Mechanics of Advanced Materials and Structures

ISSN: 1537-6494 (Print) 1537-6532 (Online) Journal homepage: https://www.tandfonline.com/loi/umcm20

# Mroz' Local Response Moduli as a Means for Comparing Constitutive Equations in Plasticity

S. K. Jain

**To cite this article:** S. K. Jain (2009) Mroz' Local Response Moduli as a Means for Comparing Constitutive Equations in Plasticity, Mechanics of Advanced Materials and Structures, 16:8, 632-640, DOI: 10.1080/15376490903165709

**To link to this article:** https://doi.org/10.1080/15376490903165709

Published online: 02 Nov 2009.

*Mechanics of Advanced Materials and Structures*, 16:632–640, 2009

ISSN: 1537-6494 print / 1537-6532 online
DOI: 10.1080/15376490903165709



# Mroz′ Local Response Moduli as a Means for Comparing Constitutive Equations in Plasticity

**S. K. Jain**
*Department of Civil Engineering, Jaypee University of Information Technology, Waknaghat, India*

**Linear and nonlinear relations of time independent plasticity are compared and their local response for varying stress or strain increment orientation is studied. The applicability of incremental relations to stability and post-critical response analysis can be assessed from such comparative study. In particular, the endochronic model providing nonlinear incremental relations is considered.**

**Keywords** constitutive equations, plasticity theory, non-associative model, endochronic theory, flow localization, corner flow rule

## 1. INTRODUCTION

The present note is devoted to a comparative analysis of some incrementally linear and nonlinear constitutive equations used in modeling elasto-plastic material response (of metals, soils, concrete, rocks, etc.). Usually the model identification is performed by means of experimental data for simple proportional loading tests. In most cases subsequent verification is carried out for a selected set of loading programs for which the data are readily available in the literature. In general, the domain of applicability of any formulated constitutive model is not precisely known. It should be expected, however, that this domain (specified by a class of loading programs for which the model should provide reliable predictions) should increase with the number of material parameters used in the model formulation.

An important class of problems is associated with plastic buckling and postbuckling response of structures, plastic flow localization in materials, or in general, with failure mode predictions. The local material response under changing incremental loading direction then acquires an unusual importance in an accurate prediction of the critical stress. It is well known that for plastic bifurcation analysis the critical flow theory predicts higher critical stress than the deformation theory or the relations provided by the corner flow rule. Similarly, in flow localization problems of thin metal sheets under biaxial stretching, flow theory predictions of critical strain considerably exceed the experimental observations and the predictions of the $J_2$-deformation theory. A variety of nonlinear incremental laws have been formulated in the past in order to improve critical stress predictions in bifurcation analyses. For example, Mroz and Zienkiewicz [1], Darve et al. [2], Kolymbas [3], and Christofferson and Hutchinson [4] proposed nonlinear incremental formulations resulting from the assumptions of yield corner existence along the primary loading path. On the other hand, the endochronic plasticity model of Valanis [5] also provides an incrementally nonlinear formulation. It is of considerable interest therefore to perform a comparative study of various constitutive formulations from the viewpoint of their directional properties under varying stress increment orientations.

## 2. DIRECTIONAL MODULI

We shall limit our analysis to small strains, and thus neglect configuration changes in specifying stress or strain increments (or rates). For any specified stress increment $d\sigma$, consider the predicted strain increment $d\varepsilon = d\varepsilon^e + d\varepsilon^p$, and its projection $d\varepsilon_\sigma$ on $d\sigma$, so that a directional modulus $K_\sigma$ can be constructed as

$$\frac{1}{K_\sigma} = \frac{d\varepsilon_\sigma}{d\sigma} = \frac{d\varepsilon \cdot d\boldsymbol{\sigma} \big/ (d\boldsymbol{\sigma} \cdot d\boldsymbol{\sigma})^{1/2}}{(d\boldsymbol{\sigma} \cdot d\boldsymbol{\sigma})^{1/2}} = \frac{d\varepsilon \cdot d\boldsymbol{\sigma}}{d\boldsymbol{\sigma} \cdot d\boldsymbol{\sigma}} = \frac{d\varepsilon^e_\sigma}{d\sigma} + \frac{d\varepsilon^p_\sigma}{d\sigma} \tag{1}$$

where a dot implies the inner-product and $d\sigma = (d\sigma \cdot d\sigma)^{1/2}$. Likewise, for a specified strain increment $d\varepsilon$, consider the predicted stress increment, $d\sigma = d\sigma^e - d\sigma^p$ and its projection on $d\varepsilon$. The corresponding directional modulus $K_\varepsilon$ can be expressed as

$$K_\varepsilon = \frac{d\sigma_\varepsilon}{d\varepsilon} = \frac{d\sigma \cdot d\varepsilon}{d\varepsilon \cdot d\varepsilon} = \frac{d\sigma^e_\varepsilon}{d\varepsilon} - \frac{d\sigma^p_\varepsilon}{d\varepsilon}. \tag{2}$$

Such directional moduli were introduced by Mroz [6] for studying local properties of constitutive laws. Further studies

Received 8 February 2008; accepted 28 October 2008.
Address correspondence to S. K. Jain, Department of Civil Engineering, Jaypee University of Information Technology, Waknaghat (Solan) H.P. 173215 India. E-mail: sk.jain@juit.ac.in

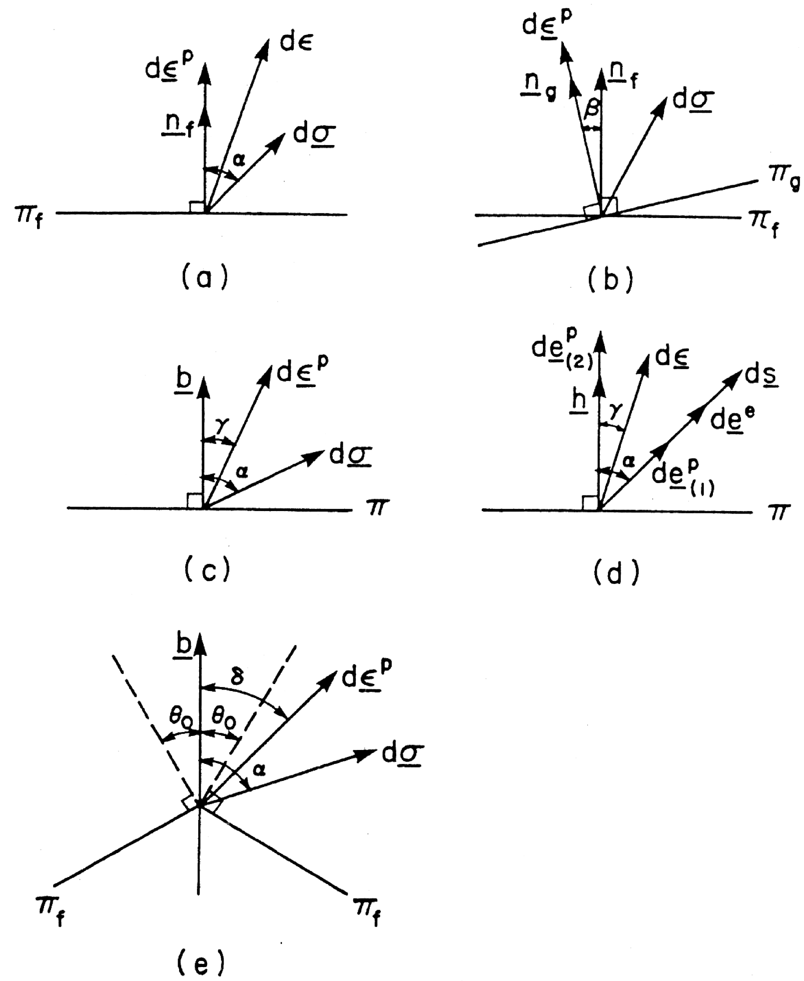


FIG. 1. Diagrams used in establishing local directional properties of various plasticity models.

on these moduli have been made by Runesson and Mroz [7] and Jain [8]. We note here that the product of the two moduli produces the tangent modulus, $K_t = K_\sigma K_\varepsilon = d\sigma/d\varepsilon$.

## 3. LINEAR INCREMENTAL RELATIONS

When Eq. (1) is referred to the deviatoric stress space and the associated flow rule is applied, that is

$$d\varepsilon = d\varepsilon^e + d\varepsilon^p = \frac{\mathrm{d}\mathbf{s}}{2G} + \frac{1}{K_p} n_f \left(\mathrm{d}\mathbf{s} \cdot \boldsymbol{n}_\mathrm{f}\right), \tag{3}$$

then

$$\frac{1}{K_\sigma} = \begin{cases} \dfrac{1}{2G} + \dfrac{1}{K_p}\cos^2\alpha; & 0 < \alpha < \pi/2 \\ \dfrac{1}{2G}; & \pi/2 \le \alpha \le \pi. \end{cases} \tag{4}$$

Where $\mathbf{s}$ denotes the stress deviator, $\alpha$ is the angle between the unit vector $n_f$ normal to the regular yield surface and the stress increment vector $\mathrm{d}\sigma$ as shown in Figure 1a, and $K_p$ is the normalized plastic hardening modulus. Similarly, for the non-associated flow rule

$$d\varepsilon^p = \frac{1}{K_p} \boldsymbol{n}_g \left(\mathbf{s} \cdot \boldsymbol{n}_g\right). \tag{5}$$

Where $\boldsymbol{n}_g$ is the unit flow vector and $\boldsymbol{n}_f$ is the unit vector normal to the regular yield surface, Eq. (1) produces

$$\frac{1}{K_\sigma} = \begin{cases} \dfrac{1}{2G} + \dfrac{1}{K_p}\cos\alpha\cos(\alpha-\beta); & \beta < \alpha < \pi/2 + \beta \\ \dfrac{1}{2G}; & \pi/2 + \beta \le \alpha \le \pi + \beta \end{cases} \tag{6}$$

Where $\cos\beta = \boldsymbol{n}_f \cdot \boldsymbol{n}_g$ is $\alpha$ denotes the angle between $\boldsymbol{n}_g$ and $d\boldsymbol{\sigma}$, as shown in Figure 1b.

The constitutive equations of $J_2$-deformation theory are expressed as,

$$\mathbf{e} = \begin{cases} \dfrac{\mathbf{s}}{2G_s}, & \text{for } F = (3J_2)^{1/2} - \sigma_p,\ \mathbf{s}\cdot d\mathbf{s} > 0 \\ \dfrac{\mathbf{s}}{2\mathrm{G}}, & \text{for } F \le 0 \text{ or } F = 0,\ \mathbf{s}\cdot d\mathbf{s} < 0 \end{cases} \tag{7}$$

where $\mathbf{e}$ denotes the deviatoric strain tensor and $G_s$ is the secant modulus. Eqs. (7) can be differentiated to obtain the incremental relations,

$$d\boldsymbol{e} = \begin{cases} \dfrac{d\mathbf{s}}{2G_s} + \dfrac{1}{2}\left(\dfrac{1}{G_t} - \dfrac{1}{G_s}\right)\dfrac{\mathbf{s}\cdot d\mathbf{s}}{s^2}\mathbf{s}, & \text{for } F = 0,\ \mathbf{s}\cdot d\mathbf{s} > 0 \\ \dfrac{d\mathbf{s}}{2\mathrm{G}}, & \text{for } F < 0 \text{ or } F = 0,\ \mathbf{s}\cdot d\mathbf{s} < 0 \end{cases} \tag{8}$$

where $G_t$ is the tangent modulus of the uniaxial hardening curve. This model does not satisfy the continuity condition for neutral loading and its applicability is limited to admissible cones satisfying local uniqueness condition [9]. The directional modulus now becomes

$$\frac{1}{K_\sigma} = \frac{1}{2G_s} + \frac{1}{2}\left(\frac{1}{G_t} - \frac{1}{G_s}\right)\cos^2\alpha \tag{9}$$

where $\alpha$ is now the angle between $\mathbf{s}$ and $d\mathbf{s}$.

## 4. NONLINEAR INCREMENTAL RELATIONS

Consider now a nonlinear relation following from the representation discussed by Kolymbas [3]:

$$d\mathbf{e} = \frac{d\mathbf{s}}{\mathrm{A}} + \frac{1}{\mathrm{B}}\mathbf{b}\,(d\mathbf{s}\cdot d\mathbf{s})^{1/2} \tag{10}$$

where $\mathbf{b}$ is the unit vector specifying the direction of plastic flow and is a function of stress and actual state variables. Note that there is no loading-unloading condition associated with Eq. (10) and the second term predicts a strain increment always directed along the vector $\mathbf{b}$ as shown in Figure 1c.

Consider the case when $d\mathbf{s}$ is directed along $\mathbf{b}$ or $-\mathbf{b}$. Equation (10) then provides, respectively

$$d\mathbf{e} = \frac{d\mathbf{s}}{\mathrm{A}} + \frac{1}{\mathrm{B}}d\mathbf{s} = \left(\frac{1}{\mathrm{A}} + \frac{1}{\mathrm{B}}\right)d\mathbf{s}, \quad d\mathbf{s}\cdot\mathbf{b} > 0 \tag{11a}$$

$$d\mathbf{e} = \frac{d\mathbf{s}}{\mathrm{A}} + \frac{1}{\mathrm{B}}d\mathbf{s} = \left(\frac{1}{\mathrm{A}} - \frac{1}{\mathrm{B}}\right)d\mathbf{s}, \quad d\mathbf{s}\cdot\mathbf{b} < 0. \tag{11b}$$

Denoting the tangent moduli of loading and unloading shear stress-strain curves by $G_t$ and G, we have

$$\frac{1}{2G_t} = \left(\frac{1}{\mathrm{A}} + \frac{1}{\mathrm{B}}\right), \quad \frac{1}{2G_t} = \left(\frac{1}{\mathrm{A}} - \frac{1}{\mathrm{B}}\right). \tag{12}$$

Equation (12) can be used in determining A and B from local loading-unloading tests along $\mathbf{b}$-direction.

$$\frac{1}{\mathrm{A}} = \frac{1}{4}\left(\frac{1}{G_t} + \frac{1}{\mathrm{G}}\right), \quad \frac{1}{\mathrm{B}} = \frac{1}{4}\left(\frac{1}{G_t} - \frac{1}{\mathrm{G}}\right) \tag{13}$$

The properties of similar nonlinear relations were discussed by Mroz and Zienkiewicz [1] who indicated that for small stress cycles there is always a progressive strain ratcheting when tangent moduli take on the values $G_t$ and G alternatingly. Relations such as those of Eq. (11a) were earlier applied successfully to soils by Duncan and Chang [10].

The constitutive relation Eq. (10) can be inverted to provide nonlinear stress increment-strain increment relations as

$$d\mathbf{s} = \mathrm{A}d\varepsilon - \frac{\mathrm{A}}{\mathrm{B}}\mathbf{b}d\mathbf{s} = \mathrm{A}d\varepsilon - \frac{\mathrm{A}}{\mathrm{B}}\mathbf{b}\mathrm{M}(\gamma)\,d\varepsilon \tag{14}$$

where $\mathrm{M}(\gamma)$, providing a relation between $d\mathbf{s}$ and $d\varepsilon$, can be established from Eq. (10) as follows. Multiplying both sides of Eq. (10) by $\mathbf{b}$ and squaring both sides, we obtain

$$\begin{aligned} \mathbf{b}\cdot d\varepsilon &= \frac{1}{\mathrm{A}}(\mathbf{b}\cdot d\mathbf{s}) + \frac{1}{\mathrm{B}}d\mathbf{s} \\ d\varepsilon^2 = d\varepsilon\cdot d\varepsilon &= \left(\frac{1}{\mathrm{A}^2} + \frac{1}{\mathrm{B}^2}\right)ds^2 + \frac{2}{\mathrm{AB}}(\mathbf{b}\cdot d\mathbf{s})d\mathbf{s}. \end{aligned} \tag{15}$$

From Eq. (15) it follows that

$$\begin{aligned} ds &= \frac{\mathrm{AB}}{\mathrm{B}^2 - \mathrm{A}^2}\left[-(\mathbf{b}\cdot d\varepsilon) + \sqrt{(\mathbf{b}\cdot d\varepsilon)^2 + d\varepsilon^2\left(\mathrm{B}^2 - \mathrm{A}^2\right)}\right] \\ &= \frac{\mathrm{AB}}{\mathrm{B}^2 - \mathrm{A}^2}\left(-\cos\gamma + \sqrt{\cos^2\gamma + \mathrm{B}^2 - \mathrm{A}^2}\right)d\varepsilon \\ &= \mathrm{M}(\gamma)\,d\varepsilon \end{aligned} \tag{16}$$

where $\gamma$ is the angle between $d\varepsilon$ and $\mathbf{b}$, and thus, $\cos\gamma = d\varepsilon\,.\,\mathbf{b}$.

The directional modulus is easily calculated from Eq. (10)

$$\frac{1}{K_\sigma} = \frac{1}{\mathrm{A}} + \frac{1}{\mathrm{B}}\cos\alpha \tag{17}$$

Similarly, from Eq. (14) it follows that

$$K_\varepsilon = \mathrm{A} - \cos\gamma \frac{\mathrm{A}^2}{\mathrm{B}^2 - \mathrm{A}^2}\left(\sqrt{\cos^2\gamma + \mathrm{B}^2 - \mathrm{A}^2} - \cos\gamma\right). \quad (18)$$

### 4.1. Endochronic Plasticity Model

We now consider the endochronic plasticity model providing an integral relationship between stress and plastic strain, cf. Valanis [5].

$$s_{ij} = \int_0^z \rho(z - z') \frac{de_{ij}^p}{dz'} dz' \quad (19)$$

The incremental form associated with Eq. (19) is

$$ds_{ij} = \rho(0)\, de_{ij}^p - h_{ij}(z)\, \mathrm{dz} \quad (20)$$

where

$$h_{ij}(z) = -\int_0^z \frac{\partial\rho(z - z')}{\partial(z - z')} \frac{de_{ij}^p(z')}{dz'} dz' \quad (21)$$

and

$$\mathrm{dz} = \frac{de^p}{f(\zeta)}, \quad d\zeta = de^p = \left(de_{ij}^p de_{ij}^p\right)^{1/2}. \quad (22)$$

Equation (20) can be rewritten in the form

$$\mathrm{d}\mathbf{e}^p = \frac{\mathrm{d}\mathbf{s}}{\rho(0)} + \frac{1}{\rho(0)}\mathbf{h}(z)\mathrm{dz} = de_{(1)}^p + de_{(2)}^p \quad (23)$$

It is seen that the plastic strain increment is composed of two terms, one quasi-elastic following the orientation of d**s**, the other of fixed direction of **h** as shown in Figure 1d. Also, the similarity between Eq. (10) and Eq. (23) becomes evident if we add the elastic term to relation (23) to have

$$\mathrm{de} = de^e + de^p = \left(\frac{1}{2\mu} + \frac{1}{\rho_o}\right)\mathrm{d}\mathbf{s} + \frac{1}{\rho_o}\mathbf{h}(z)\mathrm{dz} \quad (24)$$

where $\mu$ is the shear modulus and $\rho_o = \rho(0)$.

To establish a relationship between stress and strain increments, we will need to express dz either in terms of ds or de depending on whether a stress-controlled or a strain-controlled scheme is desired [11]. Let us first express dz in terms of d**s**. Squaring both sides of Eq. (23), we obtain

$$\rho_o (de^p)^2 = (\mathrm{ds})^2 + h^2 (\mathrm{dz})^2 + 2\,(\mathrm{d}\mathbf{s} \cdot \mathbf{h})\,\mathrm{dz}$$

or,

$$(\rho_o^2 f^2 - h^2)\mathrm{dz}^2 - 2(\mathrm{ds} \cdot \mathbf{h})\mathrm{dz} - \mathrm{ds}^2 = 0 \quad (25)$$

Solving this quadratic equation for dz, we obtain

$$\mathrm{dz} = \frac{\mathrm{ds} \cdot h}{\rho_o^2 f^2 - h^2} + \frac{\left[(ds \cdot h)^2 + ds^2\left(\rho_o^2 f^2 - h^2\right)\right]^{\frac{1}{2}}}{\rho_o^2 f^2 - h^2} \quad (26)$$

where $h = (h \cdot h)^{1/2}$ and $\mathrm{ds} = (\mathrm{ds} \cdot \mathrm{ds})^{1/2}$. Eq. (26) provides a homogeneous relation between dz and ds. Keeping in conformity to our treatment of forgoing models, we will define the angle between d**s** and **h** as $\alpha$, so the Eq. (26) can be written as

$$\mathrm{dz} = \frac{h\cos\alpha + \left[h^2\cos^2\alpha + \rho_o^2 f^2 - h^2\right]^{\frac{1}{2}}}{\rho_o^2 f^2 - h^2} ds = K(\alpha)\, ds. \quad (27)$$

In order to express dz in terms of de, let us present Eq. (23) in the form

$$\rho_o de^p = 2\mu\,(\mathrm{d}\mathbf{e} - \mathrm{de}^p) - \mathbf{h}\,\mathrm{dz} \quad (28)$$

or,

$$(\rho_o + 2\mu)\mathrm{de}^p = 2\mu\,\mathrm{d}\mathbf{e} - \mathbf{h}\,\mathrm{dz} \quad (29)$$

Squaring both sides of Eq. (29), one obtains the quadratic equation for dz:

$$(\rho_o + 2\mu)^2 f^2 dz^2 = 4\mu^2 de^2 + h^2 dz^2 - 4\mu(\mathrm{d}\mathbf{e} \cdot \mathbf{h})\mathrm{dz} \quad (30)$$

which produces

$$\mathrm{dz} = -\frac{2\mu\,(\mathrm{d}\mathbf{e} \cdot \mathbf{h}) + 2\mu\{(\mathrm{de} \cdot \mathrm{h})^2 + de^2[\left(\rho_o + 2\mu\right)^2 f^2 - h^2]\,\}^{1/2}}{(\rho_o + 2\mu)^2 f^2 - h^2}.$$

Denoting the angle between d**e** and **h** as $\gamma$, we have

$$\mathrm{dz} = -\frac{2\mu}{(\rho_o + 2\mu)^2 f^2 - h^2} \times\{\, h\cos\gamma - [h^2\cos\gamma + (\rho_o + 2\mu)^2 f^2 - h^2]^{1/2}\}\,\mathrm{d}\mathbf{e} = \mathrm{L}(\gamma)\,\mathrm{d}\mathbf{e}. \quad (31)$$

In view of Eqs. (27) and (31), the incremental relations of Eqs. (23) and (20) take the form

$$de_{ij} = \left(\frac{1}{\rho_o} + \frac{1}{2\mu}\right) ds_{ij} + \frac{1}{\rho_o} h_{ij} \mathrm{K}(\alpha)\, \mathrm{d}\mathbf{s} \tag{32}$$

$$ds_{ij} = \frac{2\mu\rho_o}{2\mu + \rho_o} de_{ij} - \frac{2\mu}{2\mu + \rho_o} h_{ij} \mathrm{L}(\gamma)\mathrm{d}\mathbf{e} \tag{33}$$

where

$$K(\alpha)\, \mathrm{d}\mathbf{s} = \mathrm{L}(\gamma)\, \mathrm{d}\mathbf{e}. \tag{34}$$

It is seen that the endochronic relations (32) and (33) are analogous in form to the relations (10) and (14).

If the kernel function ρ(z) is assumed to be of the form

$$\rho(\mathrm{z}) = \sum_{r=1}^{n} A_r e^{-\alpha_r z}, \quad \rho(z - z') = \sum_{r=1}^{n} A_r e^{-\alpha_r (z - z')} \tag{35}$$

then

$$\rho_o = \rho(0) = \sum_{r=1}^{n} A_r = A_1 + A_2 + A_3 + - - - - - - + A_n \tag{36}$$

and

$$h_{ij} = \sum_r A_r \alpha_r \int e^{-\alpha_r (z - z')} \frac{de_{ij}^p(z')}{dz'} dz' = \sum_{r=1}^{n} \alpha_r s_{ij}^{(r)} \tag{37}$$

and the incremental relations can be written in an alternative form as

$$de_{ij} = \left(\frac{1}{\rho_o} + \frac{1}{2\mu}\right) ds_{ij} + \frac{1}{\rho_o} \sum_{r=1}^{n} \alpha_r s_{ij}^{(r)} \mathrm{K}(\alpha)\, \mathrm{d}\mathbf{s} \tag{38}$$

$$ds_{ij} = \frac{2\mu\rho_o}{2\mu + \rho_o} de_{ij} - \frac{2\mu}{2\mu + \rho_o} \sum_{r=1}^{n} \alpha_r s_{ij}^{(r)} \mathrm{L}(\gamma)\, \mathrm{d}\mathbf{e} \tag{39}$$

where

$$s_{ij}^{(r)} = \int_o^z A_r e^{-\alpha_r (z - z')} \frac{de_{ij}^p(z')}{dz'} dz' \tag{40}$$

Equations (38) and (39) help us visualize the basic nature of the endochronic relations. In Eq. (38), the total strain increment is seen to be composed of an elastic strain increment (the first term) and a plastic strain increment (the second term) resulting from n slip mechanisms at the microscopic level. The term $s_{ij}^{(r)}$ is an internal variable corresponding to the rth slip mechanism. Likewise, Eq. (39) is composed of an elastic stress increment and n relaxation mechanisms.

The directional moduli can be determined from relations (32) and (33) as follows.

$$\frac{1}{K_\sigma} = \frac{1}{\rho_o} + \frac{1}{2\mu} + \frac{1}{\rho_o} h \cos\alpha\, \mathrm{K}(\alpha) \tag{41}$$

$$K_\varepsilon = \frac{2\mu\rho_o}{2\mu + \rho_o} - \frac{2\mu}{2\mu + \rho_o} h \cos\gamma\, \mathrm{L}(\gamma) \tag{42}$$

## 4.2. Nonlinear Corner Flow Rule

Referring to Figure 1e, assume that at the loading point the normals to the yield surface $\pi_f$ constitute a cone of the vertex angle $\theta_0$. The symmetry axis of this cone is specified by the vector **b**. For a stress increment $\mathrm{d}\boldsymbol{\sigma}$ inclined at an angle $\alpha$ to **b**, the corresponding plastic strain increment $\mathrm{d}\varepsilon^p$ is oriented at an angle $\delta$ to **b**. The increment $\mathrm{d}\varepsilon^p$ vanishes when the stress increment is directed either tangentially to $\pi_f$ or is in the interior of the elastic domain. It can generally be written

$$\mathrm{d}\varepsilon_{ij}^p = \left(\frac{1}{\mathrm{K}} ds_{ij} + \frac{1}{\mathrm{L}} b_{ij} ds \cos\alpha\right) \mathrm{f}(\alpha), \quad 0 < \alpha \le \theta_0 + \frac{\pi}{2} \tag{43}$$

where K and L are material parameters and the function f(α) is chosen such that $\mathrm{f}(\alpha) = 1$ for $\alpha = 0$ (proportional loading paths) and $\mathrm{f}(\alpha) = 0$ whenever $\alpha = \theta_0 + \pi/2$ (neutral loading paths). For instance, we can assume

$$\mathrm{f}(\alpha) = \cos\frac{\pi}{2}\left(\frac{\alpha}{\frac{\pi}{2} + \theta_0}\right). \tag{44}$$

A similar flow rule was considered by Christoffersen and Hutchinson [4]. Adding the elastic strain, Eq. (43) can be written as

$$\mathrm{d}\varepsilon_{ij} = \left(\frac{1}{\mathrm{M}} ds_{ij} + \frac{1}{\mathrm{L}} b_{ij} ds \cos\alpha\right) \mathrm{f}(\alpha) \tag{45}$$

where $\frac{1}{\mathrm{M}} = \frac{1}{2\mathrm{G}} + \frac{1}{\mathrm{K}}$.

Just as for Eq. (10), the material parameters for Eq. (45) are identified from a uniaxial test, and the directional modulus is expressed as

$$\frac{1}{K_\sigma} = \left(\frac{1}{\mathrm{M}} + \frac{1}{\mathrm{L}} \cos^2\alpha\right) \mathrm{f}(\alpha). \tag{46}$$

## 5. DIRECTIONAL MODULI VARIATION: A COMPARATIVE STUDY

For comparing the variations of direction moduli for different models, we selected the uniaxial stress-strain curve of cyclically stabilized OFHC copper, given in Lamba and Sidebottom [12], mainly because for a stabilized material the endochronic equations can be directly integrated, and thus, the computation and the determination of material parameters become a simple matter.

For a monotonic uniaxial test on a stabilized material

$$\sigma_x = \sqrt{\frac{3}{2}} \sum_{r=1}^{n} \frac{A_r}{\alpha_r}(1 - e^{-\alpha_r z}); \quad z = \sqrt{3/2}\ |\varepsilon_X^p| \tag{47}$$

whence

$$\frac{d\sigma_x}{d\varepsilon_x^p} = E_t^p = \frac{3}{2} \sum_{r=1}^{n} A_r e^{-\alpha_r z} = \frac{3}{2}\rho\,(z). \tag{48}$$

The following material parameters generate a stress-strain response shown in Figure 2 which is very close to the experimental one.

$$\begin{aligned} &A_1 = 8 \times 10^5\ ksi; \quad A_2 = 1.604 \times 10^4\ ksi, \\ &A_3 = 1.206 \times 10^3\ ksi, \quad A_4 = 0; \quad \alpha_1 = 1 \times 10^5, \\ &\alpha_2 = 1.38 \times 10^3, \quad \alpha_3 = 2.05 \times 10^2, \quad \alpha_4 = 0; \\ &E = 15300\ ksi, \quad \upsilon = 0.33, \quad G = 5752\ ksi \end{aligned} \tag{49}$$

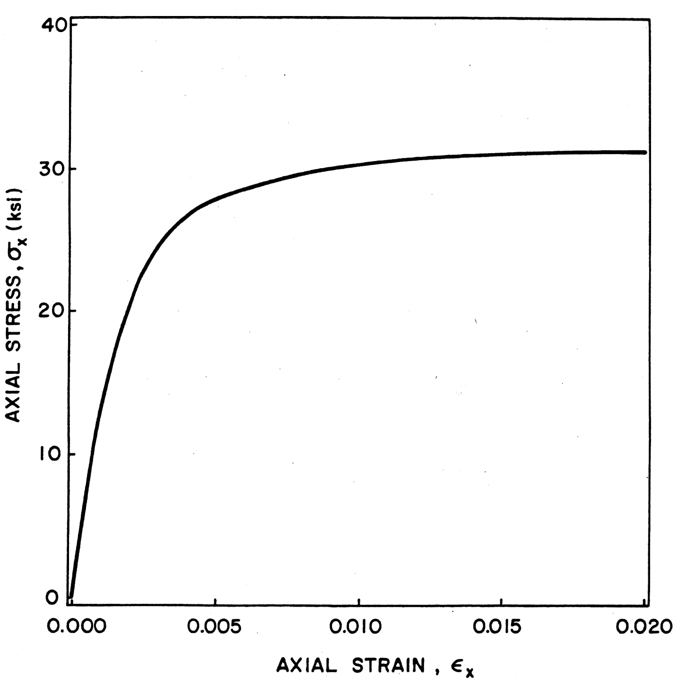


FIG. 2. Uniaxial stress-strain curve of stabilized OFHC copper.

Nearly the same values of the parameters are reported in Hsu et al. [13, 14] for the cyclic stress-strain curve of OFHC copper.

For the uniaxial case, Eq. (21) leads to

$$h_{11} = -2h_{22} = -2h_{33} = \sqrt{\frac{2}{3}} \sum_{r=1}^{n} A_r(1 - e^{-\alpha_r z})$$

therefore

$$h = (h_{ij}h_{ij})^{\frac{1}{2}} = \sum_{r=1}^{n} A_r\left(1 - e^{-\sqrt{3/2}\alpha_r \varepsilon_x^p}\right). \tag{50}$$

The variation of the directional modulus $K_\sigma$, Eq. (37), is shown in Figure 3 for different values of uniaxial strain, $\varepsilon_x$. As seen in the figure, $K_\sigma$ is nearly equal to the elastic modulus in the beginning of the uniaxial test for all values of $\alpha$. For large values of $\varepsilon_x$, during the final stages of the test, $K_\sigma$ nearly remains zero until $\alpha = \pi/2$, whereupon it experiences a jump to attain a value of the elastic modulus and stays at that constant value until $\alpha = \pi$. We found this to be true for strains as high as 24%. Such a variation, in the sense of plasticity theory, corresponds to an elastic-perfectly plastic behavior.

What is striking in Figure 3 is the fact that the behavior of the model is nearly elastic for $\alpha = \pi/2$ to $\pi$ and that there is a smooth transition between the elastic and the elastic-plastic domains.

We will now use the stress-strain curve of Figure 2 to compute the variation of the direction modulus K for various models. Utilizing a kinematically hardening von-Mises yield surface, the associated flow rule, Eq. (3), gives for the uniaxial test

$$K_p = \frac{2}{3}\frac{d\sigma_X}{d\varepsilon_X^p} = \frac{2}{3}E_t^p. \tag{51}$$

The expression of $E_t^p$ in Eq. (48) can again be used with the parameters of Eq. (49) provided the first term of the series with $A_1$ and $\alpha_1$ is dropped. (This term gives rise to an apparent elastic domain in the endochronic theory.) Thus,

$$K_p = \sum_{r=2}^{n} A_r e^{-\alpha_r z} \tag{52}$$

Such an expression for $K_p$ is also obtained if a hereditary hardening rule [15, 16] is used in the associated flow theory, as shown in Appendix A. Also to be noted is the fact that the formulation of the endochronic theory with a yield surface as discussed in Valanis [5] and in Watanabe and Atluri [17] gives rise to an identical associated flow model discussed above. This is shown in Appendix B by a direct application of the consistency condition.

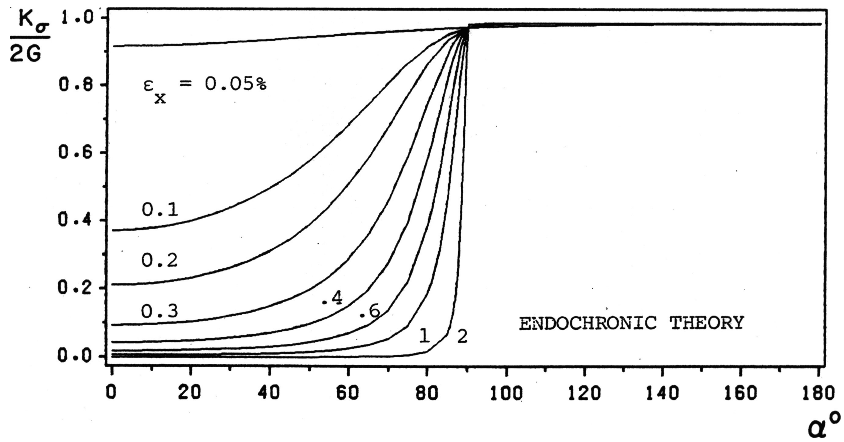


FIG. 3. Diagram illustrating the variation of normalized directional modulus with the direction of stress increment vector.

Let us compare different models for a strain level, $\varepsilon_x = 0.003$ with $\varepsilon_x^p = 0.00139$.

$$
\begin{aligned}
E_t^p &= \frac{3}{2}\left(A_2 e^{-\alpha_2 z} + A_3 e^{-\alpha_3 z}\right) = \frac{3}{2}(1531 + 850.7) = 3573\ ksi,\\
K_p &= 2381.7\ ksi,\ G_t^p = E_t^p/3 = 1191\ ksi,\ G_t = 986.7\ ksi;\\
E_x^p &= 17,840\ ksi,\ G_s^p = 5947\ ksi,\ G_s = 2924\ ksi;\\
E_t &= 2986.6\ ksi,\ \mathrm{A} = 3368.9\ ksi,\ \mathrm{B} = 4764\ ksi;\\
\mathrm{K} &= \mathrm{L} = 4764\ ksi.
\end{aligned}
$$

For the corner flow rule, we computed the value of θ utilizing the following expression of $\beta_e$ given in Mroz [9].

$$
\beta_e = \sin^{-1}\left[\frac{\sqrt[2]{G/G_s}}{1 + G/G_s}\right] = 71^\circ \tag{53}
$$

$$
\theta_o = 180^\circ - \left(90^\circ + \beta_e\right) = 19^\circ
$$

The variation of K for different models is shown in Figure 4. The figure demonstrates the extent of plasticity present in a model for different directions of stress-increment vector. In the associative theory of plasticity (or the endochronic theory with a yield surface) no plastic deformations are possible for $\alpha = \pi/2$ to $\pi$. This is clearly seen in the figure. What is surprising is that the endochronic theory with no distinct yield domain shows similar characteristics to those of associated flow model. The deformation theory model indicates plasticity for all values of α.

The corner flow rule appears very attractive inasmuch as the relative magnitude of the plastic deformations, indicated by the ratio $K_\sigma/2\mathrm{G}$, can be adjusted at will for different directions of stress increment vector. The non-linear incremental relation, Eq. (10), which appears to be a limiting case of corner flow rule, generates a decreasing amount of plasticity for increasing values of $\alpha$ · An elastic response is obtained only for the completely reversed loading directions indicated by $\alpha = 180^0$.

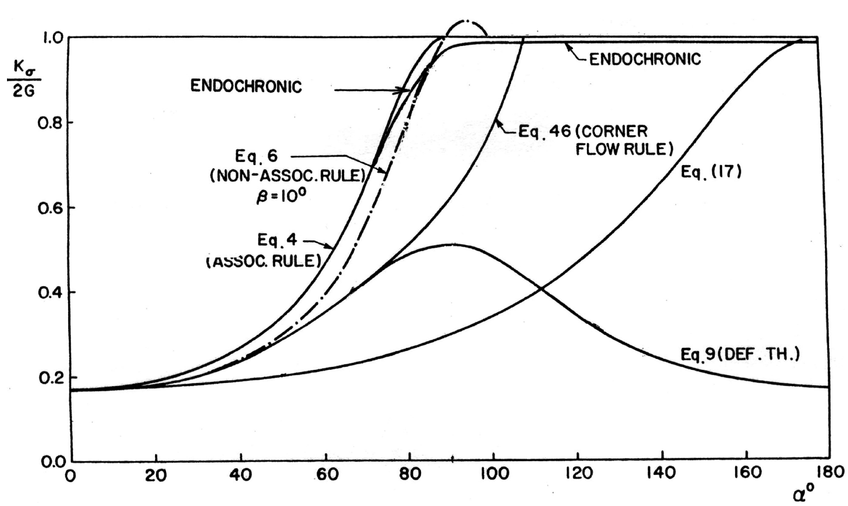


FIG. 4. Variation of normalized directional modulus for various plasticity models.

## 6. CONCLUSIONS

Local directional properties of some incrementally linear and nonlinear constitutive equations of time independent plasticity are explored. It is shown that Mroz' local response moduli provide us with a tool for comparing different models at a fundamental level. Especially, a use of the moduli shows the amount of plasticity present in a model for tangential loading directions (in the sense of associative theory of plasticity) which can be a decisive factor in the suitability of a constitutive model for a boundary value problem at hand.

A model which can be tailored so as to extract a desired amount of plasticity for varying orientations of the stress increment vector, is stated to be a superior model.

It is proven that the $J_2$-flow theory of plasticity is a limiting case of the endochronic theory, but it is not clear if the small amount of plasticity that is ever present in the endochronic theory for all directions of stress increment vector, is its limitation or a blessing. The success of the endochronic theory [8] may be due to an infinitesimal amount of plasticity, discovered in this investigation, for tangential loading direction. Whatever the case may be, the finding brings out a fundamental difference between the $J_2$-flow theory and the endochronic theory.

## ACKNOWLEDGEMENTS

Some very useful discussions the author has had with Professor Z. Mroz, Professor Emeritus at the Institute of Fundamental Technological Research, Warsaw, Poland, are gratefully acknowledged. Without his kind assistance, this work would not have been in existence.

## REFERENCES

1. Z. Mroz and O.C. Zienkiewicz, Uniform Formulation of Constitutive Equations for Clays and Sands, Mechanics of Engineering Materials, John Wiley, 1984.
2. F. Darve, M. Boulong and R. Chambon, Loi Rehologique Incrementale des Sols, J. de Mecanique, vol. 17, pp. 679–716, 1978.
3. D.A. Kolymbas, A Rate-dependent Constitutive Equation for Soils, Mech. Res. Comm.,vol. 4, pp. 367–372, 1977.
4. J. Christoffersen and J.W. Hutchinson, A Class of Phenomenological Corner Theories of Plasticity, J. Mech. Phys. Solids, vol. 27, pp. 465–487, 1979.
5. K.C. Valanis, Fundamental Consequences of a New Intrinsic Time Measure: Plasticity as a Limit of the Endochronic Theory, Arch. Mech., vol. 32, pp. 171–191, 1980.
6. Z. Mroz, On Forms of Constitutive Laws for Elastic-Plastic Solids, Arch. Mech. Stos., vol. 18, pp. 3–33, 1966.
7. K. Runesson and Z. Mroz, A Note on Nonassociated Plastic Flow Rules, Int. J. Plast., vol. 5, pp. 639–658, 1989.
8. S.K. Jain, Five Parameter Endochronic Solid, Mech. Adv. Matl. Struct., vol. 14(6), pp. 453–463, 2007.
9. Z. Mroz, Admissible Loading Paths in the Deformation Theories of Plasticity, Arch. Mech. Stos., vol. 16, pp. 1091–1102, 1964.
10. J.M. Duncan and C.Y. Chang, Nonlinear Analysis of Stress and Strain in Soils, J. Soil Mech. Fdn. Div., ASCE, vol. 96, pp. 1629–1653,1970.
11. H. Murakami and H.E. Read, A Second Order Numerical Scheme for Integrating the Endochronic Plasticity Equations, Comput. Struct., vol. 31(5), pp. 663–672, 1989.
12. H.S. Lamba and O.M. Sidebottom, Biaxial Cyclic Hardening of Annealed Copper Cylinder Evaluated by Deformation Plasticity Theories, T&A.M. Report No. 406, University of Illinois, Urbana, 1976.
13. S.Y. Hsu, S.K. Jain, and O.H. Griffin, A Procedure for Determining Endochronic Material Functions and Verification of Endochronic Theory for Nonproportional Loading Paths, Tech. Rep. VPI-E89-25, Coll. Engrg., Virginia Tech, Blacksburg, 1989.
14. S.Y. Hsu, S.K. Jain, and O.H. Griffin, Verification of Endochronic Theory for Nonproportional Loading Paths, J. Engrg. Mech., ASCE, vol. 117, pp. 110–131, 1991.
15. Z. Mroz and S.K. Jain, On Incremental and Hereditary Descriptions of Hardening Evolutions of Plasticity, Constitutive Laws for Engineering Materials: Theory and Applications (Eds. C.S. Desai et al.), Proc. 3rd Int. Conf., Tucson AZ, ASME Press, NY, 147–152, 1991.
16. S.P. Engelstad, S.K. Jain, and J.N. Reddy, On the Application of the Incremental Theory of Plasticity with Endochronic Hardening Rule, Computational Plasticity: Fundamentals and Applications, Proc. 3rd Intl. Conf., Barcelona, Spain, 271-282, 1992.
17. O. Watanbe and S.N. Atluri, Constitutive Modeling of Cyclic Plasticity and Creep using an Internal Time Concept, Int. J. Plast., vol. 2, 1986.
18. G. Backhaus, ZAMM, vol. 56, pp. 337–348, 1976.

## APPENDIX A

Consider the following formalism of the associative theory of plasticity for a kinematically hardening material.

$$\text{Yield function, } f\left(\sigma_{ij}-\alpha_{ij}\right)=\sigma_0 \tag{i}$$

$$\text{flow rule, } de^p_{ij}=\frac{1}{K_p}n_{ij}(n_{kl}d\sigma_{kl}) \tag{ii}$$

$$\text{consistency condition, } (d\sigma_{ij}-d\alpha_{ij})n_{ij}=0 \tag{iii}$$

$$\text{translation rule, } \alpha_{ij}=\int_0^z \rho_1(z-z')\frac{de^p_{ij}}{dz'}dz' \tag{iv}$$

where, $z=(de^p_{ij}de^p_{ij})^{\frac{1}{2}}$. Differentiating Eq. (iv) we obtain,

$$d\alpha_{ij}=\rho(0)de^p_{ij}-h_{ij}dz \tag{v}$$

$$\text{where, } h_{ij}=-\int_0^z \frac{\partial p_1(z-z')}{\partial z}\frac{de^p_{ij}(z')}{dz'}dz' \tag{vi}$$

Substituting Eqs. (ii) and (v) into Eq. (iii), we obtain,

$$K_p=\rho(0)-n_{ij}h_{ij} \tag{vii}$$

Assuming a von-Mises yield function, this equation reduces to, for a monotonic uniaxial test,

$$K_p=\sum_{r=2}^{n}A_r e^{-\alpha_r z}=\rho_1(z)=\frac{2}{3}\frac{d\sigma_x}{d\varepsilon^p_x} \tag{viii}$$

Integrating this equation beyond initial yield, we obtain the expression for the stress-plastic strain curve.

$$\sigma_X = \sigma_0 + \sqrt{3/2}\sum_{r=2}^{n}\frac{A_r}{\alpha_r}\left[1-\exp\left(-\sqrt{3/2}\alpha_r\varepsilon_X^p\right)\right] \quad \text{(ix)}$$

The hereditary translation rule, Eq. (iv) appears to have been first used by Backhaus [18] within the formalism of associative theory of plasticity.

## APPENDIX B

Considering the endochronic equation

$$s_{ij} = \int_0^z \rho(z-z')\frac{de_{ij}^p}{dz'}dz' \quad \text{(x)}$$

and assuming the kernel ρ(z) to be of the form

$$\rho(z) = A_0\delta(z) + \rho_1(z) \quad \text{(xi)}$$

where $\delta(z)$ represents a dirac-delta function, we obtain

$$s_{ij} = A_0\frac{de_{ij}^p}{dz} + \alpha_{ij} \quad \text{(xii)}$$

where $\alpha_{ij}$ is as given in Eq. (iv).

From Eq. (xi) we can obtain the following function which has been interpreted by Valanis [5] to form an elastic domain:

$$\left[\frac{3}{2}(s_{ij}-\alpha_{ij})(s_{ij}-\alpha_{ij})\right]^{\frac{1}{2}} = \sqrt{3/2}A_0 = \sigma_0 \quad \text{(xiii)}$$

Eq. (xi) can be rearranged to have

$$de_{ij}^p = n_{ij}dz; \quad n_{ij} = (s_{ij}-\alpha_{ij})\Big/\sqrt{2/3}\sigma_0 \quad \text{(xiv)}$$

Applying the consistency condition, Eq. (iii), it is easily found that

$$dz = \frac{n_{ij}d\sigma_{ij}}{\rho(0) - h_{ij}n_{ij}}. \quad \text{(xv)}$$

Thus, we find that, for the von-Mises yield function, Appendices A and B provide identical answers.

## APPENDIX C. NOMENCLATURE

Bold faced small letters and bold faced Greek letters are vector quantities. Superscripts "e" and "p" respectively denote elastic and plastic components. Subscript "s" implies secant and "t" implies tangent. Prefix "d" stands for an infinitesimal increment.

| | |
|---|---|
| A, B | material parameters, Eq. (13) |
| **b** or $b_{ij}$ | unit vector denoting the direction of plastic flow or the direction of stress deviator |
| $e_{ij}$ | strain deviator |
| $de^p = ds = \left(de_{ij}^p de_{ij}^p\right)^{\frac{1}{2}}$, | norm of plastic strain increment vector |
| f | hardening function in endochronic theory, Eq. 19 |
| μ or G | shear modulus |
| **h** or $h_{ij}$ | defined in Eq. (22) |
| $h = (h_{ij}h_{ij})^{\frac{1}{2}}$, | the norm of **h** |
| L, M | material functions |
| $K_\sigma$, $K_\varepsilon$ | directional modulii defined in Eq. (1) and (2) |
| $K_p$ | normalized plastic hardening modulus, Eq. (3) |
| $\mathbf{n}_f$ or $\mathbf{n}_{ij}$ | unit vector normal to the yield surface |
| **s** or $s_{ij}$ | stress deviator |
| z | intrinsic or endochronic time scale, Eq. (22) |
| α | angle between the stress increment vector and the direction of plastic flow (the plastic strain increment vector or the stress deviator vector). |
| γ | angle between the strain increment vector and the direction of plastic flow. |
| ε or $\varepsilon_{ij}$ | strain deviator |
| **σ** or $\sigma_{ij}$ | stress tensor |
| $d\varepsilon_\sigma$ | component of strain increment vector in the direction of stress increment vector, Eq. (1) |
| $\rho_1(z)$, ρ(z) | material functions in endochronic theories with and without a yield surface, respectively. |