On Some Parametric Generalized Measures of Fuzzy Information, Directed Divergence and Information Improvement

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ABSTRACT
In the present communication, we have introduced two new parametric generalizations of some existing measures of fuzzy information and two parametric directed divergence measures with the proof of their validity. We studied some measures of total ambiguity and new generalized measures of fuzzy information improvement. Further, particular cases of fuzzy entropy and directed divergence measures are also discussed.

General Terms
Fuzzy Information Theory, Fuzzy Set Theory.

Keywords
Fuzzy Sets, Fuzzy information measure, Fuzzy Directed Divergence, Total ambiguity, Fuzzy Information improvement measure.

1. INTRODUCTION
A fuzzy subset [1] A in U (universe of discourse) is characterized by a *membership function* $\mu_A : U \to [0,\ 1]$ which represents the *grade of membership* of $x \in U$ in A as follows:

$$\mu_A(x) = \begin{cases} 0, & \text{if } x \notin A \text{ and there is no ambiguity,} \\ 1, & \text{if } x \in A \text{ and there is no ambiguity,} \\ 0.5, & \text{if there is max. ambiguity whether } x \in A \text{ or } x \notin A \end{cases}$$

In fact $\mu_A(x)$ associates with each $x \in U,$ a grade of membership in the set A. When $\mu_A(x)$ is valued in $\{0, 1\}$, it is the characteristic function of a crisp (i.e. nonfuzzy) set.

Two fuzzy sets A and B are said to be *fuzzy-equivalent if* $\mu_B(x_i) =$ either $\mu_A(x_i)$ or $1 - \mu_A(x_i)$ for each value of i. It is clear that fuzzy-equivalent sets have the same entropy, but two sets may have the same fuzzy entropy without being fuzzy equivalent. From the fuzziness point of view there is no essential difference between fuzzy equivalent sets.
A fuzzy set A^* is called a *sharpened* version of A if the following conditions are satisfied:

$$\mu_{A^*}(x_i) \le \mu_A(x_i), \text{ if } \mu_A(x_i) \le 0.5; \quad \forall i$$

and

$$\mu_{A^*}(x_i) \ge \mu_A(x_i), \text{ if } \mu_A(x_i) \ge 0.5; \quad \forall i.$$

It may be noted that if $x_1, x_2, \ldots, x_n$ are members of the universe of discourse, then all $\mu_A(x_1), \mu_A(x_2), \ldots, \mu_A(x_n)$ lie between 0 and 1, but these are not probabilities because their sum is not unity. However,

$$\Phi_A(x_i) = \frac{\mu_A(x_i)}{\sum_{i=1}^{n} \mu_A(x_i)}; \quad i = 1, 2, \ldots, n \tag{1}$$

is a probability distribution.

Kaufman [2] defined entropy of a fuzzy set A having *n* support points by

$$H(A) = -\frac{1}{\log n} \sum_{i=1}^{n} \Phi_A(x_i) \log \Phi_A(x_i). \tag{2}$$

Since $\mu_A(x)$ and $1 - \mu_A(x)$ gives the same degree of fuzziness, therefore, analogous to the entropy due to Shannon [3], De Luca and Termini [4] suggested the following measure of fuzzy entropy:

$$H(A) = -\sum_{i=1}^{n} \left[\mu_A(x_i) \log \mu_A(x_i) + (1 - \mu_A(x_i)) \log (1 - \mu_A(x_i))\right]. \tag{3}$$

De Luca and Termini [4] introduced a set of four properties and these properties are widely accepted as a criterion for defining any new fuzzy entropy. In fuzzy set theory, the entropy is a measure of fuzziness which expresses the amount of average ambiguity/difficulty in making a decision whether an element belongs to a set or not. So, a measure of average fuzziness $H(A)$ in a fuzzy set should have at least the following properties to be valid fuzzy entropy:

- P1 (*Sharpness*): $H(A)$ is minimum if and only if A is a crisp set i.e., $\mu_A(x) = 0$ or $1; \forall x.$
- P2 (*Maximality*): $H(A)$ is maximum if and only if A is most fuzzy set i.e., $\mu_A(x) = 0.5; \forall x.$
- P3 (*Resolution*): $H(A) \ge H(A^*)$, where A^* is sharpened version of A.
- P4 (*Symmetry*): $H(A) = H(\bar{A})$, where $\bar{A}$ is the complement of A i.e., $\mu_{\bar{A}}(x_i) = 1 - \mu_A(x_i)$.

Later on Bhandari and Pal [5] made a survey on information measures on fuzzy sets and gave some new measures of fuzzy ntropy. Analogous to Rényi's [6] entropy they have suggested the following measure:

$$H_\alpha(A) = \frac{1}{1-\alpha}\sum_{i=1}^{n}\log\left[\mu_A^\alpha(x_i) + (1-\mu_A(x_i))^\alpha\right];\ \alpha \neq 1,\ \alpha > 0 \tag{4}$$

and analogous to Pal and Pal's [7] exponential entropy they introduced

$$H_e(A) = \frac{1}{n\sqrt{e}-1}\sum_{i=1}^{n}\log\left[\mu_A(x_i)e^{1-\mu_A(x_i)} + (1-\mu_A(x_i))\,e^{\mu_A(x_i)} - 1\right]. \tag{5}$$

Kapur [8] has given measure of fuzzy entropy analogous to Havrda and Charvat's [9] entropy as

$$H^\alpha(A) = \frac{1}{1-\alpha}\sum_{i=1}^{n}\left[\mu_A^\alpha(x_i) + (1-\mu_A(x_i))^\alpha - 1\right]. \tag{6}$$

Boekee and Lubbe [10] defined and studied $R-$norm information measure of the distribution P for $R \in \mathbb{R}^+$ as given by

$$H_R(P) = \frac{R}{R-1}\left[1 - \left(\sum_{i=1}^{n} p_i^R\right)^{\frac{1}{R}}\right]; \quad R > 0,\ R \neq 1. \tag{7}$$

The measure (7) is a real function from Δ_n to $\mathbb{R}^+$ and is called $R-$norm information measure. The most important property of this measure is that when $R \to 1,$ it approaches to Shannon's entropy and in case

$$R \to \infty, H_R(P) \to (1 - \max p_i);\ i = 1, 2, \ldots, n.$$

Analogous to measure (7), Hooda [11] proposed and characterized the following fuzzy information measure:

$$H_R(A) = \frac{R}{R-1}\left[\sum_{i=1}^{n} 1 - \left(\mu_A^R(x_i) + (1-\mu_A(x_i))^R\right)^{\frac{1}{R}}\right]; R > 0,\ R \neq 1. \tag{8}$$

Kullback and Leibler [12] obtained the measure of directed divergence of probability distribution $P = (p_1, p_2, \ldots, p_n)$ from the probability distribution $Q = (q_1, q_2, \ldots, q_n)$ as

$$D(P:Q) = \sum_{i=1}^{n} p_i \log\frac{p_i}{q_i}. \tag{9}$$

Kullback [13] suggested the measure of symmetric divergence as

$$J(P:Q) = \sum_{i=1}^{n}(p_i - q_i)\log\frac{p_i}{q_i}. \tag{10}$$

Let A and B be two standard fuzzy sets with same supporting points $x_1, \ldots, x_n$ and with fuzzy vectors $\mu_A(x_1), \ldots, \mu_A(x_n)$ and $\mu_B(x_1), \mu_B(x_2), \ldots, \mu_B(x_n)$.

The simplest measure of fuzzy directed divergence as suggested by Bhandari and Pal [5], is

$$I(A,B) = \sum_{i=1}^{n}\left[\mu_A(x_i)\log\frac{\mu_A(x_i)}{\mu_B(x_i)} + (1-\mu_A(x_i))\log\frac{(1-\mu_A(x_i))}{(1-\mu_B(x_i))}\right] \tag{11}$$

and the analogous symmetric divergence measure by

$$J(A,B) = I(A,B) + I(B,A),$$

which on simplification gives

$$J(A,B) = \sum_{i=1}^{n}\left[(\mu_A(x_i) - \mu_B(x_i)\right]\log\frac{\mu_A(x_i)(1-\mu_A(x_i))}{\mu_B(x_i)(1-\mu_B(x_i))}. \tag{12}$$

It is important to notice that if we take $B = A_F$ (the most fuzzy set) i.e., $\mu_B(x_i) = 0.5;\ \forall\, i$ then from (11) and (3) we have

$$I(A, A_F) = n\log 2 - \left[-\sum_{i=1}^{n}\mu_A(x_i)\log\mu_A(x_i) + (1-\mu_A(x_i))\log(1-\mu_A(x_i))\right]$$

or

$$I(A, A_F) = n\log 2 - H(A). \tag{13}$$

Kapur [8] suggested that (13) will hold whatever be the measure of directed divergence we use. Thus from every measure of directed divergence for fuzzy sets we can deduce a corresponding measure of entropy for a fuzzy set.

In literature, a number of measures of fuzzy entropy analogous to the various information measures have been proposed in order to combine the fuzzy set theory and its application to the entropy concept as fuzzy information measurements.

In the present paper, two new generalized parametric measure of fuzzy information has been proposed and validated in Section 2. Their particular cases have also been studied. In Section 3, we have proposed and discussed two parametric fuzzy directed divergence measures. In Section 4, new generalized measures of total ambiguity and fuzzy information improvement are defined and discussed.

2. PARAMETRIC FUZZY INFORMATION MEASURE

Hooda and Bajaj [14] proposed a generalized $R-$norm measure of fuzzy information given by

$$H_R^\alpha(A) = \frac{R}{R+\alpha-2}\left[\sum_{i=1}^{n} 1 - \left[(\mu_A(x_i))^{\frac{R}{2-\alpha}} + (1-\mu_A(x_i))^{\frac{R}{2-\alpha}}\right]^{\frac{2-\alpha}{R}}\right]; \tag{14}$$

where $0 < \alpha \leq 1,\ R > 0,\ R + \alpha \neq 2$ and proved its validity.

Similarly, for parameter β, $R-$norm measure of fuzzy information is given by

$$H_R^\beta(A) = \frac{R}{R+\beta-2}\left[\sum_{i=1}^{n} 1 - \left[(\mu_A(x_i))^{\frac{R}{2-\beta}} + (1-\mu_A(x_i))^{\frac{R}{2-\beta}}\right]^{\frac{2-\beta}{R}}\right]; \tag{15}$$

where $0 < \beta \leq 1,\ R > 0,\ R + \beta \neq 2.$

We propose a parametric fuzzy information measure which is a linear combination of $H_R^\alpha(A)$ and $H_R^\beta(A)$ and is given by

$$\begin{aligned} H_{R,\rho,\eta}^{\alpha,\beta}(A) \quad = \quad & \rho\frac{R}{R+\alpha-2}\left[\sum_{i=1}^{n} 1 - \left[(\mu_A(x_i))^{\frac{R}{2-\alpha}} + (1-\mu_A(x_i))^{\frac{R}{2-\alpha}}\right]^{\frac{2-\alpha}{R}}\right] \\ & +\eta\frac{R}{R+\beta-2}\left[\sum_{i=1}^{n} 1 - \left[(\mu_A(x_i))^{\frac{R}{2-\beta}} + (1-\mu_A(x_i))^{\frac{R}{2-\beta}}\right]^{\frac{2-\beta}{R}}\right]; \end{aligned} \tag{16}$$

where $\rho, \eta > 0,\ R > 0,\ 0 < \alpha, \beta \leq 1,\ R + \alpha \neq 2,\ R + \beta \neq 2.$

In order to prove that $H_{R,\rho,\eta}^{\alpha,\beta}(A)$ is a valid fuzzy information measure, we shall show that four properties (P1 - P4) i.e. sharpness, maximality, resolution and symmetry are satisfied.

P1: (*Sharpness*) $H(A)$ is minimum if and only if A is a crisp set.

$$H_{R,\rho,\eta}^{\alpha,\beta}(A)=0$$

$$\Rightarrow \rho\frac{R}{R+\alpha-2}\left[\sum_{i=1}^{n}1-\left[(\mu_A(x_i))^{\frac{R}{2-\alpha}}+(1-\mu_A(x_i))^{\frac{R}{2-\alpha}}\right]^{\frac{2-\alpha}{R}}\right]$$
$$+\eta\frac{R}{R+\beta-2}\left[\sum_{i=1}^{n}1-\left[(\mu_A(x_i))^{\frac{R}{2-\beta}}+(1-\mu_A(x_i))^{\frac{R}{2-\beta}}\right]^{\frac{2-\beta}{R}}\right]=0;$$

$$\Rightarrow \left[\sum_{i=1}^{n}1-\left[(\mu_A(x_i))^{\frac{R}{2-\alpha}}+(1-\mu_A(x_i))^{\frac{R}{2-\alpha}}\right]^{\frac{2-\alpha}{R}}\right]=0$$

and

$$\left[\sum_{i=1}^{n}1-\left[(\mu_A(x_i))^{\frac{R}{2-\beta}}+(1-\mu_A(x_i))^{\frac{R}{2-\beta}}\right]^{\frac{2-\beta}{R}}\right]=0;$$

which is satisfied in case $\mu_A(x_i)=0$ or $1,\ \forall i=1,\ 2,\ldots,\ n.$

Conversely, if A be a non-fuzzy set, then either $\mu_A(x_i)=0$ or $\mu_A(x_i)=1$. It implies

$$(\mu_A(x_i))^{\nu_1}+(1-\mu_A(x_i))^{\nu_1}=1$$

and

$$(\mu_A(x_i))^{\nu_2}+(1-\mu_A(x_i))^{\nu_2}=1$$

where
$\nu_1=\frac{R}{2-\beta},\ \nu_2=\frac{R}{2-\beta},\ \nu_1>0,\ \nu_1\neq 1,\ \nu_2>0,,\ \nu_2\neq 1$

for which $H_{R,\rho,\eta}^{\alpha,\beta}(A)=0$. Hence $H_{R,\rho,\eta}^{\alpha,\beta}(A)=0$ if and only if A is non-fuzzy set or crisp set.

P2: (*Maximality*) $H(A)$ is maximum if and only if A is most fuzzy set.

For maximal $H_{R,\rho,\eta}^{\alpha,\beta}(A)$, $\frac{\partial H_{R,\rho,\eta}^{\alpha,\beta}(A)}{\partial\mu_A(x_i)}=0$. Differentiating $H_{R,\rho,\eta}^{\alpha,\beta}(A)$ with respect to $\mu_A(x_i)$, then we have

$$\frac{\partial H_{R,\rho,\eta}^{\alpha,\beta}(A)}{\partial\mu_A(x_i)}=-\rho\lambda[(\mu_A(x_i))^{\nu_1}+(1-\mu_A(x_i))^{\nu_1}]^{\frac{1-\nu_1}{\nu_1}}\times\left[(\mu_A(x_i))^{\nu_1-1}-(1-\mu_A(x_i))^{\nu_1-1}\right]-\eta\mu[(\mu_A(x_i))^{\nu_2}+(1-\mu_A(x_i))^{\nu_2}]^{\frac{1-\nu_2}{\nu_2}}\times\left[(\mu_A(x_i))^{\nu_2-1}-(1-\mu_A(x_i))^{\nu_2-1}\right].$$

Let $0\leq\mu_A(x_i)<0.5$, then following four cases arise:

Case 1: $R>2-\beta$ and $R>2-\alpha$

We have $\lambda>0,\ \nu_1>1,\ \mu>0,\ \nu_2>1$ and
$(\mu_A(x_i))^{\nu_1-1}-(1-\mu_A(x_i))^{\nu_1-1}<0,$
$(\mu_A(x_i))^{\nu_2-1}-(1-\mu_A(x_i))^{\nu_2-1}<0$ which implies that
$\frac{\partial H_{R,\rho,\eta}^{\alpha,\beta}(A)}{\partial\mu_A(x_i)}>0.$

Case 2: $R>2-\beta$ and $R<2-\alpha$

We have $\lambda<0,\ \nu_1<1,\ \mu>0,\ \nu_2>1$ and
$(\mu_A(x_i))^{\nu_1-1}-(1-\mu_A(x_i))^{\nu_1-1}>0,$
$(\mu_A(x_i))^{\nu_2-1}-(1-\mu_A(x_i))^{\nu_2-1}<0$ which implies that

$\frac{\partial H_{R,\rho,\eta}^{\alpha,\beta}(A)}{\partial\mu_A(x_i)}>0.$

Case 3: $R<2-\beta$ and $R>2-\alpha$

We have $\lambda<0,\ \nu_1<1,\ \mu>0,\ \nu_2>1$ and
$(\mu_A(x_i))^{\nu_1-1}-(1-\mu_A(x_i))^{\nu_1-1}>0,$
$(\mu_A(x_i))^{\nu_2-1}-(1-\mu_A(x_i))^{\nu_2-1}<0$ which implies that

$\frac{\partial H_{R,\rho,\eta}^{\alpha,\beta}(A)}{\partial\mu_A(x_i)}>0.$

Case 4: $R<2-\beta$ and $R<2-\alpha$

We have $\lambda<0,\ \nu_1<1,\ \mu<0,\ \nu_2<1$ and
$(\mu_A(x_i))^{\nu_1-1}-(1-\mu_A(x_i))^{\nu_1-1}>0,$
$(\mu_A(x_i))^{\nu_2-1}-(1-\mu_A(x_i))^{\nu_2-1}>0$ which implies that
$\frac{\partial H_{R,\rho,\eta}^{\alpha,\beta}(A)}{\partial\mu_A(x_i)}>0.$

Hence, $H_{R,\rho,\eta}^{\alpha,\beta}(A)$ is an increasing function of $\mu_A(x_i)$ whenever $0\leq\mu_A(x_i)<0.5$.

Similarly, it can be proved that $H_{R,\rho,\eta}^{\alpha,\beta}(A)$ is a decreasing function of $\mu_A(x_i)$ whenever $0.5<\mu_A(x_i)\leq 1$.

It is evident that $\frac{\partial H_{R,\rho,\eta}^{\alpha,\beta}(A)}{\partial\mu_A(x_i)}=0$, when $\mu_A(x_i)=0.5$. Hence $H_{R,\rho,\eta}^{\alpha,\beta}(A)$ is a concave function and it has a global maximum at $\mu_A(x_i)=0.5$. Therefore, $H_{R,\rho,\eta}^{\alpha,\beta}(A)$ is maximum if and only if A is the most fuzzy set.

P3: (*Resolution*) $H(A)\geq H(A^*)$**,** where A^* is sharpened version of A.

Since $H_{R,\rho,\eta}^{\alpha,\beta}(A)$ is an increasing function of $\mu_A(x_i)$ in the interval [0, 0.5) and decreasing function in the interval (0.5, 1], therefore

$\mu_{A^*}(x_i)\leq\mu_A(x_i)\Rightarrow H_{R,\rho,\eta}^{\alpha,\beta}(A^*)\leq H_{R,\rho,\eta}^{\alpha,\beta}(A)$ in [0, 0.5)

and

$\mu_{A^*}(x_i)\geq\mu_A(x_i)\Rightarrow H_{R,\rho,\eta}^{\alpha,\beta}(A^*)\leq H_{R,\rho,\eta}^{\alpha,\beta}(A)$ in (0.5, 1].

Taking the above equations together, we get

$$H_{R,\rho,\eta}^{\alpha,\beta}(A^*)\leq H_{R,\rho,\eta}^{\alpha,\beta}(A).$$

P4: (*Symmetry*) $H(A)=H(\bar{A})$**,** where $\bar{A}$ is the complement of A.

It may be noted that from the definition of $H_{R,\rho,\eta}^{\alpha,\beta}(A)$ and $\mu_{\bar{A}}(x_i)=1-\mu_A(x_i)$, we conclude that

$H_{R,\,\rho,\,\eta}^{\alpha,\,\beta}\left(\bar{A}\right) = H_{R,\,\rho,\,\eta}^{\alpha,\,\beta}\left(A\right)$. Hence, $H_{R,\,\rho,\,\eta}^{\alpha,\,\beta}\left(A\right)$ satisfies all the properties of fuzzy entropy. Therefore, $H_{R,\,\rho,\,\eta}^{\alpha,\,\beta}\left(A\right)$ is a valid measure of fuzzy entropy.

Particular Cases:

It may be noted that if $\alpha = 1,\ \beta = 1,\ \rho = \frac{1}{2},\ \eta = \frac{1}{2}$, then

- (16) reduces to (8).
- (16) reduces to (3) if $R \to 1$.
- (16) reduces to $\sum_{i=1}^{n} \left[1 - \max\left\{\mu_A(x_i),\ 1 - \mu_A(x_i)\right\}\right]$, if $R \to \infty$.

Further, Hooda [11] proposed a generalized measure of fuzzy information given by

$$H_{\alpha}^{\beta}\left(A\right) = \frac{1}{2^{1-\beta}-1} \sum_{i=1}^{n} \left[\left(\mu_A^{\alpha}(x_i) + (1 - \mu_A(x_i))^{\alpha}\right)^{\frac{\beta-1}{\alpha-1}} - 1\right] \tag{17}$$

Similarly, for parameter γ and δ, we consider the following measure of fuzzy information given by

$$H_{\gamma}^{\delta}\left(A\right) = \frac{1}{2^{1-\delta}-1} \sum_{i=1}^{n} \left[\left(\mu_A^{\gamma}(x_i) + (1 - \mu_A(x_i))^{\gamma}\right)^{\frac{\delta-1}{\gamma-1}} - 1\right] \tag{18}$$

Now we propose a new measure of fuzzy information given by

$$\begin{aligned} H_{\alpha,\,\gamma,\,\lambda}^{\beta,\,\delta,\,\mu}\left(A\right) &= \frac{\lambda}{2^{1-\beta}-1} \sum_{i=1}^{n} \left[\left(\mu_A^{\alpha}(x_i) + (1 - \mu_A(x_i))^{\alpha}\right)^{\frac{\beta-1}{\alpha-1}} - 1\right] \\ &+ \frac{\mu}{2^{1-\delta}-1} \sum_{i=1}^{n} \left[\left(\mu_A^{\gamma}(x_i) + (1 - \mu_A(x_i))^{\gamma}\right)^{\frac{\delta-1}{\gamma-1}} - 1\right] \\ &= \lambda \cdot H_{\alpha}^{\beta}\left(A\right) + \mu \cdot H_{\gamma}^{\delta}\left(A\right). \end{aligned} \tag{19}$$

On the similar background, it can be shown that $H_{\alpha,\,\gamma,\,\lambda}^{\beta,\,\delta,\,\mu}\left(A\right)$ follows the four axioms (P1 - P4) of fuzzy entropy. Hence, $H_{\alpha,\,\gamma,\,\lambda}^{\beta,\,\delta,\,\mu}\left(A\right)$ is a valid measure of fuzzy information.

3. PARAMETRIC FUZZY DIRECTED DIVERGENCE MEASURE

Bajaj and Hooda [15], defined and proved the validity of the following measure of fuzzy directed divergence:

$$I_{\alpha}^{\beta}(A, B) = \frac{1}{2^{\beta-1}-1} \sum_{i=1}^{n} \left[\left(\mu_A^{\alpha}(x_i)\mu_B^{1-\alpha}(x_i) + (1 - \mu_A(x_i))^{\alpha}(1 - \mu_B(x_i))^{1-\alpha}\right)^{\frac{\beta-1}{\alpha-1}} - 1\right]; \tag{20}$$

where $\alpha \neq 1,\ \ \alpha > 0,\ \beta > 0,\ \beta \neq 1.$

Similarly, for the parameter γ and δ, we consider the following measure of fuzzy directed divergence:

$$I_{\gamma}^{\delta}\left(A,\ B\right) = \frac{1}{2^{\delta-1}-1} \sum_{i=1}^{n} \left[\left(\mu_A^{\gamma}(x_i)\mu_B^{1-\gamma}(x_i) + (1 - \mu_A(x_i))^{\gamma}(1 - \mu_B(x_i))^{1-\gamma}\right)^{\frac{\delta-1}{\gamma-1}} - 1\right]; \tag{21}$$

where $\gamma \neq 1,\ \gamma > 0,\ \delta > 0,\ \delta \neq 1.$

Now we consider the following measure of fuzzy directed divergence which is the linear combination of (20) and (21):

$$I_{\alpha,\gamma,\lambda}^{\beta,\delta,\mu}\left(A,\ B\right) = \lambda \cdot I_{\alpha}^{\beta}\left(A, B\right) + \mu \cdot I_{\gamma}^{\delta}\left(A, B\right) \tag{22}$$

where

$\lambda,\ \mu,\ \alpha,\ \beta,\ \gamma,\ \delta > 0,\ \alpha \neq 1,\ \beta \neq 1,\ \gamma \neq 1,\ \delta \neq 1.$

It may be easily verified that (22) is a valid four parametric measure of fuzzy directed divergence.

Particular Case:

Let $B = A_F$, the most fuzzy set, i.e. $\mu_B(x_i) = 0.5\ \forall x_i$, then

$$\begin{aligned} I_{\alpha,\gamma,\lambda}^{\beta,\delta,\mu}\left(A,\ B\right) &= \frac{\lambda}{2^{\beta-1}-1} \sum_{i=1}^{n} \left[\left(\mu_A^{\alpha}(x_i)(0.5)^{1-\alpha} + (1 - \mu_A(x_i))^{\alpha}(1 - (0.5))^{1-\alpha}\right)^{\frac{\beta-1}{\alpha-1}} - 1\right] \\ &+ \frac{\lambda}{2^{\delta-1}-1} \sum_{i=1}^{n} \left[\left(\mu_A^{\gamma}(x_i)(0.5)^{1-\gamma} + (1 - \mu_A(x_i))^{\gamma}(1 - (0.5))^{1-\gamma}\right)^{\frac{\delta-1}{\gamma-1}} - 1\right] \\ &= \frac{\lambda}{2^{\beta-1}-1} \cdot \frac{1}{2^{1-\beta}} \sum_{i=1}^{n} \left[\left(\mu_A^{\alpha}(x_i) + (1 - \mu_A(x_i))^{\alpha}\right)^{\frac{\beta-1}{\alpha-1}} - 2^{1-\beta}\right] \\ &+ \frac{\mu}{2^{\delta-1}-1} \cdot \frac{1}{2^{1-\delta}} \sum_{i=1}^{n} \left[\left(\mu_A^{\gamma}(x_i) + (1 - \mu_A(x_i))^{\gamma}\right)^{\frac{\delta-1}{\gamma-1}} - 2^{1-\delta}\right] \end{aligned}$$

which on simplification gives,

$$I_{\alpha,\gamma,\lambda}^{\beta,\delta,\mu}\left(A,\ B\right) = n\left(\lambda + \mu\right) - H_{\alpha,\gamma,\lambda}^{\beta,\delta,\nu}\left(A\right).$$

Hood and Bajaj [14] proposed and proved the following $R-$ norm fuzzy directed divergence measure:

$$I_R\left(A,\ B\right) = \frac{R}{R-1} \sum_{i=1}^{n} \left[\left(\mu_A^{R}(x_i)\mu_B^{1-R}(x_i) + (1 - \mu_A(x_i))^{R}(1 - \mu_B(x_i))^{1-R}\right)^{\frac{1}{R}} - 1\right] \tag{23}$$

We propose a new parametric $R-$norm fuzzy directed divergence measure given by

$$I_R^{\alpha}\left(A,\ B\right) = \frac{R}{R+\alpha-2} \sum_{i=1}^{n} \left[\left(\mu_A^{\frac{R}{2-\alpha}}(x_i)\mu_B^{1-\frac{R}{2-\alpha}}(x_i) + (1 - \mu_A(x_i))^{\frac{R}{2-\alpha}}(1 - \mu_B(x_i))^{1-\frac{R}{2-\alpha}}\right)^{\frac{2-\alpha}{R}} - 1\right] \tag{1}$$

where $R > 0,\ R + \alpha \neq 2,\ 0 < \alpha \leq 1.$

The validity of the above proposed directed divergence measure can be analytically proved on the similar pattern as it has been carried out for $I_{\alpha}^{\beta}(A, B)$ by Bajaj and Hooda [15].

Similarly, for parameter β, we consider the following measure of $R-$norm fuzzy directed divergence:

$$I_R^{\beta}\left(A,\ B\right) = \frac{R}{R+\beta-2} \sum_{i=1}^{n} \left[\left(\mu_A^{\frac{R}{2-\beta}}(x_i)\mu_B^{1-\frac{R}{2-\beta}}(x_i) + (1 - \mu_A(x_i))^{\frac{R}{2-\beta}}(1 - \mu_B(x_i))^{1-\frac{R}{2-\beta}}\right)^{\frac{2-\beta}{R}} - 1\right] \tag{25}$$

where $R > 0,\ R + \beta \neq 2,\ 0 < \beta \leq 1.$

Further, we propose the following measure of fuzzy directed divergence which is the linear combination of (24) and (25):

$$\begin{aligned} I_{R,\,\rho,\,\eta}^{\alpha,\,\beta}\left(A,\ B\right) &= \rho \cdot I_R^{\alpha}\left(A,\ B\right) + \eta \cdot I_R^{\beta}\left(A, B\right) \\ &= \frac{\rho \cdot R}{R+\alpha-2} \sum_{i=1}^{n} \left[\left(\mu_A^{\frac{R}{2-\alpha}}(x_i)\mu_B^{1-\frac{R}{2-\alpha}}(x_i) + (1 - \mu_A(x_i))^{\frac{R}{2-\alpha}}(1 - \mu_B(x_i))^{1-\frac{R}{2-\alpha}}\right)^{\frac{2-\alpha}{R}} - 1\right] \\ &+ \frac{R}{R+\beta-2} \sum_{i=1}^{n} \left[\left(\mu_A^{\frac{R}{2-\beta}}(x_i)\mu_B^{1-\frac{R}{2-\beta}}(x_i) + (1 - \mu_A(x_i))^{\frac{R}{2-\beta}}(1 - \mu_B(x_i))^{1-\frac{R}{2-\beta}}\right)^{\frac{2-\beta}{R}} - 1\right] \end{aligned} \tag{26}$$

where

$\rho > 0,\ \eta > 0,\ R > 0,\ 0 < \alpha,\ \beta \leq 1,\ R + \alpha \neq 2,\ R + \beta \neq 2.$

It may be easily verified that (26) is valid measure of fuzzy directed divergence.

Particular Case:

Let $B = A_F$, the most fuzzy set, i.e. $\mu_B(x_i) = 0.5\ \forall x_i$, then

$$I^{\alpha,\beta}_{R,\rho,\eta}(A,\ A_F) = \frac{\rho\cdot R}{R+\alpha-2}\sum_{i=1}^{n}\left[\left(\mu_A^{\frac{R}{2-\alpha}}(x_i)(0.5)^{1-\frac{R}{2-\alpha}} + (1-\mu_A(x_i))^{\frac{R}{2-\alpha}}(1-0.5)^{1-\frac{R}{2-\alpha}}\right)^{\frac{2-\alpha}{R}} - 1\right]$$
$$+ \frac{\eta\cdot R}{R+\beta-2}\sum_{i=1}^{n}\left[\left(\mu_A^{\frac{R}{2-\alpha}}(x_i)(0.5)^{1-\frac{R}{2-\beta}} + (1-\mu_A(x_i))^{\frac{R}{2-\beta}}(1-0.5)^{1-\frac{R}{2-\beta}}\right)^{\frac{2-\beta}{R}} - 1\right]$$
$$= \frac{\rho\cdot R\cdot(0.5)^{\frac{2-\alpha}{R}-1}}{R+\alpha-2}\sum_{i=1}^{n}\left[\left(\mu_A^{\frac{R}{2-\alpha}}(x_i)\right)^{1-\frac{R}{2-\alpha}} + (1-\mu_A(x_i))^{\frac{R}{2-\alpha}}\right)^{\frac{2-\alpha}{R}} - (0.5)^{1-\frac{2-\alpha}{R}}\right]$$
$$+ \frac{\eta\cdot R\cdot\left((0.5)^{\frac{2-\beta}{R}-1}\right)}{R+\beta-2}\sum_{i=1}^{n}\left[\left(\mu_A^{\frac{R}{2-\alpha}}(x_i) + (1-\mu_A(x_i))^{\frac{R}{2-\beta}}\right)^{\frac{2-\beta}{R}} - (0.5)^{1-\frac{2-\beta}{R}}\right]$$
$$= \frac{R}{R+\alpha-2}(\rho' - n\cdot\rho) + \frac{R}{R+\beta-2}(\eta' - n\cdot\eta) - H^{\alpha,\beta}_{R,\rho',\eta'}(A)$$

where $\rho' = \rho\cdot(0.5)^{\frac{2-\alpha}{R}-1}$ and $\eta' = \eta\cdot(0.5)^{\frac{2-\beta}{R}-1}$.

4. MEASURES OF TOTAL AMBIGUITY AND INFORMATION IMPROVEMENT

The uncertainty is the state of being uncertain (not certain to occur) which gives rise to fuzziness and ambiguity. Ambiguity can be viewed in non-specificity (indistinguishable alternatives) and conflict (distinguishable alternatives) while fuzziness can be viewed as lack of distinction between a set and its complement and vagueness is non-specific knowledge about lack of distinction. Thus the measure of total fuzzy ambiguity can be obtained by taking the sum of measure of fuzzy directed divergence and corresponding measure of fuzzy entropy. From (3) and (11), we have

$$TA = -\sum_{i=1}^{n}\mu_A(x_i)\log\mu_A(x_i) - \sum_{i=1}^{n}(1-\mu_A(x_i))\log(1-\mu_A(x_i))$$
$$+\sum_{i=1}^{n}\left[\mu_A(x_i)\log\frac{\mu_A(x_i)}{\mu_B(x_i)} + (1-\mu_A(x_i))\log\frac{(1-\mu_A(x_i))}{(1-\mu_B(x_i))}\right]$$

or

$$TA = -\sum_{i=1}^{n}\mu_A(x_i)\log\mu_B(x_i) - \sum_{i=1}^{n}(1-\mu_A(x_i))\log(1-\mu_B(x_i)). \quad (27)$$

Total ambiguity is a fuzzy measure of inaccuracy analogous to Kerridge [16] inaccuracy and is related to two fuzzy sets. It is not symmetric as we get something different if we interchange the role of the fuzzy sets A and B.

Corresponding to fuzzy information measure (19) and the proposed fuzzy directed divergence (22), total ambiguity is given by

$$TA = H^{\beta,\delta,\mu}_{\alpha,\gamma,\lambda}(A) + I^{\beta,\delta,\mu}_{\alpha,\gamma,\lambda}(A,\ B)$$

and corresponding to fuzzy information measure (16) and the proposed fuzzy directed divergence (26), total ambiguity is given by

$$TA = H^{\alpha,\beta}_{R,\rho,\eta}(A) + I^{\alpha,\beta}_{R,\rho,\eta}(A, B).$$

Let P and Q be observed and predicted distribution respectively of a random variable. Let $R = (r_1, r_2, \ldots, r_n)$ be the revised probability distribution of Q, then

$$D(P:Q) - D(P:R) = \sum_{i=1}^{n} p_i \log\frac{r_i}{q_i}$$

which is which is known as Theil's [17] measure of information improvement and has found wide applications in economics, accounts and financial management.

If we subtract directed divergence of fuzzy set A and fuzzy set C from the directed divergence of fuzzy set A and fuzzy set B, we get the reduction in ambiguity in revising B to C and obtain a measure, which can be called fuzzy information improvement measure.

Now suppose that the correct fuzzy set is A and originally our estimate for it was the fuzzy set B and that was revised to fuzzy set C. The original ambiguity was $I^{\beta,\delta,\mu}_{\alpha,\gamma,\lambda}(A,\ B)$ and the final ambiguity is $I^{\beta,\delta,\mu}_{\alpha,\gamma,\lambda}(A,\ C)$, so the reduction in ambiguity is

$$I^{\beta,\delta,\mu}_{\alpha,\gamma,\lambda}(A,\ B,\ C) = I^{\beta,\delta,\mu}_{\alpha,\gamma,\lambda}(A,\ B) - I^{\beta,\delta,\mu}_{\alpha,\gamma,\lambda}(A,\ C).$$

Similarly, we have

$$I^{\alpha,\beta}_{R,\rho,\eta}(A,\ B,\ C) = I^{\alpha,\beta}_{R,\rho,\eta}(A,\ B) - I^{\alpha,\beta}_{R,\rho,\eta}(A,\ C).$$

The measure $I^{\beta,\delta,\mu}_{\alpha,\gamma,\lambda}(A,\ B,\ C)$ and $I^{\alpha,\beta}_{R,\rho,\eta}(A,\ B,\ C)$ can be called as parametric fuzzy information improvement measures.

5. CONCLUSION

The two new proposed parametric generalizations of existing measures of fuzzy information and directed divergence are valid. Some other measures of total ambiguity and new generalized measures of fuzzy information improvement have further been characterized. Particular cases of fuzzy entropy, directed divergence, symmetric divergence are also discussed.

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