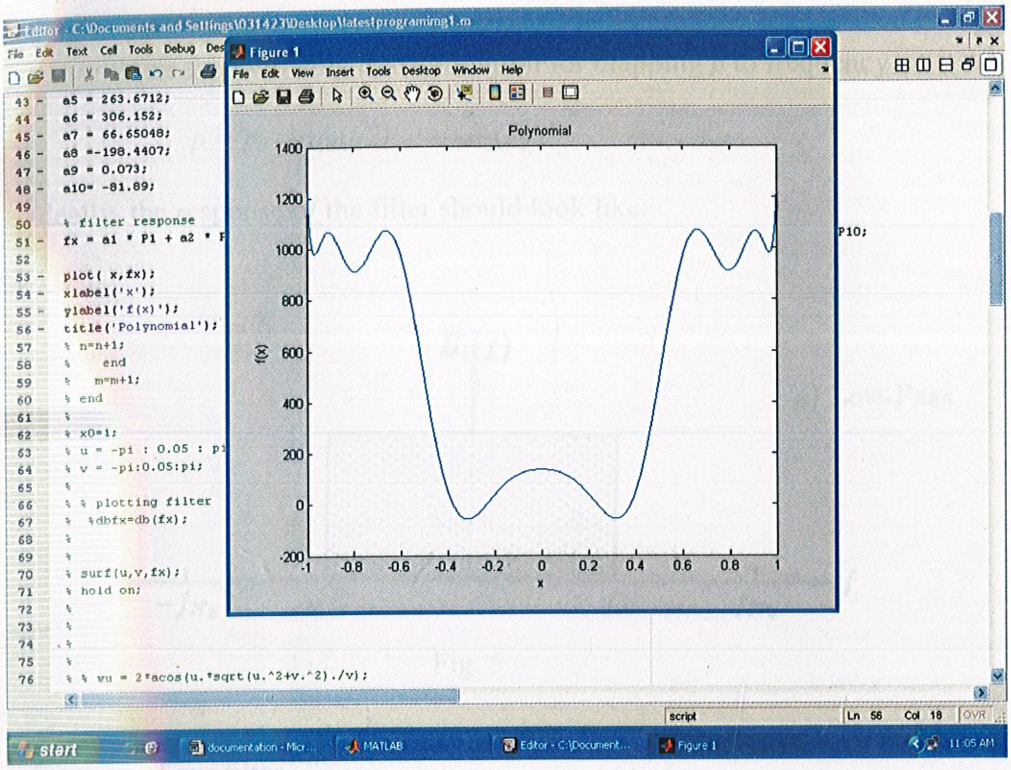

Fig. 4

Graph (4) does not follow graph (3) faithfully because simply having a set of orthogonal functions and having a procedure for evaluating coefficients does not guarantee that the series so developed can represent an arbitrary function. Such can well be the case even when the orthogonal set consists of an infinite number of independent functions. When, however, the orthogonal set does indeed include all the functions necessary to allow and error free expansion of an arbitrary function then the set is said to be complete. The set of orthogonal functions that we have developed is hence, incomplete.

Step 4:

It is known that a desirable filter function can be obtained by equispacing the cut off frequencies of a filter on the appropriate arc of the unit circle. A pattern obtained by equispacing the cut off frequencies indicates how a better pattern can be obtained. For a given width of passband, the first side lobe can be decreased by moving the second cutoff frequency closer to the first. Of course, this increases the second side lobe, but that is permissible as long as it does not exceed the passband. The optimum transfer function is obtained when all the side lobes have the same level. We use the following result, obtained from the Tchebysheff polynomials, used in design and synthesis problems, to optimize our filter

We will use the following transformation for mapping ρ to frequency ω.

$$\rho = \rho_0 \cos(\omega /2) \quad \text{-------(7)}$$

Ideally, the response of the filter should look like:

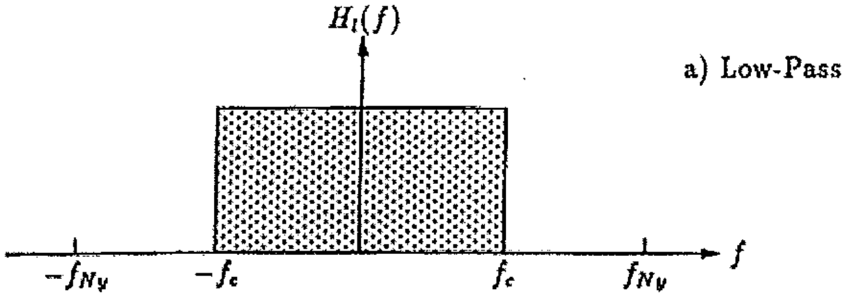

Fig. 5

Thus we get $f(\omega)$ by substituting equation (7) into equation (6). Here $f(\omega)$ is the transfer function of our filter.

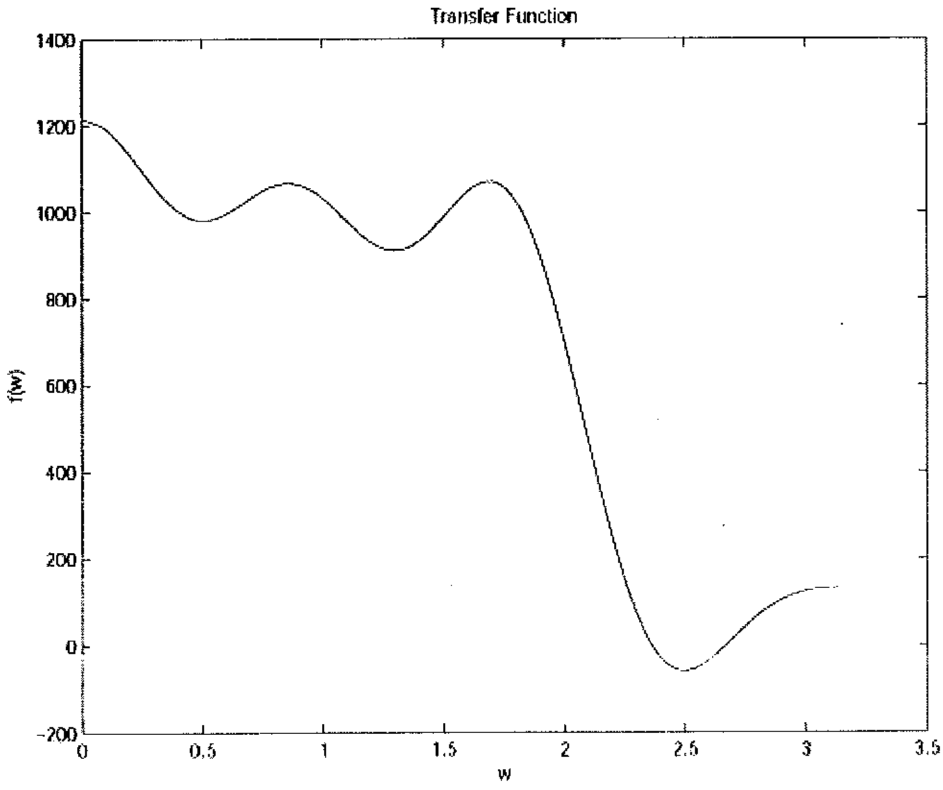

Fig. 6

Step 5:

We choose u and v as the frequency components of ω.

Where,

$$\omega^2 = u^2 + v^2 \quad \text{.............-(8)}$$

Substituting ω in equation (7), we get

$$\rho = \cos \frac{\sqrt{u^2 + v^2}}{2} \quad \text{.............(9)}$$

We use this value of ρ in the polynomials. Using these polynomials in equation (6), we get the three dimensional filter response.

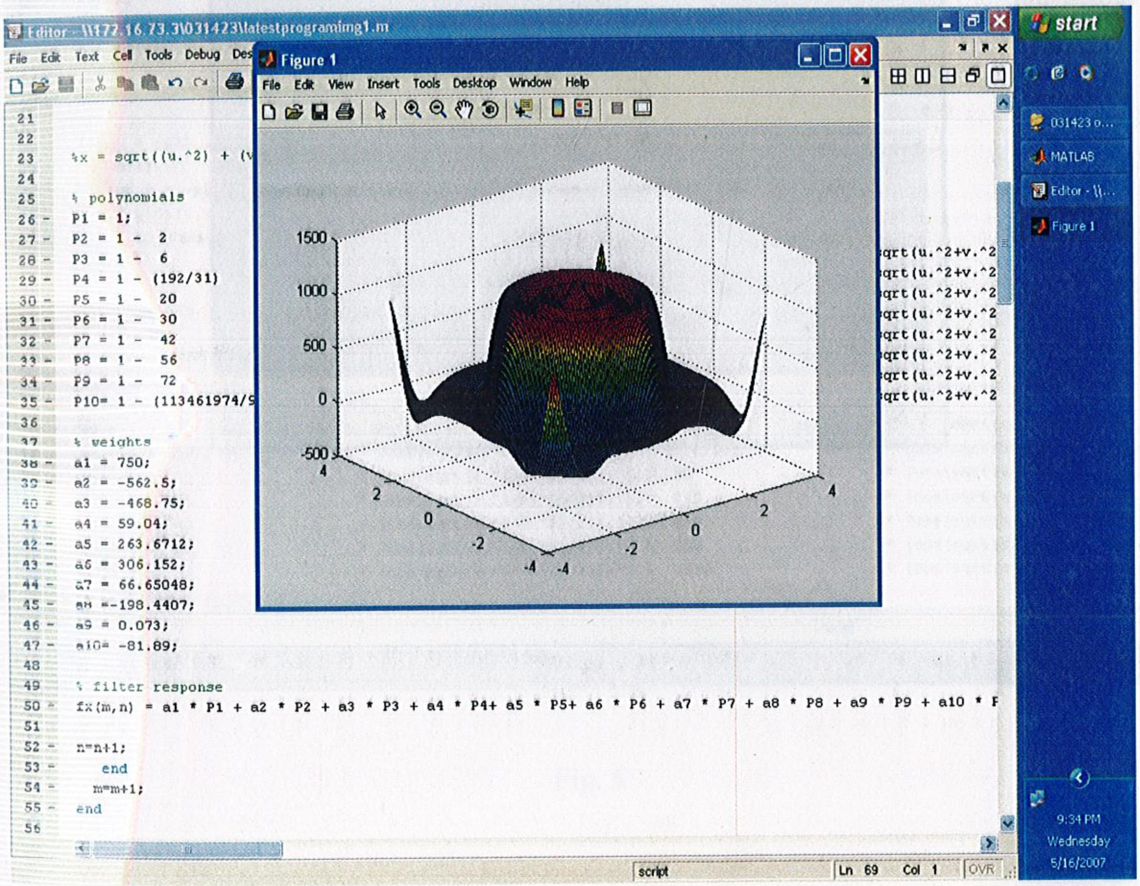

Fig. 7

Step6:
We choose the following image and pass it through the filter.

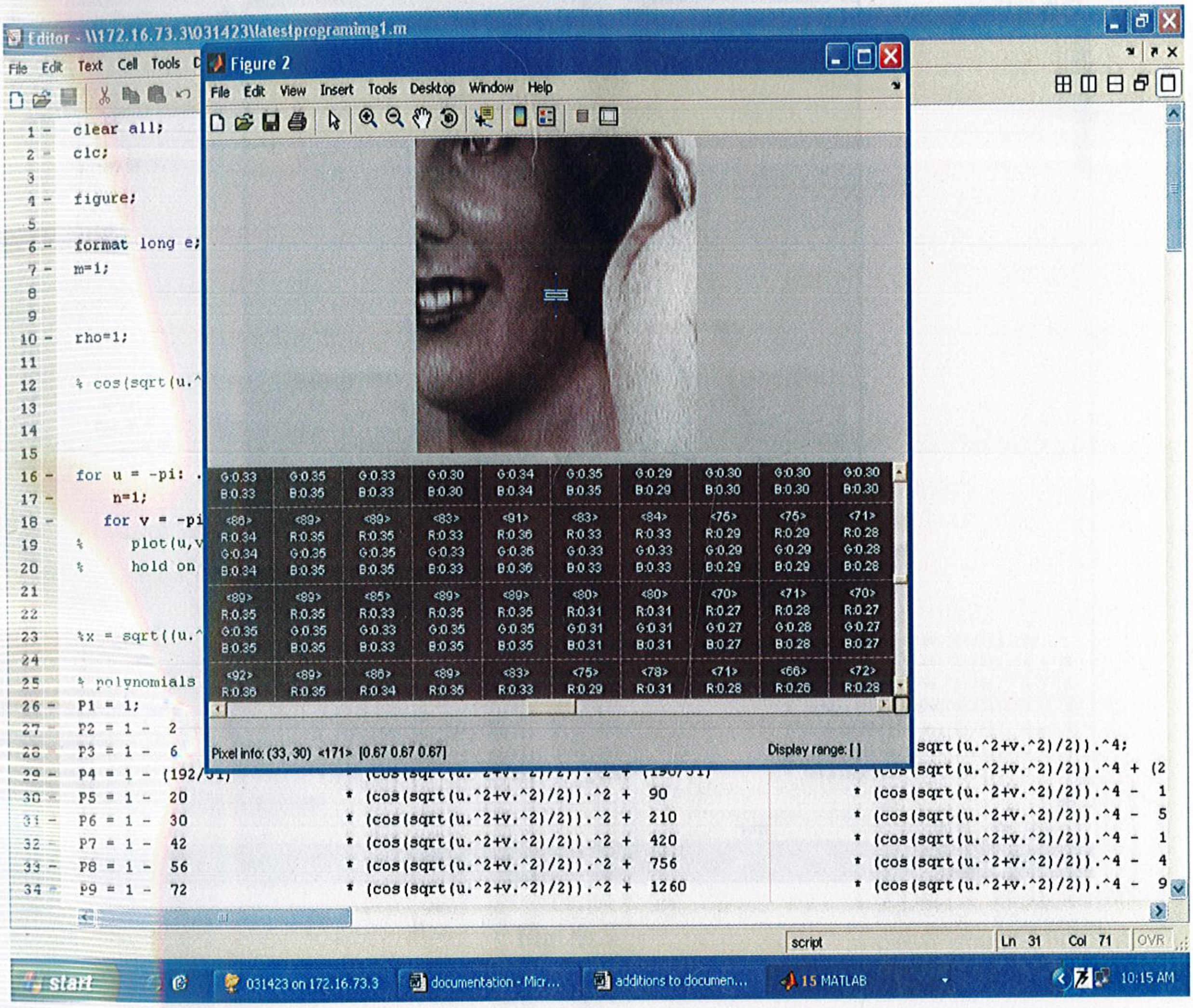

Fig. 8

Now, we add Gaussian noise to this image. The image now looks like:

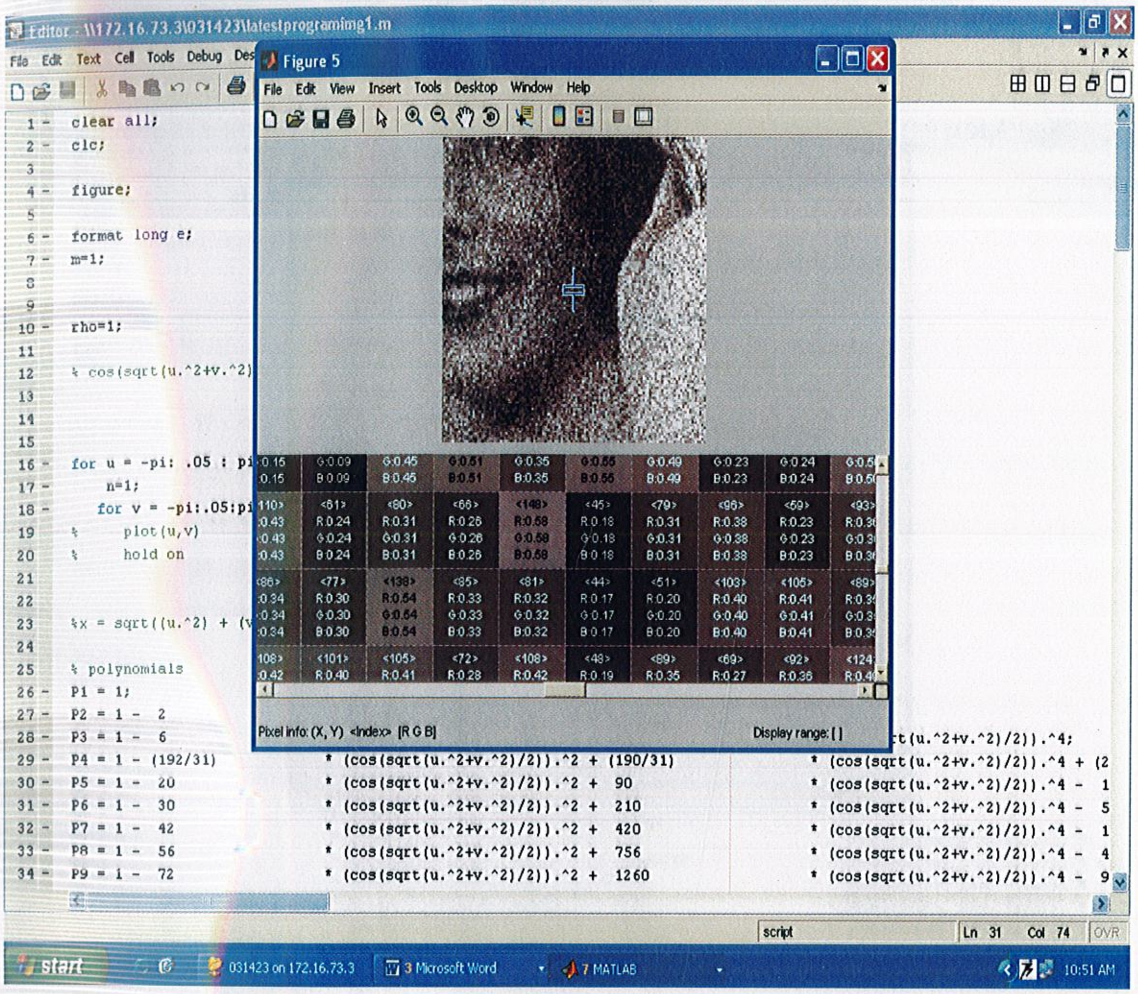

Fig. 9

Now we can apply the filter on the noisy image. After passing through the filter the image looks like-

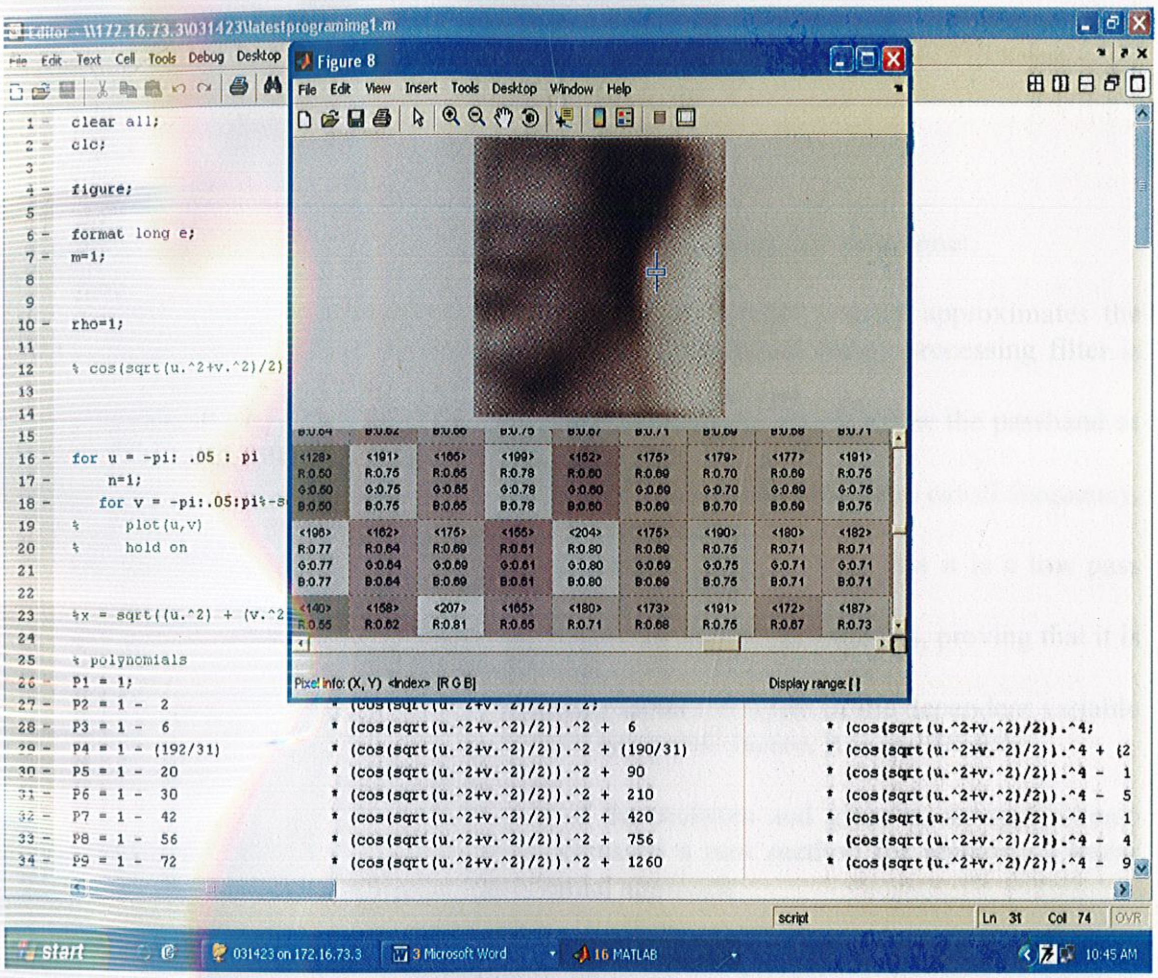

Fig. 10

We observe a significant reduction in noise in the image. The image is getting blurred because the filter used passes only the lower frequencies. Blurring of the image can be corrected afterwards. Till now, the filter has reduced a major portion of the noise that was added to the image.

CONCLUSION

We have drawn the following conclusions based on the work we have done:

(1) The filter obtained from the orthogonal polynomials closely approximates the ideal filter. Thus, our approach towards designing an image processing filter is correct.
(2) The transition band is narrow and the side lobes are much below the passband of the filter. Hence, the filter is an optimum one.
(3) The filter removes the noise added to the filter that is above the cutoff frequency. Hence, it is serving the purpose it was designed for.
(4) The output image from the filter is blurred, proving again that it is a low pass filter.
(5) The output image is not distorted in terms of spatial co-ordinates, proving that it is a linear phase filter.
(6) The polynomials used did not have any factors in terms of the dependent variable in the denominator. So, it cannot have any poles. Hence, it is an FIR filter.

This work provides a mathematical view of the problem and gives us an approximate solution to the problem, while we have discussed a new method for realizing a linear phase FIR filter through orthogonal polynomial approach.

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